

GENERALIZED JOUANOLOU DUALITY, WEAKLY GORENSTEIN RINGS, AND APPLICATIONS TO BLOWUP ALGEBRAS

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ABSTRACT. We provide a generalization of Jouanolou duality that is applicable to a plethora of situations. The environment where this generalized duality takes place is a new class of rings, that we introduce and call weakly Gorenstein rings. As a consequence, we obtain a new general framework to investigate blowup algebras. We use our results to study and determine the defining equations of the Rees algebra for certain families of ideals.

1. INTRODUCTION

1.1. Generalized Jouanolou duality and weakly Gorenstein rings.

In a series of seminal papers Jouanolou studied elimination theory through the lens of modern algebraic geometry ([33–38]). One of his main tools was a new duality, nowadays dubbed *Jouanolou duality*. Let S be a positively graded Noetherian T -algebra, with graded irrelevant ideal \mathfrak{m} . When S is a (not necessarily flat) complete intersection over T and $\dim(S) = \dim(T)$, Jouanolou proved that, up to shift in degree, there are graded S -isomorphisms $H_{\mathfrak{m}}^i(S) \cong {}^*\mathrm{Ext}_T^i(S, T)$ for all $i \geq 0$, where ${}^*\mathrm{Ext}$ denotes the graded Ext functor.

The fundamental problem in elimination theory is to compute the image of a projection (see [16, §14.1]), like the natural projection $\pi: X = \mathrm{Proj}(S) \rightarrow \mathrm{Spec}(T)$. The scheme-theoretic image of π is given by the closed subscheme $\mathrm{Spec}(T/\mathfrak{a})$ with $\mathfrak{a} = \mathrm{Ker}(T \xrightarrow{\mathrm{nat}} H^0(X, \mathcal{O}_X))$. We say that \mathfrak{a} is the *resultant ideal*. We have the four-term exact sequence $0 \rightarrow H_{\mathfrak{m}}^0(S)_0 \rightarrow T = S_0 \xrightarrow{\mathrm{nat}} H^0(X, \mathcal{O}_X) \rightarrow H_{\mathfrak{m}}^1(S)_0 \rightarrow 0$. This shows that the scheme-theoretic image of π is given by

$$\mathrm{Im}(\pi) = \mathrm{Spec}(T/\mathfrak{a}) \quad \text{with} \quad \mathfrak{a} = H_{\mathfrak{m}}^0(S)_0.$$

Therefore, Jouanolou duality gives an effective method to compute the image of a projection. Indeed, if we have the isomorphism $H_{\mathfrak{m}}^0(S) \cong {}^*\mathrm{Hom}_T(S, T)(-\delta)$, we can compute the torsion part $H_{\mathfrak{m}}^0(S)_0$ (which involves the multiplicative structure of S) via the T -module $\mathrm{Hom}_T(S_{\delta}, T)$ (which can be computed as the kernel of the transpose of a presentation matrix of S_{δ} as a T -module). Using his duality, Jouanolou proved many beautiful formulas involving resultants ([33–38]). The expository references [5, 7, 8, 15] show how Jouanolou’s work (in particular, his duality) is still relevant in modern elimination theory.

Our goal is to extend this duality to algebras that are not complete intersections, in fact not even Gorenstein. To this end, we introduce a new generalization of Gorenstein rings. We call these rings *weakly Gorenstein* rings. As in the classical case of Gorenstein rings where local duality takes place,

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our new notion of weakly Gorenstein rings is the natural environment where Jouanolou duality exists and can be generalized. Assume S is Cohen-Macaulay and ω_S is a graded canonical module of S . Let $\mathfrak{a} \subset S$ be a homogeneous ideal and $i \geq 0$ be an integer. We say that S is *i-weakly Gorenstein with respect to \mathfrak{a}* if there exists a fixed homogeneous element $y \in \omega_S$ that generates ω_S generically and locally in codimension at most i at every prime that contains \mathfrak{a} . More precisely, we require that

$$\dim(\text{Supp}(\omega_S/Sy)) < \dim(S) \quad \text{and} \quad \dim(\text{Supp}(\omega_S/Sy) \cap V(\mathfrak{a})) < \dim(S) - i.$$

