Math 428 Midterm Exam

Each problem is worth 25 points.

1. Assume that

$$S(x) = \sum_{n=1}^{\infty} B_n \sin nx$$

is the sum of the Fourier sine series on $[0, \pi]$ for the function $f(x)$ that is equal to one on the interval $[0, \pi]$.

a) Write an integral formula for B_n and evaluate the integral.

b) Graph the function that the Fourier sine series converges to on $[-2\pi, 2\pi]$, being careful at jumps and endpoints.

c) Evaluate the sum

$$\sum_{n=1}^{\infty} {B_n}^2.$$

Hint: The Fourier sine series is equal to the full Fourier series of the function on $[-\pi, \pi]$ you graphed in part (b).

2. Given a continuous *even* real valued function $g(x)$ on $[-\pi, \pi]$, what trigonometric polynomial of the form

$$T_N(x) = A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_N \cos Nx$$

minimizes

$$\int_{-\pi}^{\pi} |g(x) - T_N(x)|^2 \, dx?$$

Explain.

3. Show that the three functions x, $\cos 2x$, and 1 are orthogonal on $[-\pi, \pi]$. (Simple facts about even and odd functions save work here.) If

$$f(x) = c_0 + c_1 x + c_2 \cos 2x,$$

write integral formulas for the coefficients c_n involving $f(x)$ and the given functions.

4. Show that the function given in polar coordinates as $r^n \cos n\theta$ is a polynomial in x and y in Cartesian coordinates.
Hint: DeMoivre's formula is useful here.

Formula sheet, OVER

Important Formulas

Fourier sine series for $f(x)$ on $[0,\pi]$ is $\displaystyle\sum_{n=1}^{\infty} B_n \sin nx$ where

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx$$

Full Fourier series for $f(x)$ on $[-\pi,\pi]$ is $\displaystyle a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

The complex version is $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$ where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ dx$$

Facts: $c_0 = a_0$, and if $n \geq 1$: $c_n = \frac{a_n - ib_n}{2}$ and $c_{-n} = \frac{a_n + ib_n}{2}$

Bessel's inequalities (Parseval's identities with equality)

$$2a_0^2 + \sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$