

MATH 530 Qualifying Exam

January 1998

Notation: $D_r(a)$ denotes the disk, $\{z \in \mathbb{C} : |z - a| < r\}$.

1. (15 pts) Evaluate the integral $\int_{-\infty}^0 \frac{x^2}{x^4 + x^2 + 1} dx$.
2. (15 pts) Find a one-to-one analytic map from $D_1(0) \cap \{x + iy : x, y > 0\}$ onto $D_1(0)$.
3. (25 pts) Let \mathcal{F} denote the set of analytic functions f on $D_1(0)$ such that $|f(z)| < 1$ for all $z \in D_1(0)$, $f(0) = 0$, and $f'(0) = 0$. Prove that if $f \in \mathcal{F}$, then $|f(z)| \leq |z|^2$ for all $z \in D_1(0)$. Let $M = \sup\{|f''(0)| : f \in \mathcal{F}\}$. Find all functions, if any, in \mathcal{F} such that $|f''(0)| = M$.
4. (15 pts) How many zeroes does the polynomial

$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.

5. (15 pts) Prove that a harmonic function cannot have an isolated zero.
6. (15 pts) Let $C_1(0)$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and let f be a function that is analytic on $D_r(0)$ for some $r > 1$. Prove that if $f(C_1(0)) \subset C_1(0) \setminus \{1\}$, then f is a constant function.