Our first main result says that a generalization of Jouanolou duality holds for weakly Gorenstein rings. Indeed, let $B = T[x_1, \dots, x_d]$ be a positively graded polynomial ring mapping onto S and assume that S is a perfect B -module of codimension c . If S is $(i+1)$ -weakly Gorenstein with respect to \mathfrak{m} , then there is a graded S -isomorphism

$$H_{\mathfrak{m}}^i(S) \cong {}^* \text{Ext}_T^{i+c-d}(S, T)(\deg(y))$$

(see [Theorem 2.8](#)). When $i = 0$ and S is standard graded with $c = d$, and $\delta := -\deg(y)$, we prove that there is an isomorphism $\circ : H_{\mathfrak{m}}^0(S)_{\delta} \xrightarrow{\cong} T$ and that the multiplication maps $H_{\mathfrak{m}}^0(S)_j \otimes_T S_{\delta-j} \rightarrow H_{\mathfrak{m}}^0(S)_{\delta}$ followed by \circ are perfect pairings inducing isomorphisms $H_{\mathfrak{m}}^0(S)_j \xrightarrow{\cong} \text{Hom}_T(S_{\delta-j}, T)$ as above (see [Theorem 2.12](#)). We provide explicit inverses of these isomorphisms, which we construct from any suitable element Δ in the annihilator of the diagonal ideal, the kernel of the multiplication map $S \otimes_T S \rightarrow S$ (see [Theorem 2.18](#)). The significance of such explicit inverses is that they reduce the computation of $H_{\mathfrak{m}}^0(S)$, as an ideal in S , to the computation of ${}^* \text{Hom}_T(S, T)$. Jouanolou achieved this by means of Morley forms, which our construction generalizes. We will elaborate on this for the special case of symmetric algebras, in the second part of the Introduction.

Surprisingly, many classes of algebras and ideals of interest satisfy the weakly Gorenstein condition. We prove that determinantal rings tend to have this property (see [Theorem 3.3](#)), and so do symmetric algebras as long as they are Cohen-Macaulay (see [Theorem 4.6](#)). Our approach to showing that symmetric algebras have the weakly Gorenstein property is by computing explicitly the canonical module. The formula for the canonical module is interesting in its own right, and yields several applications.

We now describe a family of symmetric algebras that satisfy the weakly Gorenstein property, their canonical modules, and some related results that we obtain along the way. Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring, and $I = (f_1, \dots, f_n) \subset R$ be an ideal minimally generated by n elements and of codimension $g = \text{ht}(I) \geq 2$. One says that I has the property F_0 if for any $\mathfrak{p} \in V(I)$ the minimal number of generators satisfies $\mu(I_{\mathfrak{p}}) \leq \dim(R_{\mathfrak{p}}) + 1$. For any $k \geq 0$, one says that I has the sliding depth property SD_k if the depth of the i -th Koszul homology with respect to the sequence f_1, \dots, f_n is at least $d - n + i + k$ for all $i \geq 0$. The condition F_0 is necessary for the Cohen-Macaulayness of $\text{Sym}(I)$, and thus unavoidable to show the weakly Gorenstein condition. Under the additional assumptions that R is Gorenstein with infinite residue field and that I satisfies SD_1 , in [Theorem 4.6](#), we compute explicitly the canonical module of $\text{Sym}(I)$ and prove that $\text{Sym}(I)$ is d -weakly Gorenstein with respect to the ideal $\mathfrak{m}\text{Sym}(I)$.

These results are applicable in a wide range of situations since the condition SD_1 is satisfied by several classes of ideals (for instance, strongly Cohen-Macaulay ideals, and so in particular, perfect ideals of codimension two and perfect Gorenstein ideals codimension three). Our formula for the canonical

module of symmetric algebras coincides with a formula for the canonical module of certain Rees algebras (see [22]), which leads to interesting consequences in [Corollary 4.10](#).

Our main tool to compute $\omega_{\text{Sym}(I)}$ is a new complex that mends one of the main drawbacks of the approximation complex \mathcal{Z}_\bullet . The approximation complex \mathcal{Z}_\bullet is ubiquitous in the study of blowup algebras, and it provides a resolution of the symmetric algebra in many cases of relevance. However, the fact that it is made up of Koszul syzygies, which are typically not free, can be a non trivial obstacle. To remedy this problem we introduce a halfway resolution that refines \mathcal{Z}_\bullet . In [Proposition 4.5](#), we introduce the new complex \mathcal{L}_\bullet that consists of free modules in the last $g - 1$ positions and that coincides with \mathcal{Z}_\bullet in the remaining positions. The complex \mathcal{L}_\bullet is acyclic when \mathcal{Z}_\bullet is.

Furthermore, these halfway free resolutions lead to actual free resolutions of the symmetric algebra for special families of ideals such as almost complete intersections and perfect ideals of deviation two (see [Theorem 4.11](#)). The construction of the complex \mathcal{L}_\bullet is notable since computing free resolutions of symmetric algebras is a problem of tall order.

