TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

ITERATED SOCLES AND INTEGRAL DEPENDENCE IN REGULAR RINGS

ALBERTO CORSO, SHIRO GOTO, CRAIG HUNEKE, CLAUDIA POLINI, AND BERND ULRICH

Abstract. Let R be a formal power series ring over a field, with maximal ideal $\mathfrak{m}$, and let I be an ideal of R. We study iterated socles of I, that is, ideals of the form $I :_R \mathfrak{m}^s$ for positive integers s. We are interested in iterated socles in connection with the notion of integral dependence of ideals. In this article we show that iterated socles are integral over I, with reduction number at most one, provided $s \leq \mathrm{o}(I_1(\varphi_d)) - 1$, where $\mathrm{o}(I_1(\varphi_d))$ is the order of the ideal of entries of the last map in a minimal free R-resolution of R/I. In characteristic zero, we also provide formulas for the generators of iterated socles whenever $s \leq \mathrm{o}(I_1(\varphi_d))$. This result generalizes previous work of Herzog, who gave formulas for the socle generators of any homogeneous ideal I in terms of Jacobian determinants of the entries of the matrices in a minimal homogeneous free R-resolution of R/I. Applications are given to iterated socles of determinantal ideals with generic height. In particular, we give surprisingly simple formulas for iterated socles of height two ideals in a power series ring in two variables. The generators of these socles are suitable determinants obtained from the Hilbert-Burch matrix.

1. Introduction

The *socle* of a module M over a local ring R with maximal ideal $\mathfrak{m}$ is the submodule $0 :_M \mathfrak{m}$, the unique largest R-submodule that has the structure of a module over the residue field $k = R/\mathfrak{m}$. Socle generators of modules are as important as the minimal generators of the module, to which they are (in some sense) dual, but, in general, they are much harder to find. In this article we will usually assume that $R = k[\![x_1, \ldots, x_d]\!]$ is a formal powers series ring over a field k and M will often be a cyclic module R/I. We pull back the socle of $M = R/I$ to the ideal $I :_R \mathfrak{m}$ in R and by abuse of language call this ideal the socle of I.

The computation of the socle is well understood in the complete intersection case: If I is generated by a regular sequence $f_1, \ldots, f_d$ contained in the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_d)$, standard linkage theory gives that $I :_R \mathfrak{m} = (I, \det C)$, where C is a square transition matrix that writes the f's in terms of the x's. Moreover, if the characteristic of the field k is 0 and the f's are homogeneous polynomials,

2010 *Mathematics Subject Classification.* Primary 13B22, 13D02, 13N15; Secondary 13C40, 13D07.

Key words and phrases. Socle of a local ring, Jacobian ideals, integral dependence of ideals, free resolutions, determinantal ideals.

The third author was partially supported by NSF grant DMS-1259142. The fourth author was partially supported by NSF grant DMS-1202685 and NSA grant H98230-12-1-0242. The last author was partially supported by NSF grant DMS-1205002 and as a Simons Fellow.

one can take as C the Jacobian matrix of the f's, by Euler's formula. Hence the socle is generated by I together with the determinant of the Jacobian matrix, $I :_R \mathfrak{m} = (I, |\partial f_i/\partial x_j|)$. The same formula holds in the non-graded case, although the result is much less obvious [23, 21, 18]. The converse also holds. Namely, the socle of the Artinian algebra $A = R/I$ is its Jacobian ideal, $0 :_A \mathfrak{m} = \mathrm{Jac}(A)$, if and only if A is a complete intersection [21]. In other words, only for complete intersections one can expect a simple formula based on derivatives of the generators of the ideal.

In the complete intersection case the generators of the ideal immediately give the matrices in a free R-resolution of R/I, by means of the Koszul complex. Thus a natural generalization is to ask whether, in general, one can obtain formulas for the socle using derivatives of the entries of all the matrices in a free resolution. If I is a homogeneous ideal of a power series ring over a field of characteristic 0, Herzog gave such a formula in terms of Jacobian determinants of the entries of the matrices in a homogeneous minimal resolution [11]. His formula suffices to deduce, for instance, that if I is an ideal of maximal minors having generic height then the socle is contained in the ideal of next lower minors. This gives rather strong restrictions on where the socle can sit. Recently, Herzog's result has been used in the study of Golod ideals [12].

Iterated (or *quasi*) *socles* of a module are defined as socles modulo socles. After s iterations one obtains a submodule that can be more easily described as $0 :_M \mathfrak{m}^s$. For instance, if I is a homogeneous ideal in a polynomial ring $R = k[x_1, \ldots, x_d]$, then the largest ideal defining the subscheme $V(I) \subset \mathbb{P}_k^{d-1}$ is the saturation of I, which is in fact an iterated socle, $I :_R \mathfrak{m}^s$. It is of great interest to understand the difference between I and its saturation. For example, if a subscheme is defined by the saturated ideal I, a hyperplane section is defined by I together with the linear form corresponding to the hyperplane, but this ideal is in general not saturated anymore. We can apply our results to give precise formulas for the saturated ideals defining hyperplane sections of subschemes of low Castelnuovo-Mumford regularity (see Remark 3.9). Iterated socles also appear in the study of the scheme $\mathrm{Gor}(T)$ of Gorenstein Artin algebras having fixed Hilbert function T. In particular, Iarrobino makes use of Loewy filtrations, which are defined by means of subquotients of iterated socles [15, 16]. The present work also led us to define what we feel is a powerful concept, *distance*, that can be used as an effective substitute for the Castelnuovo-Mumford regularity in the local case. We explore this notion in [5], where we apply it to characterize the Cohen-Macaulayness and Gorensteinness of associated graded rings and to explore a conjecture on Loewy lengths of Avramov, Buchweitz, Iyengar, and Miller [2].

In this paper we study iterated socles from several perspectives. Many natural questions arise. For instance, Herzog's socle formula is extremely valuable. Are there similar explicit formulas for iterated socles? Another question deals with integral closures of ideals. Integral closure plays a crucial role in the study of Hilbert functions, in intersection theory, and in equisingularity theory, for instance. Since the integral closure of an ideal is difficult to compute, one would like to find at least a large part of it. An obvious place to look for integral elements are iterated socles, which immediately leads to our main motivating question:

For which values of s is $I :_R \mathfrak{m}^s$ contained in the integral closure of I?

Still another problem is to relate iterated socles to other ideals derived from I, when more is known about the structure of I. One example is the result of Herzog mentioned above. If I is determinantal, its socle lies in the ideal of next lower size minors. In particular, iterated socles are contained in the ideal of yet lower size minors. Can more be said?

We provide almost complete answers to these questions.

One cannot expect a positive answer to our main motivating question if s is too large. One obstruction for being in the integral closure arises from the order. Recall that the *order* of an ideal I in a Noetherian local ring $(R, \mathfrak{m})$ is defined as $\mathrm{o}(I) = \sup\{t \,|\, I \subset \mathfrak{m}^t\}$. If R is regular, the powers of the maximal ideal are integrally closed, hence $\mathrm{o}(I) = \mathrm{o}(\overline{I})$, where $\overline{I}$ denotes the integral closure of I. This means that passing to the integral closure of an ideal cannot lower the order, at least when R is regular.

There are several past results dealing with our main motivating question. The first one is due to Burch [4]. In the same paper where she proves the Hilbert-Burch theorem, she also shows that if R is not regular and I has finite projective dimension then the entire socle lies in the integral closure of I. Stronger results have been proved for complete intersection ideals, see for instance [9, 7, 8, 20, 25, 26]. The result of Wang [25] says that if $(R, \mathfrak{m})$ is a regular local ring of dimension $d \geq 2$ and I is a complete intersection then $(I : \mathfrak{m}^s)^2 = I(I : \mathfrak{m}^s)$ provided $s \leq \mathrm{o}(I) - 1$. In other words, the iterated socle is not only integral over the ideal but also has reduction number at most one. Little is known for socles, or iterated socles, of ideals that are not complete intersections. The integral dependence (with reduction number at most one) of the socle of a Gorenstein ideal contained in the square of the maximal ideal has been proved in [6]. A connection between iterated socles and adjoints of ideals has been established by Lipman [19].

We now explain our results in more detail. In Section 2 we unify and generalize the previously known results about integral dependence with reduction number at most one. Most notably, we are able to eliminate the assumption that the ideal be a complete intersection. The main result of Section 2 deals with iterated socles of any ideal I in a power series ring R in $d \geq 2$ variables over a field; we prove that the iterated socle is integral over I with reduction number at most one as long as $s \leq \mathrm{o}(I_1(\varphi_d)) - 1$, where $(F_\bullet, \varphi_\bullet)$ is a minimal free R-resolution of R/I (see Theorem 2.4). Notice that if R/I is an Artinian Gorenstein ring then $I_1(\varphi_d) = I$ by the symmetry of the resolution, and the above inequality for s simply becomes the condition $s \leq \mathrm{o}(I) - 1$ required by Wang and other authors. Thus our result recovers [9, 7, 8, 6, 20, 25] in the case of a regular ambient ring and provides, at the same time, a vast generalization. A somewhat surprising feature of the proof is that the precise knowledge of the module structure of the quotient $(I :_R \mathfrak{m}^{s+1})/I$ suffices to deduce the equality $(I :_R \mathfrak{m}^s)^2 = I(I :_R \mathfrak{m}^s)$ back in the ring R. The structural information we use is the fact that this quotient is a direct sum of copies of the canonical module $\omega_{R/\mathfrak{m}^{s+1}}$, which in turn embeds into the module of polynomials in the inverse variables in the sense of Macaulay's inverse systems (see Corollary 2.2).

