MA 16100
Study Guide - Exam # 2

(1) Average rate of change of $y = f(x)$ over the interval $[x_1, x_2]$: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$; average velocity. Definition of derivative of $y = f(x)$ at a: $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ or, equivalently, $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$; interpretation of derivative:

$$f'(a) = \begin{cases} \text{slope of tangent line the graph of } y = f(x) \textit{ at } a \\ \text{velocity at time } a \\ \text{(instantaneous) rate of change of } f \text{ at } a \end{cases}$$

(2) Derivative as a function: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$; differentiable functions (i.e., $f'(x)$ exists); higher order derivatives.

(3) **<u>Differentiation Formulas</u>**

(1) $\dfrac{d(c)}{dx} = 0$ (2) $\dfrac{d\,(x^n)}{dx} = nx^{n-1}$ (3) $\dfrac{d\,(e^x)}{dx} = e^x$

(4) $\dfrac{d\,(\sin x)}{dx} = \cos x$ (5) $\dfrac{d\,(\cos x)}{dx} = -\sin x$ (6) $\dfrac{d\,(\tan x)}{dx} = \sec^2 x$

(7) $\dfrac{d\,(\sec x)}{dx} = \sec x\,\tan x$ (8) $\dfrac{d\,(\csc x)}{dx} = -\csc x\,\cot x$ (9) $\dfrac{d\,(\cot x)}{dx} = -\csc^2 x$

(10) $\dfrac{d\,(a^x)}{dx} = a^x(\ln a)$ (11) $\dfrac{d\,(\ln x)}{dx} = \dfrac{1}{x}$ (12) $\dfrac{d\,(\log_a x)}{dx} = \dfrac{1}{x \ln a}$

(13) $\dfrac{d\left(\sin^{-1} x\right)}{dx} = \dfrac{1}{\sqrt{1-x^2}}$ (14) $\dfrac{d\,(\cos^{-1} x)}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$ (15) $\dfrac{d\,(\tan^{-1} x)}{dx} = \dfrac{1}{1+x^2}$

(4) **<u>Differentiation Rules</u>**: Suppose f and g are differentiable functions, and c is a constant.

(a) *Constant Rule* : $\dfrac{d(c)}{dx} = 0$

(b) *Sum Rule* : $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

(c) *Difference Rule* : $\dfrac{d(u-v)}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}$

(d) *Product Rule* : $\dfrac{d(uv)}{dx} = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$

(e) *Quotient Rule* : $\dfrac{d\left(\dfrac{u}{v}\right)}{dx} = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$

(5) *Special Trig Limits* : $\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$ $\boxed{\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1}$ $\boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0}$.

(6) **<u>CHAIN RULE</u>** : If g is differentiable at x and f is differentiable at $g(x)$, then the composition function $f \circ g$ is differentiable at x and its derivative is

$$f(g(x))' = f'(g(x))\, g'(x)$$

i.e., if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

(7) **<u>Implicit Differentiation</u>** : Given an equation with two variables that defines one variable as a function of the other (independent) variable, differentiate the equation with respect to the independent variable using the Chain Rule (if $y = y(x)$, then $\frac{d(y^n)}{dx} = ny^{n-1}\frac{dy}{dx}$) and then solve for the desired derivative.

(8) **<u>Logarithmic Differentiation</u>**

`Step 1` : Take the natural log of both sides of $y = f(x)$ and simplify using Law of Logarithms

`Step 2` : Differentiate implicitly w.r.t x

`Step 3` : Solve the resulting equation for $\frac{dy}{dx}$

(9) **Applications**

(a) <u>*Physics*</u>: If $s = f(t)$= position of an object moving in a straight line then $v = \frac{ds}{dt}$ = velocity; $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ = acceleration.

(b) <u>*Economics*</u>: If $C(x)$ = cost to produce x units, then $C'(x) = \frac{dC}{dx}$ = marginal cost function; Basic formula $\boxed{C(n+1) - C(n) \approx C'(n)}$, where $C(n+1) - C(n)$ = exact cost to produce the $(n+1)^{st}$ item and $C'(n)$ = marginal cost of producing n units.

(c) <u>*Biology*</u>: If $n(t)$ = population at time t, then $\frac{dn}{dt}$ = population growth rate.

(10) *Exponential Growth/Decay* : $\boxed{\frac{dy}{dt} = k\,y}$ if $k > 0$ this is the law of natural growth; while if $k < 0$ this is the law of natural decay. The solutions are all of the form $y(t) = C\, e^{kt}$.

(11) *Compound Interest* : An initial amount of A_0 dollars is invested in an account earning an interest rate of r compounded n times per year for t years will be

$$A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

If the interest is compounded continuously, then $A(t) = A_0\, e^{rt}$.

Recall that $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$.

(12) *Newton's Law of Cooling* : If $T(t)$ is the temperature of the object at time t and if T_s = temperature of the environment, then

$$\boxed{\frac{dT}{dt} = k\,(T - T_s)}$$

All solutions have the form $T(t) = T_s + C\, e^{kt}$.