Lesson 14 on 4.6

Practice Exams on Home Page
Review lectures M, W

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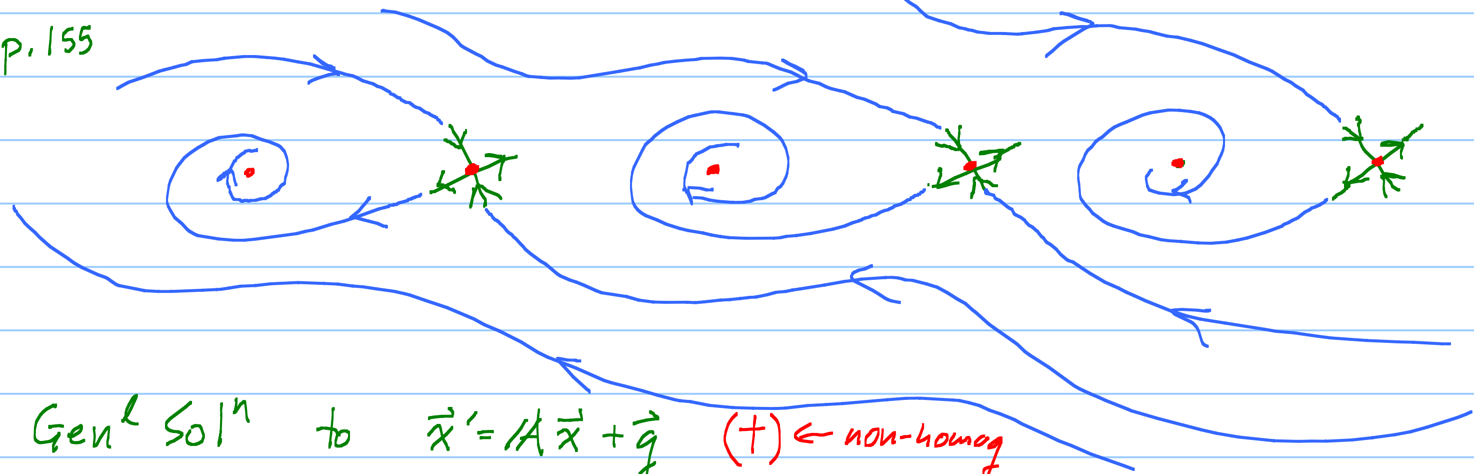

Gen^l Sol^n to $\vec{x}\,' = \mathbb{A}\vec{x} + \vec{g}$ (†) ← non-homog

$\vec{x}\,' = \mathbb{A}\vec{x}$ (∗) ← homog

Fact: Gen^l Sol^n to (†) is $\vec{x} = \underbrace{\vec{x}_c}_{\text{homog sol}^n} + \underbrace{\vec{x}_p}_{\text{part sol}^n}$

Why: Suppose $\vec{x}$ solves (†).

Key: $\vec{x} - \vec{x}_p$ solves (∗): $(\underline{\vec{x} - \vec{x}_p})' = \vec{x}\,' - \vec{x}_p\,'$

$= (\mathbb{A}\vec{x} + \vec{g}) - (\mathbb{A}\vec{x}_p + \vec{g})$

$= \mathbb{A}(\underline{\vec{x} - \vec{x}_p})$

Hence $\vec{x} - \vec{x}_p = \vec{x}_c$ for some choice of c's.

So $\vec{x} = \vec{x}_c + \vec{x}_p$ ✓

Also need to check that <u>all</u> fcns $\vec{x}_c + \vec{x}_p$ actually satisfy (†).

<u>Method of Undetermined Coeff</u>: $\vec{x}\,' = \underbrace{\mathbb{A}}_{\text{const. coeff}}\vec{x} + \underbrace{\vec{g}}_{\text{special}}$

$\vec{g}$	Use trial sol^n $\vec{x}_p =$
$\vec{a}e^{rt}$	$\vec{b}e^{rt}$ ← danger if r is eval for $\mathbb{A}$
$\vec{a}t^2$	$\vec{b}_2 t^2 + \vec{b}_1 t + \vec{b}_0$ ← danger if 0 eval
$\vec{a}_n t^n + \cdots + \vec{a}_0$	$\vec{b}_n t^n + \cdots + \vec{b}_0$

$\vec{a}\cos\omega t$ or $\vec{a}\sin\omega t$ or $\vec{a}_1\cos\omega t + \vec{a}_2\sin\omega t$	$\vec{A}\cos\omega t + \vec{B}\sin\omega t \leftarrow$ danger $\pm\omega i$ e-vals
$\vec{a}\,t^3 e^{3t}\sin 4t$	...

Danger: If any single term in the trial sol^n might solve the homog, method might bomb.