In Section 3 we generalize Herzog's formula from socles to iterated socles. We work with a power series ring $R = k[\![x_1, \ldots, x_d]\!]$ over a field k of characteristic zero and consider a finite R-module M with minimal free resolution $(F_\bullet, \varphi_\bullet)$. We provide formulas for minimal generators of the iterated socles $0:_M \mathfrak{m}^s$ in the range $s \leq \mathrm{o}(I_1(\varphi_d))$ (see Theorem 3.8, which is based on Theorems 3.4 and 3.6). Our results cannot be obtained by repeatedly applying Herzog's formulas because this would require knowing the resolutions of all the intermediate iterated socles. Our proof reduces to computing cycles in the tensor product of the resolution $F_\bullet$ and the Koszul complex built on suitable monomial complete intersections. The crucial ingredient is an ad hoc modification of the classical de Rham differential on this family of Koszul complexes, which yields a k-linear contracting homotopy in positive degrees. Our modified de Rham differential is not a derivation as in Herzog's case, it is simply a connection. Yet, this property suffices for our calculations to go through.

In Section 4 we provide applications of the formulas obtained in Section 3 to iterated socles of determinantal ideals with generic height. In other words, we obtain strong restrictions on where iterated socles of ideals of minors of matrices can sit (see Theorem 4.1). Similar results hold for ideals of minors of symmetric matrices and of Pfaffians. In particular, we give surprisingly simple formulas for the generators of iterated socles of height two ideals in a power series ring in two variables. These generators are suitable determinants obtained from the Hilbert-Burch matrix (see Theorem 4.5).

In a subsequent article [5] we consider iterated socles of ideals in non-regular local rings. We obtain substantial improvements in this situation, as we are able to quantify the contribution that comes from the non-regularity of the ring.

For unexplained terminology and background we refer the reader to [3, 22, 24].

2. Reduction number one

In this section we prove our main result on integral dependence, Theorem 2.4. We begin with an observation that will be used throughout the paper.

Proposition 2.1. *Let R be a Noetherian local ring, M a finite R-module, $N = R/J$ with J a perfect R-ideal of grade g, and write $-^\vee = \mathrm{Ext}^g_R(\_, R)$.*

(a) *There are natural isomorphisms $0:_M J \cong \mathrm{Hom}_R(N, M) \cong \mathrm{Tor}^R_g(N^\vee, M)$.*

(b) *Let $(F_\bullet, \varphi_\bullet)$ be a resolution of M by finite free R-modules and assume that $I_1(\varphi_g) + I_1(\varphi_{g+1}) \subset J$. There is a natural isomorphism*
$$\mathrm{Tor}^R_g(N^\vee, M) \cong N^\vee \otimes_R F_g\,.$$

Proof. We first prove (a). As $N = R/J$, there is a natural isomorphism $0:_M J \cong \mathrm{Hom}_R(N, M)$. Since J is perfect of grade g, we also have
$$\mathrm{Hom}_R(N, M) \cong \mathrm{Tor}^R_g(N^\vee, M)\,.$$
Indeed, let $G_\bullet$ be a resolutions of N of length g by finite free R-modules. Notice that $G^*_\bullet[-g]$ is a resolution of $N^\vee$ of length g by finite free R-modules. Hence we obtain natural isomorphisms
$$\begin{aligned}\mathrm{Tor}^R_g(N^\vee, M) &\cong H_g(G^*_\bullet[-g] \otimes_R M)\\ &\cong \mathrm{Ker}(G^*_0 \otimes_R M \longrightarrow G^*_1 \otimes_R M)\\ &\cong \mathrm{Ker}(\mathrm{Hom}_R(G_0, M) \longrightarrow \mathrm{Hom}_R(G_1, M))\\ &\cong \mathrm{Hom}_R(N, M)\,.\end{aligned}$$

As for part (b), notice that

$$\operatorname{Tor}_g^R(N^\vee, M) \cong H_g(N^\vee \otimes_R F_\bullet) \cong N^\vee \otimes_R F_g\,,$$

where the last isomorphism holds because $N^\vee \otimes_R \varphi_{g+1} = 0 = N^\vee \otimes_R \varphi_g$ by our assumption on $I_1(\varphi_{g+1})$ and $I_1(\varphi_g)$. □

The above proposition provides strong structural information about the iterated socle $0 :_M \mathfrak{m}^s$; it implies, under suitable hypotheses, that this colon is a direct sum of copies of the canonical module of $R/\mathfrak{m}^s$.

Corollary 2.2. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d, M a finite R-module, and $(F_\bullet, \varphi_\bullet)$ a minimal free R-resolution of M. One has*

$$0 :_M \mathfrak{m}^s \cong \omega_{R/\mathfrak{m}^s} \otimes_R F_d$$

for every $s \le \mathrm{o}(I_1(\varphi_d))$.

Proof. We apply Proposition 2.1 with $N = R/\mathfrak{m}^s$. □

In Proposition 2.3 below we formalize the key step in the proof of Theorem 2.4.

Proposition 2.3. *Let R be a commutative ring, $I \subset K$ ideals, x, y elements of R, and W a subset of R/I annihilated by some power of x so that $xW = yW$ generates K/I. Assume that whenever $x^t yw = 0$ in R/I for some $t > 0$ and $w \in W$, then $x^t w = 0$ or $yw = 0$ in R/I. Then*

$$K^2 = IK.$$

Proof. Let U, V be preimages in R of W and of $xW = yW$, respectively. We prove that if v_1, v_2 are in V and $x^t v_1 \equiv 0 \bmod I$ for some $t > 0$, then $v_1 v_2 \equiv v_1' v_2' \bmod IK$ for v_1', v_2' in V with $x^{t-1} v_1' \equiv 0 \bmod I$. Decreasing induction on t then shows that $v_1 v_2 \equiv 0 \bmod IK$.

We may assume that $v_1 \not\equiv 0 \bmod I$, and write $v_1 \equiv y u_1 \bmod I$, $v_2 \equiv x u_2 \bmod I$ for elements u_1, u_2 of U. Now

$$\begin{aligned} v_1 v_2 &\equiv y u_1 v_2 \mod IK \\ &\equiv y u_1 x u_2 \mod IK \\ &= x u_1 y u_2 . \end{aligned}$$

Since $x^t y u_1 \equiv x^t v_1 \equiv 0 \bmod I$ and $y u_1 \equiv v_1 \not\equiv 0 \bmod I$, it follows that $x^t u_1 \equiv 0 \bmod I$, hence $x^{t-1}(x u_1) \equiv 0 \bmod I$. Now set $v_1' = x u_1$ and $v_2' = y u_2$. □

Theorem 2.4 below greatly generalizes the results of [9, 7, 8, 6, 20, 25] in the case of a regular ambient ring. We consider iterated socles $I : \mathfrak{m}^s$ of arbitrary ideals I in an equicharacteristic regular local ring R of dimension $d \ge 2$, and we prove that the iterated socles are integral over I with reduction number at most one as long as $s \le \mathrm{o}(I_1(\varphi_d)) - 1$, where $(F_\bullet, \varphi_\bullet)$ is a minimal free R-resolution of R/I. This inequality replaces the assumption that I be a complete intersection and $s \le \mathrm{o}(I) - 1$ required in the earlier work, and hence appears to be the correct condition to fully understand and generalize [9, 7, 8, 6, 20, 25]. The bound $s \le \mathrm{o}(I_1(\varphi_d)) - 1$ is sharp, as can be seen by taking $I = \mathfrak{m}$ and $s = 1$.

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$ containing a field, I an R-ideal, and $(F_\bullet, \varphi_\bullet)$ a minimal free R-resolution of R/I. One has*

$$(I : \mathfrak{m}^s)^2 = I(I : \mathfrak{m}^s)$$

for every $s \leq \mathrm{o}(I_1(\varphi_d)) - 1$.

Proof. After completing we may assume that $R = k[\![x_1, \ldots, x_d]\!]$ is a power series ring over a field k. We wish to apply Proposition 2.3 with $K = I : \mathfrak{m}^s$. Corollary 2.2 gives an isomorphism

$$(I : \mathfrak{m}^{s+1})/I \cong \omega_{R/\mathfrak{m}^{s+1}} \otimes_R F_d \,,$$

which restricts to

$$(I : \mathfrak{m}^s)/I \cong \omega_{R/\mathfrak{m}^s} \otimes_R F_d \,.$$

We recall some standard facts about injective envelopes of the residue field and Macaulay's inverse systems. One has

$$\omega_{R/\mathfrak{m}^{s+1}} \cong E_{R/\mathfrak{m}^{s+1}}(k) \cong 0 :_{E_R(k)} \mathfrak{m}^{s+1} \subset E_R(k) \cong k[x_1^{-1}, \ldots, x_d^{-1}].$$

The R-module structure of the latter is given as follows. We use the identification of k-vector spaces $k[x_1^{-1}, \ldots, x_d^{-1}] = k[x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}]/N$, where N is the subspace spanned by the monomials not in $k[x_1^{-1}, \ldots, x_d^{-1}]$. As N is a $k[x_1, \ldots, x_d]$-submodule, the vector space $k[x_1^{-1}, \ldots, x_d^{-1}]$ becomes a module over $k[x_1, \ldots, x_d]$, and then over R since each x_i acts nilpotently. The R-submodule $\omega_{R/\mathfrak{m}^{s+1}} = 0 : \mathfrak{m}^{s+1} \subset k[x_1^{-1}, \ldots, x_d^{-1}]$ is generated by the set M of monomials of degree s in the inverse variables $x_1^{-1}, \ldots, x_d^{-1}$.

Notice that $x_1^{s+1} M = 0$ and $x_i M$ is the set of monomials of degree $s - 1$ in the inverse variables. In particular, $x_1 M = x_2 M$ generates the submodule $\omega_{R/\mathfrak{m}^s}$. Moreover, if $w = x_1^{-a_1} \cdots x_d^{-a_d} \in M$ and $x_1^t x_2 w = 0$, then $t > a_1$ or $1 > a_2$, in which case $x_1^t w = 0$ or $x_2 w = 0$. For B any R-basis of F_d consider

$$M \otimes_R B \subset \omega_{R/\mathfrak{m}^{s+1}} \otimes_R F_d \cong (I : \mathfrak{m}^{s+1})/I \subset R/I \,.$$

We may now apply Proposition 2.3 with $K = I : \mathfrak{m}^s$, $x = x_1$, $y = x_2$, and W the image of $M \otimes_R B$ in R/I. □

Corollary 2.5. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension ≥ 2 containing a field and I an R-ideal. If R/I is Gorenstein, then*

$$(I : \mathfrak{m}^s)^2 = I(I : \mathfrak{m}^s)$$

for every $s \leq \mathrm{o}(I) - 1$.

