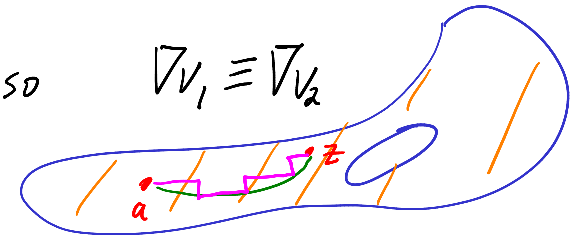
Midterm exam solutions

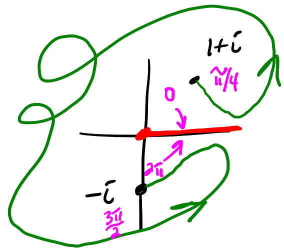
1. C-R eqns: $\begin{cases} \dfrac{\partial v_1}{\partial y} = \dfrac{\partial v_2}{\partial y} = \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v_1}{\partial x} = \dfrac{\partial v_2}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$ so $\nabla v_1 \equiv \nabla v_2$



Let $v = v_1 - v_2$. $\nabla v \equiv 0 \Rightarrow v \equiv \text{const}$ on a domain. ✓

2. Use branch $\log_0 z = \operatorname{Ln}|z| + i\theta$ where $\theta \in \arg z$ with $0 < \theta < 2\pi$.

Then $\displaystyle\int_\gamma \frac{1}{z}\,dz = \int_\gamma \frac{d}{dz}\left(\log_0 z\right) dz = \Big[\log_0 z\Big]_{1+i}^{-i}$



$= \left(\operatorname{Ln}|-i| + i\frac{3\pi}{2}\right) - \left(\operatorname{Ln}|1+i| + i\frac{\pi}{4}\right)$

$= \left(0 + i\frac{3\pi}{2}\right) - \left(\operatorname{Ln}\sqrt{2} + i\frac{\pi}{4}\right) = -\operatorname{Ln}\sqrt{2} + i\frac{5\pi}{4}$

3. Let $f = u + iv$. $e^f = e^{u+iv} = e^u e^{iv}$ is entire.

$|e^f| = \underbrace{|e^u|}_{e^u}\,\underbrace{|e^{iv}|}_{1} = e^u \cdot 1$ is bounded

because $u < 0 \Rightarrow e^u < e^0 = 1$ (because e^x is $\nearrow$).

Liouville's $\Rightarrow$ $e^f \equiv$ const. Claim: $e^f \equiv$ const $\Rightarrow$

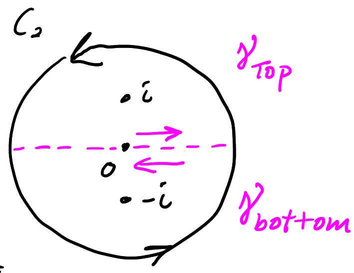
$f \equiv$ const because $|e^f| = e^u \equiv c > 0$ means $u \equiv \operatorname{Ln} c$.

C-R eqns yield $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} \equiv 0$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \equiv 0$.

So $\nabla v \equiv 0$ and we conclude v must be constant too.

So $f = u + iv$ is constant. ✓

4. $\dfrac{e^{iz}}{z^2+1} = \dfrac{e^{iz}}{(z-i)(z+i)}$



Let $f(z) = \dfrac{e^{iz}}{z+i}$, $g(z) = \dfrac{e^{iz}}{z-i}$

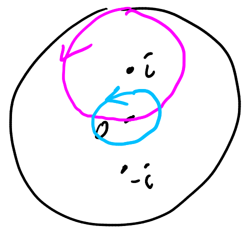
$$\frac{1}{2\pi i}\int_{\gamma_{top}} \frac{\overbrace{e^{iz}/z+i}^{f(z)}}{z-i}\,dz = f(i)$$

a) $$\int_{C_2(0)} \frac{e^{iz}}{z^2+1}\,dz = \left(\int_{\gamma_{Top}} + \int_{\gamma_{bottom}}\right)$$

$$= 2\pi i\, f(i) + 2\pi i\, g(-i) \quad \text{by Cauchy int formula}$$

$$= 2\pi i \frac{e^{i\cdot i}}{i+i} + 2\pi i \frac{e^{i\cdot(-i)}}{i-(-i)} = \pi\left(e^{-1}+e\right)$$

b) $$\int_{C_1(i)} = \underset{\substack{\uparrow \\ \text{clockwise}}}{-}\, 2\pi i\, f(i) = -2\pi i \frac{e^{i\cdot i}}{i+i} = -\frac{\pi}{e}$$



c) $\int_{C_{1/2}(0)} = 0$ by Cauchy's thm since the integrand is analytic inside and on γ.

or use partial fractions:

$$\frac{1}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}$$

$$1 = A(z+i) + B(z-i) = \underbrace{(A+B)}_{0}z + i\underbrace{(A-B)}_{1}$$

$$A = \frac{-i}{2} \qquad B = \frac{i}{2}$$