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Phase Portraits for a linear system: $\mathbf{x}' = A\,\mathbf{x}$

Given the general solution to $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$, the following describes how to sketch solutions (trajectories) of the system. A plot of the trajectories of a given homogeneous system

$$\mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}$$

is called a **Phase Portrait** of the system. To sketch the phase portrait, we need to find the corresponding eigenvalues λ_1, λ_2 and eigenvectors of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and then there are various cases.

Useful Fact: If λ_1, λ_2 are the eigenvalues of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

(i) $\det A = (ad - bc) = \lambda_1\,\lambda_2$

(ii) $\operatorname{trace} A = (a + d) = \lambda_1 + \lambda_2$

★ This fact is needed in order to use the **Stability Diagram** ← Click here

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Recall, the method to solve linear systems $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors:

The Eigenvalue Method : If λ is an eigenvalue of the matrix A and $\mathbf{v}$ a corresponding eigenvector, then $\boxed{\mathbf{x}(t) = e^{\lambda t}\mathbf{v}}$ is a solution to $\mathbf{x}' = A\mathbf{x}$ (#).

Case 1 $\lambda_1, \lambda_2, \cdots, \lambda_m$ *are real and distinct eigenvalues* (each with multiplicity 1).

If $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots \mathbf{v}^{(m)}$ are their corresponding eigenvectors $\implies$ m independent solutions to (#):

$$\mathbf{x}^{(1)}(t) = e^{\lambda_1 t}\mathbf{v}^{(1)},\ \mathbf{x}^{(2)}(t) = e^{\lambda_2 t}\mathbf{v}^{(2)},\ \cdots\ \mathbf{x}^{(m)}(t) = e^{\lambda_m t}\mathbf{v}^{(m)}$$

Case 2 $\lambda = \alpha + i\beta$ *is a complex eigenvalue.*

If $\mathbf{v}$ is a corresponding eigenvector, then 2 independent real-valued solutions to (#)

$$\mathbf{x}^{(1)}(t) = \Re e\left\{e^{\lambda t}\mathbf{v}\right\} \text{ and } \mathbf{x}^{(2)}(t) = \Im m\left\{e^{\lambda t}\mathbf{v}\right\}$$

Case 3 λ *has multiplicity 2, but only one independent eigenvector* $\mathbf{v}$ (i.e., λ has defect $d = 1$).

Then 2 independent solutions to $\mathbf{x}' = A\,\mathbf{x}$ are as follows:

$$\mathbf{x}^{(1)}(t) = e^{\lambda t}\mathbf{v} \quad ✓$$

$$\mathbf{x}^{(2)}(t) = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w} \quad ✓✓$$

where $(A - \lambda I)\,\mathbf{w} = \mathbf{v}$

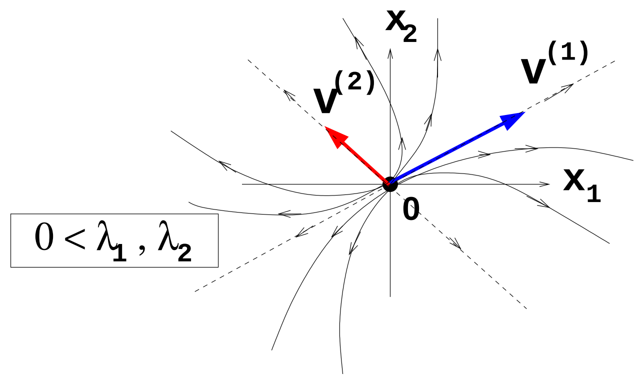
Phase Portraits: $\mathbf{x}' = A\,\mathbf{x}\ (*)$

I $\boldsymbol{\lambda_1, \lambda_2}$ are real and distinct : If $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ are eigenvectors corresponding to λ_1 and λ_2, respectively $\Rightarrow$ $\mathbf{x}^{(1)}(t) = e^{\lambda_1 t}\,\mathbf{v}^{(1)}$ and $\mathbf{x}^{(2)}(t) = e^{\lambda_2 t}\,\mathbf{v}^{(2)}$ are independent solutions and hence general solution of $(*)$ is $\boxed{\mathbf{x}(t) = C_1\,\mathbf{x}^{(1)}(t) + C_2\,\mathbf{x}^{(2)}(t)}$ and hence since $\lambda_1 < \lambda_2$:

$$\mathbf{x}(t) = \underbrace{C_1\,e^{\lambda_1 t}\,\mathbf{v}^{(1)}}_{\substack{\texttt{dominates} \\ \texttt{as } t \longrightarrow -\infty}} + \underbrace{C_2\,e^{\lambda_2 t}\,\mathbf{v}^{(2)}}_{\substack{\texttt{dominates} \\ \texttt{as } t \longrightarrow \infty}}$$

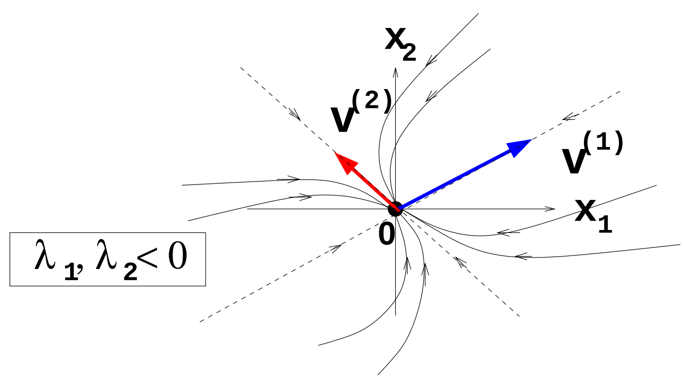
$(0,0)$ is an unstable node

(occurs when $\lambda_1 > 0$, $\lambda_2 > 0$)