Proof. The assertion follows from Theorem 2.4 and the symmetry of the minimal free R-resolution of R/I. □

For another example showing that the assumption $s \leq \mathrm{o}(I) - 1$ is needed in Corollary 2.5, let $(R, \mathfrak{m})$ be a power series ring in $d \geq 2$ variables over a field and I a generic homogeneous $\mathfrak{m}$-primary Gorenstein ideal, in the sense that its dual socle generator is a general form, say of degree $2s - 2$. One has $\mathrm{o}(I) = s$, see for instance [17, 3.31]. On the other hand $\mathrm{o}(I : \mathfrak{m}^s) = s - 1$, hence $I : \mathfrak{m}^s \not\subset \overline{I}$.

As pointed out in [10, 9, 7, 8, 6, 20, 25, 26], the case of a regular ambient ring is the worst as far as integral dependence is concerned. If the ambient ring is not regular, we are able to extend the result about integral dependence with reduction number at most one to the range $s \leq \mathrm{o}(I_1(\varphi_d))$ and we prove integral dependence with possibly higher reduction number in a considerably larger range. This work is done in [5].

3. A formula for iterated socles

The second goal of this article is to provide closed formulas for the generators of iterated socles of any finitely generated module M over a Noetherian local ring $(R, \mathfrak{m})$ of equicharacteristic zero. After completing and choosing a Cohen presentation, we may assume that $R = k[\![x_1, \dots, x_d]\!]$ is a power series ring in d variables over a field k of characteristic zero. Let $(F_\bullet, \varphi_\bullet)$ be a minimal free R-resolution of M. We will use this resolution to construct $0 :_M \mathfrak{m}^s$ in the range $s \leq \mathrm{o}(I_1(\varphi_d))$.

Our result generalizes work of Herzog [11], who treated the case where $s = 1$ and $M = R/I$ for I an ideal generated by homogeneous polynomials. We stress again that our result, even for $M = R/I$, does not follow by repeating Herzog's construction s times, because that would require knowing the resolution of the socles at each step. Our method instead produces the iterated socle in one step from the minimal free resolution of M. Our approach resembles that of Herzog, but there are serious obstacles that need to be overcome. Proposition 3.1 below allows us to reduce first to the computation of $0 :_M J$, where J is a special monomial complete intersection. We then consider the Koszul complex of this complete intersection and define on it a k-linear contracting homotopy modeled after the usual de Rham differential, which splits the Koszul differential in positive degrees. The contracting homotopy we construct is not a derivation anymore, yet it suffices to obtain explicit formulas for Koszul cycles and hence for $0 :_M J$. The formulas for general Koszul cycles are needed in the next section, where we study iterated socles of determinantal ideals.

In the setting of Proposition 3.1 below one has $\mathfrak{m}^s = \cap(x_1^{a_1}, \dots, x_d^{a_d})$. Hence the assertion of the proposition would follow if one could take the intersection out of the colon as a sum. This is indeed possible by linkage theory if $M = R/I$ is a Gorenstein ring and $s \leq \mathrm{o}(I)$. The content of the proposition is that even the weaker assumption $s \leq \mathrm{o}(I_1(\varphi_d))$ suffices. We use the notation $\underline{a} = (a_1, \dots, a_d)$ for a vector in $\mathbb{Z}^d$ and write $|\underline{a}| = \sum_{i=1}^{d} a_i$.

Proposition 3.1. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d > 0$ with a regular system of parameters $x_1, \dots, x_d$. Let M be a finite R-module, $(F_\bullet, \varphi_\bullet)$ a minimal free R-resolution of M, and s a positive integer. If $s \leq \mathrm{o}(I_1(\varphi_d))$, then*

$$0 :_M \mathfrak{m}^s = \sum_{\substack{|\underline{a}| = s+d-1 \\ a_i > 0}} 0 :_M (x_1^{a_1}, \dots, x_d^{a_d})\,.$$

Proof. One has

$$\mathfrak{m}^s = \bigcap_{\substack{|\underline{a}| = s+d-1 \\ a_i > 0}} (x_1^{a_1}, \dots, x_d^{a_d})\,.$$

Since any of the ideals $J_{\underline{a}} := (x_1^{a_1}, \dots, x_d^{a_d})$ contains $\mathfrak{m}^s$, we obtain

$$\omega_{R/J_{\underline{a}}} \cong \mathrm{Hom}_R(R/J_{\underline{a}}, \omega_{R/\mathfrak{m}^s}) \cong 0 :_{\omega_{R/\mathfrak{m}^s}} J_{\underline{a}}\,.$$

In particular, $\operatorname{ann}_R(0 :_{\omega_{R/\mathfrak{m}^s}} J_{\underline{a}}) = J_{\underline{a}}$. Therefore

$$\operatorname{ann}_R\left(\sum_{\underline{a}} 0 :_{\omega_{R/\mathfrak{m}^s}} J_{\underline{a}}\right) = \bigcap_{\underline{a}} J_{\underline{a}} = \mathfrak{m}^s\,.$$

In other words, $\sum\limits_{\underline{a}} 0 :_{\omega_{R/\mathfrak{m}^s}} J_{\underline{a}}$ is a faithful $R/\mathfrak{m}^s$-submodule of the canonical module $\omega_{R/\mathfrak{m}^s}$. As $\omega_{R/\mathfrak{m}^s}$ cannot have a proper faithful $R/\mathfrak{m}^s$-submodule, we conclude that

$$\omega_{R/\mathfrak{m}^s} = \sum_{\underline{a}} 0 :_{\omega_{R/\mathfrak{m}^s}} J_{\underline{a}}\,.$$

According to Corollary 2.2, the module $E = 0 :_M \mathfrak{m}^s$ is isomorphic to $\omega_{R/\mathfrak{m}^s} \otimes_R F_d$. Therefore

$$E = \sum_{\underline{a}} 0 :_E J_{\underline{a}}\,.$$

Finally, the inclusion $\mathfrak{m}^s \subset J_{\underline{a}}$ gives $0 :_M \mathfrak{m}^s \supset 0 :_M J_{\underline{a}}$, hence $0 :_E J_{\underline{a}} = 0 :_M J_{\underline{a}}$. $\square$

In the next discussion we set up the interpretation of cycles in tensor complexes that we will use to obtain explicit formulas for Koszul cycles and, eventually, for iterated socles.

Discussion 3.2. Let R be a Noetherian ring, let M, N be finite R-modules, and let $F_\bullet, G_\bullet$ be resolutions of M, N by finite free R-modules with augmentation maps π, ρ, respectively. The graded R-module $\operatorname{Tor}_\bullet^R(M, N)$ can be identified with these homology modules,

$$H_\bullet(F_\bullet \otimes_R N) \cong H_\bullet(F_\bullet \otimes_R G_\bullet) \cong H_\bullet(M \otimes_R G_\bullet)\,.$$

More precisely, the maps

$$\begin{array}{ccc} & F_\bullet \otimes_R G_\bullet & \\ {\scriptstyle \mathrm{id}_\bullet \otimes \rho} \swarrow & & \searrow {\scriptstyle \pi \otimes \mathrm{id}_\bullet} \\ F_\bullet \otimes_R N & & M \otimes_R G_\bullet \end{array}$$

induce epimorphisms on the level of cycles and isomorphisms on the level of homology.

Recall that an element $\alpha = (\alpha_0, \ldots, \alpha_t)$ of $[F_\bullet \otimes_R G_\bullet]_t = \bigoplus\limits_{i=0}^{t} F_i \otimes G_{t-i}$ is a cycle in $F_\bullet \otimes_R G_\bullet$ if and only if

$$(\mathrm{id}_\bullet \otimes \partial^{G_\bullet})(\alpha_i) = (-1)^{i+1}(\partial^{F_\bullet} \otimes \mathrm{id}_\bullet)(\alpha_{i+1})$$

for all i. To make the isomorphism $H_\bullet(F_\bullet \otimes_R N) \xrightarrow{\sim} H_\bullet(M \otimes_R G_\bullet)$ explicit, we take an arbitrary cycle $v \in [Z(F_\bullet \otimes_R N)]_t$. Lift v to an element $\alpha_t \in F_t \otimes G_0$ with $(\mathrm{id} \otimes \rho)(\alpha_t) = v$. Now α_t can be extended to a cycle $\alpha = (\alpha_0, \ldots, , \alpha_t) \in [Z(F_\bullet \otimes_R G_\bullet)]_t$. The image $u = (\pi \otimes \mathrm{id})(\alpha_0)$ is in $[Z(M \otimes_R G_\bullet)]_t$, and the homology class of u is the image of the homology class of v under the above isomorphism.

$$
\begin{array}{ccccccccc}
 & & & & & & \alpha_0 & \longmapsto & u \\
 & & & & & & \cap & & \cap \\
 & & & & & & F_0 \otimes G_t & \twoheadrightarrow & M \otimes G_t \\
 & & & & & & \downarrow & & \\
 & & & & \alpha_1 \in & F_1 \otimes G_{t-1} & \longrightarrow & F_0 \otimes G_{t-1} & \\
 & & & & & \downarrow & & & \\
 & & & & \cdot^{\cdot^{\cdot}} & & & & \\
 & & & & \downarrow & & & & \\
 & \alpha_{t-1} \in & F_{t-1} \otimes G_1 & \longrightarrow & F_{t-2} \otimes G_1 & & & & \\
 & & \downarrow & & & & & & \\
\alpha_t \in & F_t \otimes G_0 & \longrightarrow & F_{t-1} \otimes G_0 & & & & & \\
\downarrow & \downarrow & & & & & & & \\
v \in & F_t \otimes N & & & & & & &
\end{array}
$$

In the next discussion we construct a contracting homotopy for certain complexes $G_\bullet$ that allows us to invert the vertical differentials in the staircase of Discussion 3.2. This provides an explicit formula to pass from an element v as above to an element u.

