Lesson 23 on 6.6

21, 22 due Wed
23, 24, 25 next Wed

WebEx 8-9pm Weds.

$\mathcal{L}[f'(t)] = sF(s) - f(0)$ ← old

$\mathcal{L}[t f(t)] = -F'(s)$ ← new   Note symmetry

EX: $\mathcal{L}[t \underbrace{\sin \omega t}_{f(t)}] = -F'(s) = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$

$= -\frac{d}{ds}\left[\omega(s^2+\omega^2)^{-1}\right] = -\omega(-1)(s^2+\omega^2)^{-2}(2s)$

$= \frac{2\omega s}{(s^2+\omega^2)^2}$

EX: $\mathcal{L}[t \cos \omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2} \overset{\text{partial frac}}{=} \frac{As+B}{(s^2+\omega^2)^2} + \frac{Cs+D}{(s^2+\omega^2)}$

$= \frac{(s^2+\omega^2)-\omega^2-\omega^2}{(s^2+\omega^2)^2}$

$= \frac{1}{(s^2+\omega^2)} - \frac{2\omega^2}{(s^2+\omega^2)^2}$

↑ $\mathcal{L}[\frac{1}{\omega}\sin \omega t]$

$f(t)$	$F(s)$
$\frac{1}{2\omega^2}(\sin \omega t - \omega t \cos \omega t)$	$\frac{1}{(s^2+\omega^2)^2}$
$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2+\omega^2)^2}$

Why: $F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt$

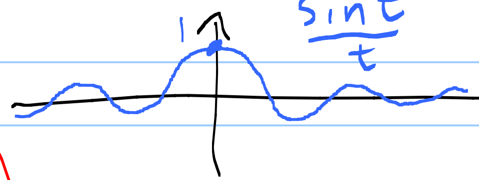
$= \int_0^\infty -t\,e^{-st} f(t)\,dt$ ✓

Fact: If $f(0)=0$ and $\lim_{t\to 0+} \frac{f(t)}{t}$ exists, ← true if f is diff'ble at 0

then $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$

Note: $\frac{f(t)}{t} = \underbrace{\frac{f(t)-\overset{0}{f(0)}}{t-0}}_{DQ}$ → Right hand derivative of f at 0.

Prob: Find $\mathcal{L}\left[\frac{\sin t}{t}\right]$



$f(t)=\sin t$

$f(0)=0$ ✓

$\frac{f(t)}{t} = \frac{\sin t}{t} \to 1$ as $t\to 0+$ ✓

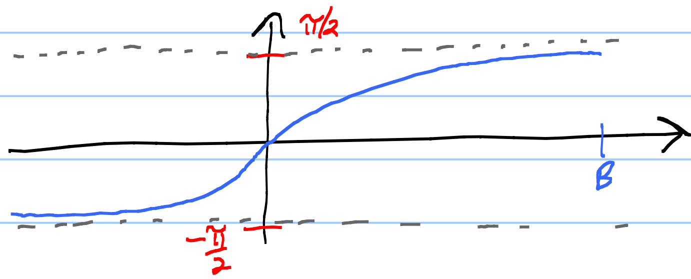
$F(s)=\mathcal{L}[\sin t] = \frac{1}{s^2+1}$

$= \int_s^\infty F(s)\,ds$

$= \int_s^\infty \frac{1}{s^2+1}\,ds$

$= \operatorname{ArcTan} s \Big|_s^\infty$

$= \lim_{B\to\infty}\left[\operatorname{ArcTan} B - \operatorname{ArcTan} s\right]$



$\lim = \frac{\pi}{2} - \operatorname{Tan}^{-1} s$

Special ODEs with variable coeff:

Laguerre Eqn: $ty'' + (1-t)y' + ny = 0$

$\mathcal{L}[ty'] = -\frac{d}{ds}\mathcal{L}[y'] = -\frac{d}{ds}\left[sY - y(0)\right]$