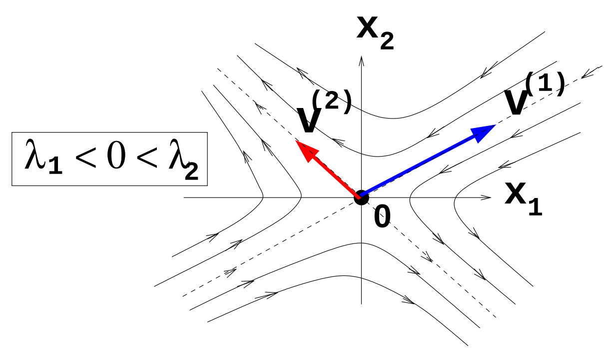
$(0,0)$ is a stable node

(occurs when $\lambda_1 < 0$, $\lambda_2 < 0$)



$(0,0)$ is a saddle point

(occurs $\iff$ $\det A = \lambda_1\lambda_2 < 0$)

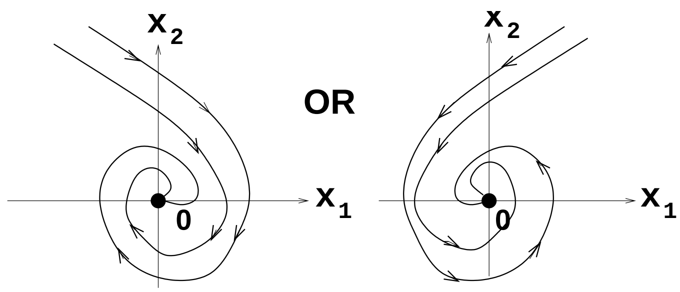


The *Eigenlines* are determined by the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in above phase portraits

II $\boldsymbol{\lambda_1 = \alpha + i\,\beta}$: If $\mathbf{v}$ is a complex e-vector corresponding to λ_1 then $\Longrightarrow$

$\mathbf{x}^{(1)}(t) = \Re e\left\{e^{\lambda_1 t}\mathbf{v}\right\}$ and $\mathbf{x}^{(2)}(t) = \Im m\left\{e^{\lambda_1 t}\mathbf{v}\right\}$ are real-valued solutions and hence general solution of $(*)$ is $\boxed{\mathbf{x}(t) = C_1\,\mathbf{x}^{(1)}(t) + C_2\,\mathbf{x}^{(2)}(t)}$.

If say $\alpha < 0$:



(Test a convenient point to decide which)

$(0,0)$ is a spiral point. If $\alpha < 0$ it is *asymptotically stable*; if $\alpha > 0$ is it said to be *asymptotically unstable*; if $\alpha = 0$ then $(0,0)$ is called a center point.

III $\boldsymbol{\lambda_1 = \lambda_2}$: If there is only *one* linearly independent eigenvector corresponding to λ_1, then solutions to $\mathbf{x}' = A\,\mathbf{x}$ are $\mathbf{x}^{(1)}(t) = e^{\lambda_1 t}\,\mathbf{v}$ and $\mathbf{x}^{(2)}(t) = t\,e^{\lambda_1 t}\,\mathbf{v} + e^{\lambda_1 t}\,\mathbf{w}$, where

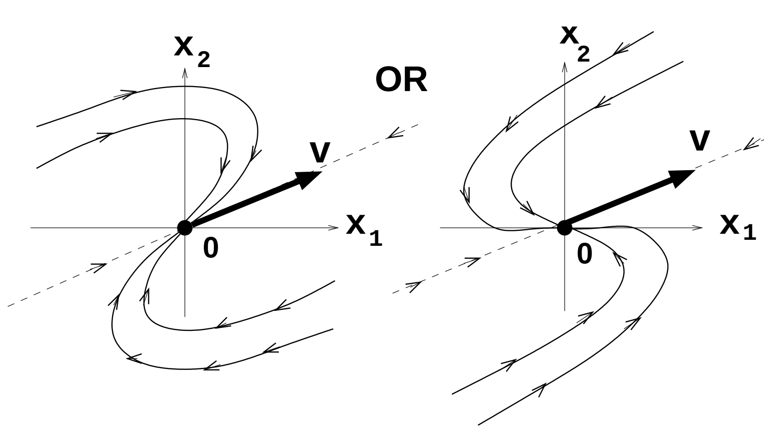
$$\begin{aligned}(A - \lambda_1\,I)\,\mathbf{v} &= \mathbf{0}\\ (A - \lambda_1\,I)\,\mathbf{w} &= \mathbf{v}\end{aligned}$$

($\mathbf{v}$ is an eigenvector of A, while $\mathbf{w}$ is a "generalized eigenvector" of A)

The general solution of the system $(*)$ is $\boxed{\mathbf{x}(t) = C_1\,\mathbf{x}^{(1)}(t) + C_2\,\mathbf{x}^{(2)}(t)}$ and hence:

$$\mathbf{x}(t) = C_1\,e^{\lambda_1 t}\,\mathbf{v} + C_2\left[\underbrace{t\,e^{\lambda_1 t}\,\mathbf{v}}_{\substack{\texttt{dominates}\\ \texttt{as } t\,\longrightarrow\,\pm\infty}} + e^{\lambda_1 t}\,\mathbf{w}\right]$$

If say $\lambda_1 < 0$:



(Test a convenient point to decide which)

$(0,0)$ is an improper node