Discussion 3.3. Let $A = k[\![y_1, \dots, y_n]\!] \supset A' = k[y_1, \dots, y_n]$, where k is a field and $y_1, \dots, y_n$ are $n > 0$ variables. We say that an A-module M is *graded* if $M \cong A \otimes_{A'} M'$ for some A'-module M' that is graded with respect to the standard grading of A'. One defines, in the obvious way, homogeneous maps and homogeneous complexes of graded A-modules. We call an element u of a graded A-module $M = A \otimes_{A'} M'$ *homogeneous* of degree d if $u = 1 \otimes u'$ for a homogeneous element u' of M' of degree d. Notice that every element u of M can be written uniquely in the form $u = \sum_{i \in \mathbb{Z}} u_i$ with u_i homogeneous of degree i.

We consider the universally finite derivation

$$d : A \longrightarrow \Omega_k(A) = F = Ae_1 \oplus \ldots \oplus Ae_n$$

with $e_i = dy_i$ homogeneous of degree 1. Let $L_\bullet = K_\bullet(y_1, \dots, y_n; A) = \bigwedge^{\bullet} F$ be the Koszul complex of $y_1, \dots, y_n$ with differential $\partial_\bullet$ mapping e_i to y_i. We extend d to a k-linear map $d_\bullet : L_\bullet \longrightarrow L_\bullet[1]$, homogeneous with respect to the internal and the homological degree, by setting $d(av) = d(a) \wedge v$ for $a \in A$ and $v = e_{\nu_1} \wedge \ldots \wedge e_{\nu_\ell}$ a typical basis element in the Koszul complex. After restriction to any graded strand $L_{\bullet_m}$ with respect to the internal grading, one has

$$\partial_\bullet d_\bullet + d_\bullet \partial_{\bullet |L_{\bullet_m}} = m \, \mathrm{id}_{L_{\bullet_m}} , \tag{3.1}$$

as can be seen from the Euler relation.

If $\operatorname{char} k = 0$ we define

$$\widetilde{d}_\bullet : L_{\bullet_{>0}} \longrightarrow L_{\bullet_{>0}}[1]$$

by $\widetilde{d}_\bullet(\eta) = \sum_{m>0} \frac{1}{m} d_\bullet(\eta_m)$, where η_m denotes the degree m component of η in the internal grading. The equality (3.1) shows that $\widetilde{d}_\bullet$ is a k-linear contracting homotopy of $L_{\bullet >0}$.

Now let $R = k[\![x_1, \ldots, x_n]\!]$ be another power series ring and let $a_1, \ldots, a_n$ be positive integers. We consider the subring $A = k[\![y_1, \ldots, y_n]\!]$ where $y_i = x_i^{a_i}$ are homogeneous of degree 1. Write $V = \bigoplus_{0 \le \nu_i < a_i} k\, x_1^{\nu_1} \cdots x_n^{\nu_n}$. One has $R \cong V \otimes_k A$ as A-modules. Thus R is a free A-module, which we grade by giving the elements of V degree 0. With $K_\bullet$ denoting the Koszul complex $K_\bullet(y_1, \ldots, y_n; R)$ we obtain isomorphisms of complexes of graded A-modules

$$K_\bullet \cong R \otimes_A L_\bullet \cong V \otimes_k L_\bullet \,.$$

We define $\nabla_\bullet : K_\bullet \longrightarrow K_\bullet[1]$ by $\nabla_\bullet = V \otimes_k d_\bullet$, which is a k-linear homogenous map with respect to the internal and the homological degree. We notice that $\nabla_0 : R \longrightarrow R \otimes_A \Omega_k(A)$ is no longer a derivation but only a connection of A-modules, which means it satisfies the product rule if one of the factors is in A. Again, if $\operatorname{char} k = 0$ we define

$$\widetilde{\nabla}_\bullet : K_{\bullet >0} \longrightarrow K_{\bullet >0}[1]$$

by $\widetilde{\nabla}_\bullet(\eta) = \sum_{m>0} \frac{1}{m} \nabla_\bullet(\eta_m)$. Alternatively, one has the description $\widetilde{\nabla}_\bullet = V \otimes_k \widetilde{d}_\bullet$. Hence by the discussion above, $\widetilde{\nabla}_\bullet$ is a k-linear contracting homotopy of $K_{\bullet >0}$.

We now describe the maps $\nabla_\bullet$ and $\widetilde{\nabla}_\bullet$ more explicitly. Let $S[[x]]$ be a power series ring in one variable over a commutative ring S and let a be a positive integer. We consider the continuous S-linear map $\frac{d}{dx^a} : S[[x]] \to S[[x]]$ with

$$\frac{d}{dx^a}(x^b) = \left\lfloor \frac{b}{a} \right\rfloor x^{b-a} \,.$$

We write $S[[y]]$ for the power series subring $S[[x^a]]$ and U for the free S-module $U = \bigoplus_{0 \le \nu < a} S\, x^\nu$. We have $S[[x]] \cong U \otimes_S S[[y]]$ and $\frac{d}{dx^a} = U \otimes_S \frac{d}{dy}$. If $R = k[[x_1, \ldots, x_n]]$ is a power series ring in several variables and $a_1, \ldots, a_n$ are positive integers as above, we write $\frac{\partial}{\partial x^{a_i}}$ for $\frac{d}{dx^{a_i}}$ with $S = k[[x_1, \ldots, \widehat{x_i}, \ldots, x_n]]$. To describe the maps $\nabla_\bullet$ and $\widetilde{\nabla}_\bullet$, let $r \in R$ and let $v = e_{i_1} \wedge \ldots \wedge e_{i_\ell}$ be a basis element in the Koszul complex $K_\bullet$ of internal degree ℓ. It turns out that

$$\nabla(rv) = \sum_{i=1}^{n} \frac{\partial r}{\partial x^{a_i}}\, e_i \wedge v \,,$$

and if $\operatorname{char} k = 0$ and $rv \in K_{\bullet >0}$,

$$\widetilde{\nabla}(rv) = \sum_{i, m>0} \frac{1}{m+\ell}\, \frac{\partial r_m}{\partial x^{a_i}}\, e_i \wedge v \,.$$

Theorem 3.4. *Let $R = k[\![x_1, \ldots, x_d]\!]$ be a power series ring in $d > 0$ variables over a field k of characteristic zero and let M be a finite R-module. For $a_1, \ldots, a_d$ positive integers, let $K_\bullet$ denote the Koszul complex $K_\bullet(x_1^{a_1}, \ldots, x_d^{a_d}; R)$ and let $\widetilde{\nabla}_\bullet$ be defined as in* Discussion 3.3. *Consider a minimal free R-resolution $(F_\bullet, \varphi_\bullet)$ of*

M and let $W_\bullet$ be a graded k-vector space with $F_i \cong W_i \otimes_k R$ (notice the grading is by homological degree only). Let t be an integer with $0<t\leq d$, let $w_1, \ldots, w_r$ be a k-basis of W_t, and assume that $(x_1^{a_1}, \ldots, x_d^{a_d}) \supset I_1(\varphi_t)$.

Then the R-module of Koszul cycles $\mathcal{Z}_t(x_1^{a_1}, \ldots, x_d^{a_d}; M)$ is minimally generated by the images in $M \otimes_R K_t$ of the r elements

$$[(\mathrm{id}_{W_\bullet} \otimes_k \widetilde{\nabla}_\bullet) \circ (\varphi_\bullet \otimes_R \mathrm{id}_{K_\bullet})]^t (w_\ell \otimes 1)\,,$$

where $w_\ell \otimes 1 \in F_t \otimes_R K_0$ and $1 \leq \ell \leq r$.

Proof. We apply Discussion 3.2 with $N = R/(x_1^{a_1}, \ldots, x_d^{a_d})$ and $G_\bullet = K_\bullet$. Notice that $[Z(M \otimes_R G_\bullet)]_t = \mathcal{Z}_t(x_1^{a_1}, \ldots, x_d^{a_d}; M)$ and $[Z(F_\bullet \otimes_R N)]_t = F_t \otimes_R R/(x_1^{a_1}, \ldots, x_d^{a_d})$ because $\varphi_t \otimes R/(x_1^{a_1}, \ldots, x_d^{a_d}) = 0$ by our assumption on $I_1(\varphi_t)$. Hence the latter R-module is minimally generated by the elements $w_\ell \otimes 1$, $1 \leq \ell \leq r$, and minimal generators of the former R-module can be obtained from these elements by applying the horizontal differential $\varphi_\bullet \otimes_R \mathrm{id}_{K_\bullet}$, taking preimages under the vertical differential $\mathrm{id}_{F_\bullet} \otimes_R \partial_\bullet^{K_\bullet}$ of Discussion 3.2, and repeating this process t times. Instead of taking preimages under $\mathrm{id}_{F_\bullet} \otimes_R \partial_\bullet^{K_\bullet} = \mathrm{id}_{W_\bullet} \otimes_k \partial_\bullet^{K_\bullet}$ we may apply the map $\mathrm{id}_{W_\bullet} \otimes_k \widetilde{\nabla}_\bullet$, because the boundaries of $K_\bullet$ are in the subcomplex $K_{\bullet>0}$ and $\widetilde{\nabla}_\bullet$ is a contracting homotopy for the latter according to Discussion 3.3. □

Corollary 3.5. *We use the assumptions of* Theorem 3.4 *with $t = d$. The R-module $0 :_M (x_1^{a_1}, \ldots, x_d^{a_d})$ is minimally generated by the images in $M \cong M \otimes_R K_d$ of the r elements*

$$[(\mathrm{id}_{W_\bullet} \otimes_k \widetilde{\nabla}_\bullet) \circ (\varphi_\bullet \otimes_R \mathrm{id}_{K_\bullet})]^d (w_\ell \otimes 1)\,,$$

where $w_\ell \otimes 1 \in F_d \otimes_R K_0$ and $1 \leq \ell \leq r = \dim_k W_d$.