$= -Y - sY' + 0$ (∗₁)

$$\mathcal{L}[t\,y''] = -\frac{d}{ds}\mathcal{L}[y''] = -\frac{d}{ds}\left[s^2 Y - s\,y(0) - y'(0)\right]$$

$$= -2sY - s^2 Y' + y(0) \qquad (*_2)$$

$$\mathcal{L}[\text{ODE}] = \left(-2sY - s^2 Y' + y(0)\right) + \left(sY - y(0)\right) - \left(-Y - sY'\right) + nY = 0$$

cancel!!

$$(s - s^2)Y' + (n+1-s)Y = 0$$

First order ODE in Y from a second order one in y!

Separable!

$$\frac{dY}{ds} = \frac{-(n+1-s)}{(s-s^2)}\,Y$$

Separate vars.

$$\frac{dY}{Y} = -\frac{(n+1-s)}{(s-s^2)}\,ds$$

Integrate:

$$\int \frac{dY}{Y} = -\int \frac{(n+1-s)}{s(1-s)}\,ds$$

$$\frac{A}{s} + \frac{B}{1-s} = \frac{n}{s-1} - \frac{n+1}{s}$$

$$\ln|Y| = n\ln(s-1) - (n+1)\ln s + C$$

Exponentiate: $e^{\ln|Y|} = |Y|$

$$|Y| = e^{\ln(s-1)^n}\, e^{\ln s^{-(n+1)}}\, e^{C}$$

$$Y = \underbrace{\pm e^{C}}_{K}\,(s-1)^n\, s^{-(n+1)}$$

$$Y = K\,\frac{(s-1)^n}{s^{n+1}}$$

Take $K=1$, $n=3$: $\quad Y = \dfrac{(s-1)^3}{s^4} = \dfrac{1}{s} - \dfrac{3}{s^2} + \dfrac{3}{s^3} - \dfrac{1}{s^4}$

So $\quad y = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3$

Hmmm. Only got y_1. Need y_2.

Reduction of order: Put Laguerre Eqn in **Standard Form:**

$$y'' + Py' + Qy = 0$$

$$y'' + \left(\frac{1}{t} - 1\right)y' + \frac{3}{t}y = 0$$

Try $y_2 = u y_1$. Need $u' = \frac{1}{y_1^2} e^{-\int P\,dt}$

$$= \frac{1}{t y_1^2} e^t$$

u has a log singularity at $t=0$.
So does y_2.

Back to: 

$$\ddot{x}_1 = -2x_1 + x_2 \quad (A)$$

$$\ddot{x}_2 = x_1 - 2x_2 \quad (B)$$

$$\begin{cases} u_1 = x_1 \\ u_2 = \dot{x}_1 \\ u_3 = x_2 \\ u_4 = \dot{x}_2 \end{cases} \qquad \begin{cases} u_1' = \dot{x}_1 = u_2 \\ u_2' = \ddot{x}_1 \overset{(A)}{=} -2x_1 + x_2 = -2u_1 + u_3 \\ u_3' = \dot{x}_2 = u_4 \\ u_4' = \ddot{x}_2 \overset{(B)}{=} u_1 - 2u_3 \end{cases}$$

$$\vec{u}\,' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} \vec{u}$$

$$(\lambda^2+1)(\lambda^2+3) = 0$$

$$\lambda = \pm i, \pm\sqrt{3}\,i$$

For $\lambda = i$: Get complex e-vect $\begin{pmatrix} 1 \\ i \\ 1 \\ i \end{pmatrix}$

Get two real sol'ns from $\left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right](\cos t + i \sin t)$

For $\lambda = \sqrt{3}\, i$: Same

$$\vec{u} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_4 \vec{u}_4 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{matrix} \leftarrow x_1 \\ \\ \leftarrow x_2 \\ \\ \end{matrix}$$

Maple demo