Variation of parameters: $\vec{x}\,' = \mathbb{A}\vec{x} + \vec{g}$

$\mathbb{A}$: coeff can depend on t; $\vec{g}$: anything

Step 1: Solve homog syst

$$\vec{x}\,' = \mathbb{A}\vec{x} \quad (*)$$

Get $\vec{x}_c = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n \leftarrow \mathbb{A}$ $n\times n$

$$\vec{x}_c = \underbrace{\left[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\right]}_{\text{Fundamental Matrix}} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$c_1, \ldots, c_n$: "parameters"

$$\boxed{\vec{x}_c = \mathbb{X}\vec{c}}$$

Step 2: Try $\vec{x}_p = \mathbb{X}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \underline{\underline{\mathbb{X}\vec{u}}}$ $\leftarrow$ turn c's into fcns

Want $\vec{x}_p\,' \overset{\text{want}}{=} \mathbb{A}\vec{x}_p + \vec{g}$

$$(\mathbb{X}\vec{u})' = \mathbb{A}(\mathbb{X}\vec{u}) + \vec{g}$$

prod. rule $\rightarrow$

$$\left(\underbrace{\mathbb{X}'\vec{u}} + \mathbb{X}\vec{u}\,'\right) = \underbrace{(\mathbb{A}\mathbb{X})\vec{u}} + \vec{g}$$

cancel!

Why: $\mathbb{X}' = [\vec{x}_1', \dots, \vec{x}_n'] = [\mathbb{A}\vec{x}_1, \dots, \mathbb{A}\vec{x}_n]$

$$= \mathbb{A}[\vec{x}_1, \dots, \vec{x}_n]$$

$$= \mathbb{A}\mathbb{X}$$

Leaving $\boxed{\mathbb{X}\vec{u}' = \vec{g}}$ ← The big formula

$\mathbb{A}$ nxn big.
Gauss elim to get $\vec{u}'$

$\mathbb{A}$ 2x2: $\boxed{\vec{u}' = \mathbb{X}^{-1}\vec{g}}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

EX: $\vec{x}' = \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-2t}$ danger! -2 is an eval

1) Homog sol'n $\vec{x}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-4t}$

$$\mathbb{X} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-4t}\right]$$

$$= \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}$$

2)

$$\mathbb{X}^{-1} = \frac{1}{-e^{-6t} - e^{-6t}}\begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}$$

$$\mathbb{X}^{-1} = \begin{bmatrix} \frac{1}{2}e^{2t} & \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{4t} & -\frac{1}{2}e^{4t} \end{bmatrix}$$

3) $$\vec{u}\,' = X^{-1}\vec{g} = X^{-1}\underbrace{\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}}_{\vec{g}}$$

$$\vec{u}\,' = \begin{pmatrix} \underline{\frac{1}{2}e^{2t}\cdot e^{-2t} + \frac{1}{2}e^{2t}\cdot 2e^{-2t}} \\ \frac{1}{2}e^{4t}\cdot e^{-2t} - \frac{1}{2}e^{4t}\cdot 2e^{-2t} \end{pmatrix} = \begin{pmatrix} 3/2 \\ -\frac{1}{2}e^{2t} \end{pmatrix}$$

$$\begin{cases} u_1' = 3/2 \\ u_2' = -\frac{1}{2}e^{2t} \end{cases}$$

Now $u_1 = \int \frac{3}{2}\,dt = \frac{3}{2}t + c_1$ ← take $c_1 = 0$ Part.!

$u_2 = \int -\frac{1}{2}e^{2t}\,dt$

$= -\frac{1}{4}e^{2t} + c_2$ ← take $c_2 = 0$

Finally $$\vec{x}_p = X\vec{u} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}\begin{pmatrix} \frac{3}{2}t \\ -\frac{1}{4}e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} \underline{\frac{3}{2}t e^{-2t} - \frac{1}{4}e^{-4t}\cdot e^{2t}} \\ \underline{\frac{3}{2}t e^{-2t} + \frac{1}{4}e^{-4t}\cdot e^{2t}} \end{pmatrix}$$

$$\underline{\underline{\vec{x}_p = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} t e^{-2t} + \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix} e^{-2t}}}$$

Hmmm. Method of Undet. Coeff.

Try $\vec{a}\,t e^{\lambda t} + \vec{b}\,e^{\lambda t}$

<u>Great tool:</u> $\cos\omega t = \mathrm{Re}\; e^{i\omega t}$

$\vec{x}\,' = A\vec{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos\omega t$ ← change $\vec{g}$ to $\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{i\omega t}$

Try $\vec{x}_p = \underbrace{\vec{b}}_{\text{complex}} e^{i\omega t}$. Get a complex sol^n!

Take $\mathrm{Re}\,\vec{x}_p$ to get sol^n to original prob.

Linearized systems v.s. Non-linear syst.

evals	Linear syst	Non-linear Syst
$\lambda_2 > \lambda_1 > 0$	Imp. Node, Unstable	Same!
$\lambda_2 < \lambda_1 < 0$	Imp. Node, A. Stable	Same!
$\lambda_2 < 0 < \lambda_1$	Saddle, Unstable	Same!
$\lambda = a \pm bi$ $a < 0$	Spiral in A. Stable CW or CCW	Same!
$a > 0$	Spiral out Unstable CW or CCW	Same!
$a = 0$ $\lambda = \pm bi$	Center, Stable	Center or Spiral in or Spiral out or
$\lambda_1 = \lambda_2$ $(\neq 0)$	Proper Node (2 e-vects) Degenerate Node 1 evect	Node or a Spiral (Stability same!)