Proof. The identification $M \cong M \otimes_R K_d$ induces an isomorphism between the modules $0 :_M (x_1^{a_1}, \ldots, x_d^{a_d})$ and $\mathcal{Z}_d(x_1^{a_1}, \ldots, x_d^{a_d}; M)$. The assertion then follows from Theorem 3.4. □

We now provide a description of the generators of $0 :_M (x_1^{a_1}, \ldots, x_d^{a_d})$ and, more generally, of the Koszul cycles in terms of Jacobian determinants, as was done by Herzog [11, Corollary 2] when $a_1 = \ldots = a_d = 1$ and M is a cyclic graded module. To do so we consider M as a module over the subring $A = k[\![y_1, \ldots, y_d]\!] \subset R$, where $y_i = x_i^{a_i}$ are homogenous of degree one. Applying Herzog's method directly would lead to generators of the Koszul cycles as A-modules, whereas using Theorem 3.4 above we obtain minimal generating sets as R-modules.

We use the assumptions and notations of Theorem 3.4. Fixing bases of W_i one obtains matrix representations $(\alpha^i_{\nu\mu})$ of the maps φ_i in the resolution $F_\bullet$. Let $\mathcal{B} \subset \mathbb{Z}^d_{\geq 0}$ be the set of all tuples $\underline{\lambda} = (\lambda_1, \ldots, \lambda_d)$ with $\lambda_i < a_i$ and recall that $\underline{x}^{\underline{\lambda}}$, $\underline{\lambda} \in \mathcal{B}$, form an A-basis of R. For $\underline{\lambda}$ and $\underline{\varepsilon}$ in $\mathcal{B}$, we let $\{\underline{\lambda} - \underline{\varepsilon}\}$ be the tuple with

$$\{\underline{\lambda} - \underline{\varepsilon}\}_j = \begin{cases} \lambda_j - \varepsilon_j & \text{if } \lambda_j - \varepsilon_j \geq 0 \\ \lambda_j - \varepsilon_j + a_j & \text{otherwise.} \end{cases}$$

Notice that $\{\underline{\lambda} - \underline{\varepsilon}\}$ is again in $\mathcal{B}$. Moreover we define $\{\{\underline{\lambda} - \underline{\varepsilon}\}\}$ by

$$\{\{\underline{\lambda} - \underline{\varepsilon}\}\}_j = \begin{cases} 0 & \text{if } \lambda_j - \varepsilon_j \geq 0 \\ 1 & \text{otherwise.} \end{cases}$$

If β is an element of R we write $\beta = \sum_{\underline{\gamma}\in\mathcal{B}} \beta_{\underline{\gamma}} \underline{x}^{\underline{\gamma}}$, where $\beta_{\underline{\gamma}} \in A$. Multiplication by β gives an A-endomorphism of R, represented by a matrix M_β with respect to the basis $\underline{x}^{\underline{\lambda}}$, $\underline{\lambda} \in \mathcal{B}$. Its $(\underline{\lambda}, \underline{\varepsilon})$-entry is

$$\beta_{\{\underline{\lambda}-\underline{\varepsilon}\}}\, \underline{y}^{\{\{\underline{\lambda}-\underline{\varepsilon}\}\}}.$$

The R-resolution $F_\bullet$ of M is also an A-resolution. As A-basis of $F_\bullet$ we choose the k-basis of $W_\bullet$ tensored with the basis $\underline{x}^{\underline{\lambda}}, \underline{\lambda} \in \mathcal{B}$. To obtain a matrix representation N_i of φ_i with respect to this A-basis, we replace each $\alpha^i_{\nu\mu}$ by the matrix $M_{\alpha^i_{\nu\mu}}$. The $(\nu, \underline{\lambda}; \mu, \underline{\varepsilon})$-entry of N_i is

$$\alpha^i_{\nu\mu,\{\underline{\lambda}-\underline{\varepsilon}\}}\, \underline{y}^{\{\{\underline{\lambda}-\underline{\varepsilon}\}\}}.$$

These entries are in A, the power series ring in the variables $y_1, \ldots, y_d$, which all have degree 1. We consider the degree m component $\alpha^i_{\nu\mu,\{\underline{\lambda}-\underline{\varepsilon}\},m}$ of $\alpha^i_{\nu\mu,\{\underline{\lambda}-\underline{\varepsilon}\}}$ and we notice that

$$\alpha^i_{\nu\mu,\{\underline{\lambda}-\underline{\varepsilon}\},m}\, \underline{y}^{\{\{\underline{\lambda}-\underline{\varepsilon}\}\}}$$

is homogeneous of degree $m + |\{\{\underline{\lambda} - \underline{\varepsilon}\}\}|$.

The formula in the next theorem involves Jacobian determinants of homogeneous elements of this form. Let t be an integer with $0 < t \le d$, consider sequences of integers $\underline{\nu} = \nu_t, \ldots, \nu_0$, $\underline{m} = m_t, \ldots, m_1$ with $m_i > 0$, $\underline{j} = j_1, \ldots, j_t$ with $1 \le j_1 < \ldots < j_t \le d$, and let $\underline{\underline{\lambda}} = \underline{\lambda}_t, \ldots, \underline{\lambda}_0$ be a sequence of tuples from $\mathcal{B}$. We define the t by t Jacobian determinant

$$D_{\underline{\nu},\underline{\underline{\lambda}},\underline{m},\underline{j}} = \left| \frac{\partial(\alpha^i_{\nu_{i-1}\nu_i,\{\underline{\lambda}_{i-1}-\underline{\lambda}_i\},m_i}\, \underline{y}^{\{\{\underline{\lambda}_{i-1}-\underline{\lambda}_i\}\}})}{\partial y_{j_k}} \right|_{\substack{1\le i\le t\\ 1\le k\le t}}.$$

We remark that in the denominator of the formula below the indices in the product and the sum are decreasing.

Theorem 3.6. *We use the assumptions of* Theorem 3.4 *and write v_j for the basis elements of W_0. The R-module of Koszul cycles $\mathcal{Z}_t(x_1^{a_1}, \ldots, x_d^{a_d}; M)$ is minimally generated by the images in $M \otimes_R K_t$ of the r elements*

$$\sum_{\underline{\nu},\underline{\underline{\lambda}},\underline{m},\underline{j}} \frac{1}{\prod_{h=t}^{1} \sum_{i=t}^{h} (m_i + |\{\{\underline{\lambda}_{i-1} - \underline{\lambda}_i\}\}|)}\, D_{\underline{\nu},\underline{\underline{\lambda}},\underline{m},\underline{j}}\, \underline{x}^{\underline{\lambda}_0} v_{\nu_0} \otimes e_{j_1} \wedge \ldots \wedge e_{j_t}\,,$$

where $\underline{\lambda}_t = \underline{0}$ and ν_t is fixed with $1 \le \nu_t \le r = \dim_k W_t$.

Proof. We claim that the displayed elements in the current theorem and in Theorem 3.4 are equal for $\nu_t = \ell$. To prove this we make use of Discussion 3.3 and the notation introduced there. In particular, let A be the subring $k[\![y_1, \ldots, y_d]\!]$ of R with $y_i = x_i^{a_i}$, V the k-vector space spanned by the monomial basis of R as an A-module, $L_\bullet$ the Koszul complex $K_\bullet(y_1, \ldots, y_d; A)$, $d_\bullet$ the de Rham differential of $L_\bullet$, and $\widetilde{d}_\bullet$ the contracting homotopy derived from it. Recall that $K_\bullet \cong V \otimes_k L_\bullet$ and $\widetilde{\nabla}_\bullet = V \otimes_k \widetilde{d}_\bullet$. It follows that

$$[(\mathrm{id}_{W_\bullet} \otimes_k \widetilde{\nabla}_\bullet) \circ (\varphi_\bullet \otimes_R \mathrm{id}_{K_\bullet})]^t (w_{\nu_t} \otimes 1) = [(\mathrm{id}_{(W_\bullet \otimes_k V)} \otimes_k \widetilde{d}_\bullet) \circ (\varphi_\bullet \otimes_A \mathrm{id}_{L_\bullet})]^t (w_{\nu_t} \otimes 1)\,.$$

Now, to compute the element on the righthand side of the equation we may consider $(F_\bullet, \varphi_\bullet)$ as an A-resolution of M with $F_i = W_i \otimes_k V \otimes_k A$. This element coincides with the one in the current theorem because $d_\bullet$, unlike $\nabla_\bullet$, is a derivation and the operators $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_d}$ commute with each other. □

To obtain an explicit description of annihilator modules we apply Theorem 3.6 with $t = d$. In this case the sequence $\underline{j}$ is necessarily $\underline{j} = 1, 2, \ldots, d$, and we write

$$D_{\underline{\nu},\underline{\underline{\lambda}},\underline{m}} = D_{\underline{\nu},\underline{\underline{\lambda}},\underline{m},\underline{j}} = \left| \frac{\partial(\alpha^i_{\nu_{i-1}\nu_i,\{\underline{\lambda}_{i-1}-\underline{\lambda}_i\},m_i}\, \underline{y}^{\{\{\underline{\lambda}_{i-1}-\underline{\lambda}_i\}\}})}{\partial y_j} \right|_{\substack{1\le i\le d\\ 1\le j\le d}} .$$

Corollary 3.7. *We use the assumptions of* Theorem 3.4 *with $t = d$ and write v_j for the basis elements of W_0. The R-module $0 :_M (x_1^{a_1}, \ldots, x_d^{a_d})$ is minimally generated by the images in M of the r elements*

$$\sum_{\underline{\nu},\underline{\underline{\lambda}},\underline{m}} \frac{1}{\prod_{h=d}^{1} \sum_{i=d}^{h} (m_i + |\{\{\underline{\lambda}_{i-1} - \underline{\lambda}_i\}\}|)}\; D_{\underline{\nu},\underline{\underline{\lambda}},\underline{m}}\; \underline{x}^{\underline{\lambda}_0} v_{\nu_0}\, ,$$

where $\underline{\lambda}_d = \underline{0}$ and ν_d is fixed with $1 \le \nu_d \le r = \dim_k W_d$.

