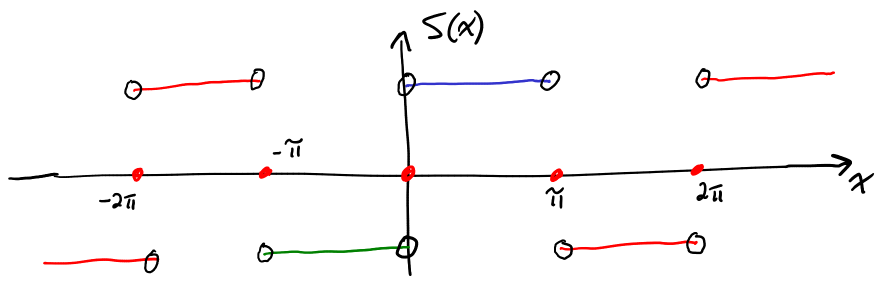
Midterm exam solutions

1. a) $$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos n\pi - \left(-\frac{1}{n}\cos 0\right)\right] = \frac{2}{\pi n}\Big(1 - \underbrace{\cos n\pi}_{(-1)^n}\Big)$$

$$= \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

b)



odd 2π-periodic extension.
$S(x)$ = midpoint of jump at jumps

c) $\sum_{n=1}^{\infty} B_n \sin nx$ is the full Fourier series for $S(x)$, which is piecewise C^1-smooth. So Parseval's identity yields

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |S(x)|^2\,dx = \sum_{n=1}^{\infty} B_n^2$$

$$\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} 1\,dx}_{=2} = \left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \cdots$$

2. Since g is even on $[-\pi,\pi]$, its Fourier series has no sine terms. The trig poly of order N that best approximates $g(x)$ in $L^2[-\pi,\pi]$ is the partial sum $S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos nx$ where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx \quad \text{and} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx.$$

3. $$\langle x, \cos 2x\rangle = \int_{-\pi}^{\pi} \underbrace{\underset{\text{odd}}{x}\ \underset{\text{even}}{\cos 2x}}_{\text{odd}}\,dx = 0$$ because $[-\pi,\pi]$ is symmetric.

$$\langle x, 1\rangle = \int_{-\pi}^{\pi} \underbrace{\underset{\substack{\uparrow\\ \text{odd}}}{x} \cdot \underset{\substack{\uparrow\\ \text{even}}}{1}}_{\text{odd}} \, dx = 0$$

$$\langle \cos 2x, 1\rangle = \int_{-\pi}^{\pi} 1\cdot \cos 2x \, dx = \left[\tfrac{1}{2}\sin 2x\right]_{-\pi}^{\pi} = \tfrac{1}{2}\left(\sin 2\pi - \sin(-2\pi)\right)$$

$$= 0 - 0 = 0$$

So the functions are $\perp$.

Multiply $f(x) = c_0 + c_1 x + c_2 \cos 2x$ by $1, x, \cos 2x$

$$\int_{-\pi}^{\pi} 1\cdot f(x)\, dx = c_0 \int_{-\pi}^{\pi} 1\cdot 1 \, dx + 0 + 0$$

$$\int_{-\pi}^{\pi} x\cdot f(x)\, dx = 0 + c_1 \int_{-\pi}^{\pi} x\cdot x \, dx + 0$$

$$\int_{-\pi}^{\pi} \cos 2x \, f(x)\, dx = 0 + 0 + c_2 \int_{-\pi}^{\pi} \cos^2 2x \, dx$$

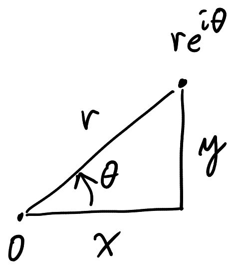
$$c_0 = \frac{\int_{-\pi}^{\pi} f(x)\, dx}{\int_{-\pi}^{\pi} 1 \, dx} \leftarrow = 2\pi$$

$$c_1 = \frac{\int_{-\pi}^{\pi} x f(x)\, dx}{\int_{-\pi}^{\pi} x^2 \, dx} \leftarrow = \frac{2\pi^3}{3}$$

$$c_2 = \frac{\int_{-\pi}^{\pi} f(x) \cos 2x \, dx}{\int_{-\pi}^{\pi} \cos^2 2x \, dx} \leftarrow = \int_{-\pi}^{\pi} \tfrac{1}{2}(1 + \cos 4x)\, dx = \pi$$

4. $$r^n \cos n\theta = \mathrm{Re}\left[r^n \underbrace{\left(\cos n\theta + i \sin n\theta \right)}_{\left(e^{i\theta}\right)^n} \right]$$

$$= \mathrm{Re}\left[\left(\underbrace{re^{i\theta}}_{=x+iy} \right)^n \right]$$



$$= \mathrm{Re}\left[(x+iy)^n \right]$$

is a polynomial in x and y because

$$= \mathrm{Re}\left[\sum_{k=0}^{n} \binom{n}{k} x^k (iy)^{n-k} \right]$$

i^{n-k}

even terms $= \pm 1$, odd terms $= \pm i$

<u>or</u> $$r^n \cos n\theta = r^n \left[\frac{e^{in\theta} + e^{-in\theta}}{2} \right] = \frac{(re^{i\theta})^n + (re^{-i\theta})^n}{2}$$

$$= \frac{(x+iy)^n + (x-iy)^n}{2}$$