Theorem 3.8. *Let $R = k[\![x_1, \ldots, x_d]\!]$ be a power series ring in $d > 0$ variables over a field k of characteristic zero, with maximal ideal $\mathfrak{m}$. Let M be a finite R-module with minimal free R-resolution $(F_\bullet, \varphi_\bullet)$ and let s be a positive integer. If $s \le \mathrm{o}(I_1(\varphi_d))$, then an explicit minimal generating set of the R-module $0 :_M \mathfrak{m}^s$ is obtained by applying the construction of* Corollary 3.5 *or 3.7 for $\underline{a} = (a_1, \ldots, a_d)$ any tuple of positive integers with $|\underline{a}| = s+d-1$ and by varying over all such tuples.*

Proof. From Proposition 3.1 we know that

$$0 :_M \mathfrak{m}^s = \sum_{\underline{a}} 0 :_M (x_1^{a_1}, \ldots, x_d^{a_d})\, ,$$

where $\underline{a} = (a_1, \ldots, a_d)$ ranges over all tuples of positive integers with $|\underline{a}| = s+d-1$. Since $(x_1^{a_1}, \ldots, x_d^{a_d}) \supset \mathfrak{m}^s \supset I_1(\varphi_d)$, we may apply Corollary 3.5 or 3.7 to obtain a generating set of $0 :_M (x_1^{a_1}, \ldots, x_d^{a_d})$ consisting of $r = \operatorname{rank} F_d$ elements. Thus we have constructed a generating set of $0 :_M \mathfrak{m}^s$ consisting of $r\binom{s+d-2}{d-1}$ elements. This is a minimal generating set because $\mu(0 :_M \mathfrak{m}^s) = r\binom{s+d-2}{d-1}$ according to Corollary 2.2. □

Remark 3.9. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring in $d \ge 2$ variables over an algebraically closed field k of characteristic zero, with homogeneous maximal ideal $\mathfrak{m}$. Let I be a homogeneous prime ideal defining the variety $V(I) \subset \mathbb{P}_k^{d-1}$, and let $x \in R_1$ be a general linear form. Fix a minimal homogeneous free resolution $(F_\bullet, \varphi_\bullet)$ of R/I over R. If $I_1(\varphi_{d-1}) \subset \mathfrak{m}^{\mathrm{reg}(A)-1}$, then Theorem 3.8 can be used to give explicit generators of the saturated ideal defining the hyperplane section $V(I) \cap V(x)$ in $V(x) \cong \mathbb{P}_k^{d-2}$. The point is that $(I, x) : \mathfrak{m}^\infty = (I, x) : \mathfrak{m}^s$ for an s that is small enough to apply the full strength of our results.

To see this, we write $(R', \mathfrak{m}') = (R/Rx, \mathfrak{m}/Rx)$, $A = R/I$, and $A' = A/Ax$. Tensoring $F_\bullet$ with R' one obtains a minimal homogeneous free resolution $(F'_\bullet, \varphi'_\bullet)$ of A' over R'. Let $\overline{A}$ denote the integral closure of A. Notice that $\overline{A}/A$ is a graded A-module concentrated in positive degrees, because A is a positively graded domain over an algebraically closed field. One sees that $H^1_{\mathfrak{m}}(A) \cong H^0_{\mathfrak{m}}(\overline{A}/A)$. On the other hand, $H^0_{\mathfrak{m}}(A')$ embeds into $H^1_{\mathfrak{m}}(A)(-1)$. We deduce that $H^0_{\mathfrak{m}}(A')$ is concentrated in degrees i, where $2 \leq i \leq \operatorname{reg}(A') = \operatorname{reg}(A)$. Therefore $\mathfrak{m}^{\operatorname{reg}(A)-1} H^0_{\mathfrak{m}}(A') = 0$, which gives $0 :_{A'} (\mathfrak{m}')^\infty = 0 :_{A'} (\mathfrak{m}')^{\operatorname{reg}(A)-1}$. On the other hand, $I_1(\varphi'_{d-1}) \subset (\mathfrak{m}')^{\operatorname{reg}(A)-1}$ by our assumption. Thus Theorem 3.8 can be applied to yield the generators of the saturation $0 :_{A'} (\mathfrak{m}')^\infty$.

The assumption $I_1(\varphi_{d-1}) \subset \mathfrak{m}^{\operatorname{reg}(A)-1}$ is obviously satisfied if $\operatorname{reg}(A) = 2$, as is the case when $V(I) \subset \mathbb{P}^{d-1}_k$ is nondegenerate of almost minimal degree [13, Theorem A].

There is another construction based on differentiating matrices in free resolutions, which has been used in the definition of Atiyah classes and characteristic classes of modules, see for instance [1]. With R, M, $(F_\bullet, \varphi_\bullet)$, and $W_\bullet$ as in Theorem 3.6, apply the universally finite derivation to the entries of each φ_i to obtain an R-linear map

$$F_i \otimes_R \bigwedge^{d-i} \Omega_{R/k} \longrightarrow F_{i-1} \otimes_R \bigwedge^{d-i+1} \Omega_{R/k}\,.$$

Composing these maps and projecting F_0 onto M yields an R-linear map

$$\Psi : F_d \longrightarrow M \otimes_R \bigwedge^{d} \Omega_{R/k}\,,$$

whose class in $\operatorname{Ext}^d_R(M, M \otimes_R \bigwedge^d \Omega_{R/k})$ only depends on M, see [1, 2.3.2]. After identifying $M \otimes_R \bigwedge^d \Omega_{R/k}$ with M, we consider the image of Ψ as a submodule of M. It is natural to try to relate this submodule to the socle of M. Indeed, $\operatorname{im}\Psi = 0 :_M \mathfrak{m}$ if $M = R/I$ is a complete intersection and $F_\bullet$ is the Koszul complex with its natural bases. This is not true in general however. For instance, let $R = k[\![x, y]\!]$ be a power series ring in the variables x, y over a field k of characteristic zero and consider the free resolution

$$F_\bullet : 0 \longrightarrow R^2 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R$$

with

$$\varphi_2 = \begin{bmatrix} x^2 & 0 \\ y^2 & x^4 \\ 0 & x^2 + y^3 \end{bmatrix} \qquad \text{and} \qquad \varphi_1 = \begin{bmatrix} x^2y^2 + y^5 & -x^4 - x^2y^3 & x^6 \end{bmatrix}.$$

In this case $\operatorname{im}\Psi$ is generated by the images in $M = R/I = H_0(F_\bullet)$ of the two elements x^5y^2 and $7xy^4 + 6x^3y$. The first element is in $I : \mathfrak{m}$, but the second is not. In fact, the second element is not even integral over I, whereas $I : \mathfrak{m} \subset \overline{I}$ according to Theorem 2.4 for instance. To see that $7xy^4 + 6x^3y$ is not integral over I, we give a grading to R by assigning to x degree 3 and to y degree 2. Now $7xy^4 + 6x^3y$ is homogeneous of degree 11 and I is generated by the homogenous element $x^2y^2 + y^5$ of degree 10 and two other homogeneous elements of degrees 12 and 18. Thus if $7xy^4 + 6x^3y$ were integral over I, it would be integral over the principal ideal generated by $x^2y^2 + y^5$, and hence would be contained in this ideal, which is not the case.

4. Applications to determinantal ideals

The formulas of Corollaries 3.5 and 3.7 are somewhat daunting. Nevertheless, they suffice to provide strong restrictions on where iterated socles of ideals of minors can sit. Such restrictions also hold for ideals of minors of symmetric matrices and of Pfaffians. In particular, this applies to any height two ideal in a power series ring in two variables. In this case the formulas for iterated socles become very simple. The generators can be expressed in terms of determinants of the original presentation matrix. All these issues will be discussed in the present section.

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field of characteristic zero, let $1 \le n \le \ell \le m$ be integers, let $I = I_n(\phi)$ be the ideal generated by the $n \times n$ minors of an $\ell \times m$ matrix ϕ with entries in $\mathfrak{m}^s$, and assume that I has height at least $(\ell - n + 1)(m - n + 1)$. One has*

$$I : \mathfrak{m}^s \subset I_{n-1}(\phi)\,.$$

Proof. We may assume that $s \ge 1$ and $I \ne R$. Recall that I is a perfect ideal of grade $(\ell-n+1)(m-n+1)$ according to [14]. We may assume that I is $\mathfrak{m}$-primary. After completing we may further suppose that $R = k[\![x_1, \ldots, x_d]\!]$ is a power series ring in $d > 0$ variables over a field k of characteristic zero.

Now let $Y = (y_{ij})$ be an $\ell \times m$ matrix of variables over R, write $S = R[\![\{y_{ij}\}]\!]$, and let $J = I_n(Y)$ be the S-ideal generated by the $n \times n$ minors of Y. We consider a minimal free S-resolution $F_\bullet$ of S/J. Since J is extended from an ideal in the ring $k[\![\{y_{ij}\}]\!]$, all matrices of $F_\bullet$ have entries in the S-ideal $I_1(Y)$ generated by the entries of Y. On the other hand, J is perfect of grade $(\ell-n+1)(m-n+1)$. Hence, regarding R as an S-module via the identification $R \cong S/I_1(Y - \phi)$, we see that $F_\bullet \otimes_S R$ is a minimal free R-resolution of R/I and the entries of $Y - \phi$ form a regular sequence on S/J. Notice that all matrices of the resolution $F_\bullet \otimes_S R$ have entries in the R-ideal $I_1(\phi) \subset \mathfrak{m}^s$. Thus according to Proposition 3.1 the assertion of the present theorem follows once we have shown that $I : (x_1^{a_1}, \ldots, x_d^{a_d}) \subset I_{n-1}(\phi)$ whenever $|\underline{a}| = s+d-1$ and $a_i > 0$. The latter amounts to proving that the module of Koszul cycles $\mathcal{Z}_d(x_1^{a_1}, \ldots, x_d^{a_d}; R/I)$ is contained in $I_{n-1}(\phi) \cdot K_d(x_1^{a_1}, \ldots, x_d^{a_d}; R/I)$. This is a consequence of the next, more general result. □

Proposition 4.2. *In addition to the assumptions of* Theorem 4.1 *suppose that $R = k[\![x_1, \ldots, x_d]\!]$ is a power series ring in $d > 0$ variables over a field k of characteristic zero. If a_i are positive integers with $\sum_{i=1}^d a_i \le s + d - 1$, then*

$$\mathcal{Z}_t(x_1^{a_1}, \ldots, x_d^{a_d}; R/I) \subset I_{n-1}(\phi) \cdot K_t(x_1^{a_1}, \ldots, x_d^{a_d}; R/I)$$

for every $t > 0$.

Proof. We adopt the notation introduced in the previous proof. Notice that $\underline{x}^{\underline{a}} = x_1^{a_1}, \ldots, x_d^{a_d}$ form a regular sequence on S/J and so do the entries $\underline{y} - \underline{\phi}$ of the matrix $Y - \phi$. Thus the Koszul complexes $K_\bullet(\underline{x}^{\underline{a}}; S/J)$ and $K_\bullet(\underline{y} - \underline{\phi}; S/J)$ are acyclic. Since the zeroth homology of the second complex is R/I, Discussion 3.2 shows that the natural map

$$K_\bullet(\underline{x}^{\underline{a}}; S/J) \otimes_{S/J} K_\bullet(\underline{y} - \underline{\phi}; S/J) \longrightarrow K_\bullet(\underline{x}^{\underline{a}}; S/J) \otimes_{S/J} R/I \cong K_\bullet(\underline{x}^{\underline{a}}; R/I)$$

induces a surjection on the level of cycles. On the other hand, we have the following isomorphisms of complexes

$$K_\bullet(\underline{x}^{\underline{a}}; S/J) \otimes_{S/J} K_\bullet(\underline{y}-\underline{\phi}; S/J) \cong K_\bullet(\underline{x}^{\underline{a}}, \underline{y}-\underline{\phi}; S/J) \cong K_\bullet(\underline{x}^{\underline{a}}, \underline{y}; S/J)\,,$$

where the first isomorphism follows from the definition of the Koszul complex. The second isomorphism uses the fact that the sequences $\underline{x}^{\underline{a}}, \underline{y}-\underline{\phi}$ and $\underline{x}^{\underline{a}}, \underline{y}$ minimally generate the same ideal in S because $I_1(\underline{\phi}) \subset \mathfrak{m}^s \subset (\underline{x}^{\underline{a}})$.

We conclude that it suffices to show that

$$\mathcal{Z}_t(\underline{x}^{\underline{a}}, \underline{y}; S/J) \subset I_{n-1}(Y) \cdot K_t(\underline{x}^{\underline{a}}, \underline{y}; S/J)$$

for every $t > 0$. To this end we apply Theorem 3.4 to the ring $S = k[\![\underline{x}, \underline{y}]\!]$, the S-module S/J with minimal free resolution $(F_\bullet, \varphi_\bullet)$, and the Koszul complex $K_\bullet = K_\bullet(\underline{x}^{\underline{a}}, \underline{y}; S)$. As observed in the previous proof, the condition $I_1(\varphi_t) \subset (\underline{y})$ is satisfied. Hence Theorem 3.4 yields the inclusion

$$\mathcal{Z}_t(\underline{x}^{\underline{a}}, \underline{y}; S/J) \subset S\,[(\mathrm{id}_{W_0} \otimes_k \widetilde{\nabla}_{t-1}) \circ (\varphi_1 \otimes_S \mathrm{id}_{K_{t-1}})](W_1 \otimes K_{t-1})\,.$$

The S-module on the right hand side is contained in $S\,\widetilde{\nabla}_{t-1}(I_n(Y) \cdot K_{t-1})$. Finally, notice that $\frac{\partial}{\partial y_{ij}}(I_n(Y)) \subset I_{n-1}(Y)$ and that $\frac{\partial}{\partial x_i^{a_i}}(I_n(Y)) \subset I_n(Y)$ because the map $\frac{\partial}{\partial x_i^{a_i}}$ is $k[\![\underline{y}]\!]$-linear. We conclude that

$$S\,\widetilde{\nabla}_{t-1}(I_n(Y) \cdot K_{t-1}) \subset I_{n-1}(Y) \cdot K_t\,,$$

as required. □

Theorem 4.1 above is sharp, as can be seen by taking R to be a power series ring in $m-n+1$ variables over a field, ϕ a matrix with linear entries, and $n = \ell$. In this case $I : \mathfrak{m} = \mathfrak{m}^n : \mathfrak{m} = \mathfrak{m}^{n-1} = I_{n-1}(\phi)$.

We now turn to perfect ideals of height two. For this it will be convenient to collect some general facts of a homological nature.

Proposition 4.3. *Let R be a Noetherian local ring, M a finite R-module, $N = R/J$ with J a perfect R-ideal of grade g, and write $-^* = \mathrm{Hom}_R(\_, R)$, $-^\vee = \mathrm{Ext}^g_R(\_, R)$.*

- (a) *If M is perfect of grade g, then there is a natural isomorphism $\mathrm{Hom}_R(N, M) \cong \mathrm{Hom}_R(M^\vee, N^\vee)$ given by $u \mapsto u^\vee$.*
- (b) *If M is perfect of grade g and $\pi : M^\vee \twoheadrightarrow N \otimes_R M^\vee$ is the natural projection, then the map $\pi^\vee$ is naturally identified with the inclusion map $0 :_M J \hookrightarrow M$.*
- (c) *Assume M has a resolution $(F_\bullet, \varphi_\bullet)$ of length g by finite free R-modules. If $I_1(\varphi_g) \subset J$, then there are natural isomorphisms*

$$N \otimes_R M^\vee \cong N \otimes_R F_g^* \quad \text{and} \quad \mathrm{Hom}_R(M^\vee, N^\vee) \cong \mathrm{Hom}_R(F_g^*, N^\vee)\,.$$

Proof. The additive contravariant functor $-^\vee$ induces an R-linear map

$$\mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Hom}_R(M^\vee, N^\vee)$$

sending u to $u^\vee$. The latter is an isomorphism since $-^{\vee\vee} \simeq \mathrm{id}$ on the category of perfect R-modules of grade g. This proves part (a).

We prove (b). Let $p : R \twoheadrightarrow R/J$ be the natural projection and notice that $\pi = p \otimes_R M^\vee$. We show that $(p \otimes_R M^\vee)^\vee$ can be naturally identified with $\mathrm{Hom}_R(p, M)$.

Let $\mathfrak{a} \subset \operatorname{ann}(M^\vee)$ be an ideal generated by an R-regular sequence of length g. There are natural identifications of maps

$$\begin{aligned} (p \otimes_R M^\vee)^\vee &= \operatorname{Ext}_R^g(p \otimes_R M^\vee, R) \\ &= \operatorname{Hom}_R(p \otimes_R M^\vee, R/\mathfrak{a}) \\ &= \operatorname{Hom}_R(p, \operatorname{Hom}_R(M^\vee, R/\mathfrak{a})) \\ &= \operatorname{Hom}_R(p, \operatorname{Ext}_R^g(M^\vee, R)) \\ &= \operatorname{Hom}_R(p, M^{\vee\vee}) \\ &= \operatorname{Hom}_R(p, M)\,, \end{aligned}$$

where the last equality uses the assumption that M is perfect of grade g. Finally notice that

$$\operatorname{Hom}_R(p, M) : \operatorname{Hom}_R(R/J, M) \longrightarrow \operatorname{Hom}_R(R, M)$$

can be identified with the inclusion map $0 :_M J \hookrightarrow M$.

As for part (c), notice that $M^\vee \cong \operatorname{coker} \varphi_g^*$. Hence the containment $I_1(\varphi_g) \subset J = \operatorname{ann}(N)$ implies that $N \otimes_R M^\vee \cong N \otimes_R F_g^*$, which is the first isomorphism in part (c). Now Hom-tensor adjointness gives the second isomorphism $\operatorname{Hom}_R(M^\vee, N^\vee) \cong \operatorname{Hom}_R(F_g^*, N^\vee)$ because $N = R/J$. □

Corollary 4.4. *In addition to the assumptions of* Proposition 4.3 *assume that M is perfect of grade g. Let $(F_\bullet, \varphi_\bullet)$ and $G_\bullet$ be resolutions of M and N of length g by finite free R-modules. Notice that $F_\bullet^*[-g]$ and $G_\bullet^*[-g]$ are resolutions of $M^\vee$ and $N^\vee$ by finite free R-modules.*

(a) *Given a linear map $v : M^\vee \longrightarrow N^\vee$, lift v to a morphism of complexes $\widetilde{v}_\bullet : F_\bullet^* \longrightarrow G_\bullet^*$, dualize to obtain $\widetilde{v}_\bullet^* : G_\bullet \longrightarrow F_\bullet$, and consider $H_0(\widetilde{v}_\bullet^*) : N = R/J \longrightarrow M$. One has*

$$0 :_M J = \{H_0(\widetilde{v}_\bullet^*)(1 + J) \mid v \in \operatorname{Hom}_R(M^\vee, N^\vee)\}\,.$$

(b) *Assume that $I_1(\varphi_g) \subset J$. Let*

$$w : M^\vee \twoheadrightarrow N \otimes_R F_g^*$$

be the composition of the epimorphism π in Proposition 4.3(b) *with the first isomorphism in* Proposition 4.3(c) *and lift w to a morphism of complexes*

$$\widetilde{w}_\bullet : F_\bullet^* \longrightarrow G_\bullet[g] \otimes_R F_g^*\,.$$

The mapping cone $C(\widetilde{w}_\bullet^*)$ is a free R-resolution of $M/(0 :_M J)$.

Proof. From Propositions 4.3(a) and 2.1(a) we have isomorphisms

$$\operatorname{Hom}_R(M^\vee, N^\vee) \xrightarrow{\sim} \operatorname{Hom}_R(N, M) \xrightarrow{\sim} 0 :_M J\,,$$

where the first map sends v to $v^\vee = H_0(\widetilde{v}_\bullet^*)$ and the second map sends u to $u(1+J)$. This proves part (a).

To show part (b), notice that

$$\widetilde{w}_\bullet^* : G_\bullet^*[-g] \otimes_R F_g \longrightarrow F_\bullet\,,$$

where $G_\bullet^*[-g] \otimes_R F_g$ and $F_\bullet$ are acyclic complexes of finite free R-modules. Moreover, $H_0(\widetilde{w}_\bullet^*) = w^\vee$. By Proposition 4.3(b) and the first isomorphism in Proposition 4.3(c), the map $w^\vee$ is injective with cokernel $M/(0 :_M J)$. It follows that the mapping cone $C(\widetilde{w}_\bullet^*)$ is a free R-resolution of $M/(0 :_M J)$. □

Theorem 4.5. *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with regular system of parameters x, y, and let I be an $\mathfrak{m}$-primary ideal minimally presented by an $n \times (n-1)$ matrix ϕ with entries in $\mathfrak{m}^s$.*

(a) *For $1 \le i \le n-1$ and $1 \le a \le s$ write the i^{th} column ϕ_i of ϕ in the form $x^{s+1-a}\eta + y^a\xi$ and let Δ_{ia} be the determinant of the $n \times n$ matrix obtained from ϕ by replacing ϕ_i with the two columns η and ξ. One has*

$$I : \mathfrak{m}^s = I + (\Delta_{ia} \mid 1 \le i \le n-1, 1 \le a \le s).$$

(b) *The ideal $I : \mathfrak{m}^s$ is minimally presented by the $[(n-1)(s+1)+1] \times (n-1)(s+1)$ matrix*

$$\psi = \left[\begin{array}{c} B \\ \hline \chi \end{array}\right].$$

The $n \times (n-1)(s+1)$ matrix B is obtained from ϕ by replacing each column ϕ_i with the $s+1$ columns $\phi_{i0}, \ldots, \phi_{is}$ defined by the equation $\phi_i = \sum_{j=0}^{s} x^{s-j}y^j\phi_{ij}$. The $(n-1)s \times (n-1)(s+1)$ matrix χ is the direct sum of $n-1$ copies of the $s \times (s+1)$ matrix

$$\begin{bmatrix} -y & x & & & \\ & -y & x & & \\ & & \ddots & \ddots & \\ & & & -y & x \end{bmatrix}.$$

Proof. We prove part (a). The R-module R/I has a minimal free R-resolution $(F_\bullet, \varphi_\bullet)$ of length 2. After a choice of bases we may assume that $\varphi_2 = \phi$. As $I_1(\varphi_2) \subset \mathfrak{m}^s$, Proposition 3.1 shows that

$$I : \mathfrak{m}^s = \sum_{1 \le a \le s} I : (x^{s+1-a}, y^a)\,.$$

We claim that $I : (x^{s+1-a}, y^a) = I + (\Delta_{ia} \mid 1 \le i \le n-1)$ for every $1 \le a \le s$.

For this we wish to apply Corollary 4.4(a) with $M = R/I$, $J = (x^{s+1-a}, y^a)$, and $G_\bullet = K_\bullet$ the Koszul complex of $-y^a, x^{s+1-a}$ with its natural bases. The second isomorphism of Proposition 4.3(c) gives $\mathrm{Hom}_R(M^\vee, N^\vee) \cong \mathrm{Hom}_R(F_2^*, \omega_{R/J})$. Let $v \in \mathrm{Hom}_R(F_2^*, \omega_{R/J})$ be the projection $\pi_i : F_2^* \twoheadrightarrow R = K_2^*$ onto the i^{th} component followed by the epimorphism $K_2^* \twoheadrightarrow \omega_{R/J}$. We lift v to a morphism of complexes $\widetilde{v}_\bullet : F_\bullet^* \longrightarrow K_\bullet^*$ with $\widetilde{v}_{-2} = \pi_i$. The R-module $\mathrm{Hom}_R(M^\vee, N^\vee) \cong \mathrm{Hom}_R(F_2^*, \omega_{R/J})$ is generated by the elements v as i varies in the range $1 \le i \le n-1$. Hence according to Corollary 4.4(a) the ideal $I : (x^{s+1-a}, y^a)$ is generated by I together with the ideals $I_1(\widetilde{v}_0^*)$. Thus it suffices to prove that each ideal $I_1(\widetilde{v}_0^*)$ is generated by Δ_{ia}.

In the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_0^* & \xrightarrow{\varphi_1^*} & F_1^* & \xrightarrow{\varphi_2^* = \phi^*} & F_2^* \\ & & \downarrow{\scriptstyle \widetilde{v}_0} & & \downarrow{\scriptstyle \widetilde{v}_{-1}} & & \downarrow{\scriptstyle \widetilde{v}_{-2}=\pi_i} \\ 0 & \longrightarrow & K_0^* & \xrightarrow{\partial_1^*} & K_1^* & \xrightarrow{\partial_2^*} & K_2^* \end{array}$$

we may choose $\widetilde{v}_{-1} = [\ \eta \mid \xi\ ]^*$ because $\partial_2^*\,\widetilde{v}_{-1} = [x^{s+1-a} \,|\, y^a] \cdot [\ \eta \mid \xi\ ]^*$ is the i^{th} row of ϕ^*, which equals $\pi_i\,\varphi_2^*$. Since v is surjective, the mapping cone $C(\widetilde{v}_\bullet)$ has homology only in degree -1. Splitting off a direct summand, dualizing, and shifting by -1 yields the acyclic complex

$$0 \longrightarrow (F_2/\pi_i^*(K_2)) \oplus K_1 \xrightarrow{\left[\begin{array}{c|c|c} \phi' & -\eta & -\xi \\ \hline 0 & \multicolumn{2}{c}{-\partial_1} \end{array}\right]} F_1 \oplus K_0 \xrightarrow{[\ \varphi_1 \,|\, -\widetilde{v}_0^*]} F_0 \ ,$$

where ϕ' is obtained from ϕ by deleting the i^{th} column. Since $I_1(\varphi_1) = I$ has height 2, the Hilbert-Burch Theorem now shows that $I_1(\widetilde{v}_0^*) = I_n([\ \phi' \mid -\eta \mid -\xi\ ]) = R\,\Delta_{ia}$, as desired.

To prove part (b) we wish to apply Corollary 4.4(b) with $M = R/I$, $J = \mathfrak{m}^s$, and $G_\bullet$ the resolution that, after a choice of bases, has differentials

$$\partial_1 = [\ x^s \quad x^{s-1}y \quad \dots \quad y^s] \qquad \text{and} \qquad \partial_2 = \begin{bmatrix} y & & & \\ -x & y & & \\ & -x & \ddots & \\ & & \ddots & y \\ & & & -x \end{bmatrix}.$$

Lift the natural epimorphism

$$w : M^\vee \twoheadrightarrow R/\mathfrak{m}^s \otimes_R F_2^*$$

to a morphism of complexes $\widetilde{w}_\bullet : F_\bullet^* \longrightarrow G_\bullet[2] \otimes_R F_2^*$ so that $\widetilde{w}_{-2} = \mathrm{id}$. In the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_0^* & \xrightarrow{\varphi_1^*} & F_1^* & \xrightarrow{\varphi_2^* = \phi^*} & F_2^* \\ & & \downarrow{\scriptstyle \widetilde{w}_0} & & \downarrow{\scriptstyle \widetilde{w}_{-1}} & & \Vert{\scriptstyle \widetilde{w}_{-2}=\mathrm{id}} \\ 0 & \longrightarrow & G_2 \otimes F_2^* & \xrightarrow{\partial_2 \otimes F_2^*} & G_1 \otimes F_2^* & \xrightarrow{\partial_1 \otimes F_2^*} & G_0 \otimes F_2^* \end{array}$$

we can choose $\widetilde{w}_{-1} = B^*$ by the definition of B. Notice that $\partial_2 \otimes F_2^* = -\chi^*$. The result now follows since $C(\widetilde{w}_\bullet^*)$ is a free R-resolution of $R/(I :_R \mathfrak{m}^s)$ according to Corollary 4.4(b). □

Remark 4.6. In the setting of Theorem 4.5 a minimal free R-resolution of $R/(I : \mathfrak{m}^s)$ is

$$0 \longrightarrow R^{(n-1)(s+1)} \xrightarrow{\psi_2} R^{(n-1)(s+1)+1} \xrightarrow{\psi_1} R\,,$$

where $\psi_2 = \psi$ and the ℓ^{th} entry of ψ_1 is the signed ℓ^{th} maximal minor of ψ. If $\ell \le n$ the ℓ^{th} maximal minor of ψ is the ℓ^{th} maximal minor of ϕ; if $\ell \ge n+1$ write $\ell = n + (i-1)s + a$ for $1 \le i \le n-1$, $1 \le a \le s$, and then the ℓ^{th} maximal minor of ψ is Δ_{ia}, where $\eta := \sum_{j=0}^{a-1} x^{a-1-j} y^j \phi_{ij}$ and $\xi := \sum_{j=a}^{s} x^{s-j} y^{j-a} \phi_{ij}$ are used in the definition of Δ_{ia}.

For the proof one uses Theorem 4.5(b), the Hilbert-Burch Theorem, and the following elementary fact about determinants that can be shown by induction on r and expansion along the last row:

If $\left[\begin{array}{c|c} \varepsilon & \mu \\ \hline 0 & \delta \end{array}\right]$ is a square matrix, where μ has columns $\mu_0, \ldots, \mu_r$ and $\delta = \begin{bmatrix} -y & x & & \\ & \ddots & \ddots & \\ & & -y & x \end{bmatrix}$, then $\det \left[\begin{array}{c|c} \varepsilon & \mu \\ \hline 0 & \delta \end{array}\right] = \det \left[\begin{array}{c|c} \varepsilon & \sum_{j=0}^{r} x^{r-j} y^j \mu_j \end{array}\right]$.

Acknowledgments. Part of this work was done at the Mathematical Sciences Research Institute (MSRI) in Berkeley, where the authors spent time in connection with the 2012-13 thematic year on Commutative Algebra, supported by NSF grant 0932078000. The authors would like to thank MSRI for its hospitality and partial support. The authors would also like to thank David Eisenbud for helpful discussions about the material of this paper.

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Department of Mathematics, University of Kentucky, Lexington, KY 40506
E-mail address: alberto.corso@uky.edu

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan
E-mail address: goto@math.meiji.ac.jp

Department of Mathematics, University of Virginia, Charlottesville, VA 22904
E-mail address: huneke@virginia.edu

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556
E-mail address: cpolini@nd.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: ulrich@math.purdue.edu