1.2. Applications to blowup algebras.

Our generalization of Jouanolou duality and the definition of weakly Gorenstein rings provide a general framework to study blowup algebras. We are particularly interested in finding the defining equations of the Rees algebra. Since Rees algebras appear as the coordinate ring of the blowup of a variety along a subvariety, the significance of finding their defining equations becomes apparent. This problem has been extensively studied by algebraic geometers, commutative algebraists, and researchers in applied areas like geometric modeling (see, e.g., [[6, 11–14, 27, 39, 41–44, 47, 48, 55](#)]).

When the symmetric algebra is a complete intersection, the classical Jouanolou duality is the standard (and the most forceful) tool to compute the defining equations of the Rees algebra (see [[6, 39, 42](#)]). We now describe and justify how our generalized Jouanolou duality can and does play a similar role when the symmetric algebra is only Cohen-Macaulay.

Let \mathbb{k} be a field, $R = \mathbb{k}[x_1, \dots, x_d]$ be a standard graded polynomial ring, $\mathfrak{m} = (x_1, \dots, x_d) \subset R$ be the graded irrelevant ideal, and $I = (f_1, \dots, f_n) \subset R$ be an ideal minimally generated by n forms of degree $D \geq 1$. Let $T = \mathbb{k}[y_1, \dots, y_n]$ be a standard graded polynomial ring and $\mathfrak{n} = (y_1, \dots, y_n) \subset T$ be the graded irrelevant ideal. Let $B = R \otimes_{\mathbb{k}} T$ be a standard bigraded polynomial ring, and consider the bihomogenous epimorphism

$$\Phi : B \twoheadrightarrow \mathcal{R}(I) = R[It] = \bigoplus_{j=0}^{\infty} I^j t^j \subset R[t], \quad x_i \mapsto x_i \text{ and } y_i \mapsto f_i t.$$

Then $\mathcal{J} = \text{Ker}(\Phi) \subset B$ is the defining ideal of $\mathcal{R}(I)$. The graph of the rational map

$$\mathcal{G} : \mathbb{P}_{\mathbb{k}}^{d-1} \dashrightarrow \mathbb{P}_{\mathbb{k}}^{n-1}$$

determined by the forms f_1, \dots, f_n is naturally given as $\Gamma = \text{BiProj}(B/\mathcal{J}) \subset \text{BiProj}(B) = \mathbb{P}_{\mathbb{k}}^{d-1} \times_{\mathbb{k}} \mathbb{P}_{\mathbb{k}}^{n-1}$.

Traditionally one considers the Rees algebra as a natural epimorphic image of the symmetric algebra $\text{Sym}(I)$ of I and one studies the kernel of this map,

$$0 \rightarrow \mathcal{A} \rightarrow \text{Sym}(I) \rightarrow \mathcal{R}(I) \rightarrow 0.$$

The kernel \mathcal{A} is the R -torsion submodule of $\text{Sym}(I)$. The defining equations of $\text{Sym}(I)$ are easily read from a minimal presentation matrix of I . An almost unavoidable constraint in the study of blowup algebras is the G_d condition. The ideal I satisfies G_d if $\mu(I_{\mathfrak{p}}) \leq \dim(R_{\mathfrak{p}})$ for all $\mathfrak{p} \in V(I)$ such that $\text{ht}(\mathfrak{p}) < d$. Furthermore, we need the conditions F_0 and SD_1 to show that $\text{Sym}(I)$ has the weakly Gorenstein property with respect to $\mathfrak{m}\text{Sym}(I)$. Under these conditions, in [Theorem 5.3](#), we provide a general framework of dualities to study blowup algebras. Let $\delta = (g-1)D - d$ and $\beta = d - g + 2$. Suppose that $n = d + 1$, $g = \text{ht}(I) \geq 2$, and $I \subset R$ satisfies the conditions G_d and SD_1 . We then show that the following six statements hold:

(i) $\mathcal{A} = H_{\mathfrak{m}}^0(\text{Sym}(I))$.

(ii) For all $0 \leq i \leq d-1$, there is an isomorphism of bigraded B -modules

$$H_{\mathfrak{m}}^i(\text{Sym}(I)) \cong {}^*\text{Ext}_{\mathbb{T}}^i(\text{Sym}(I), \mathbb{T})(-\delta, -\beta).$$

In particular, $\mathcal{A} = H_{\mathfrak{m}}^0(\text{Sym}(I)) \cong {}^*\text{Hom}_{\mathbb{T}}(\text{Sym}(I), \mathbb{T})(-\delta, -\beta)$.

(iii) For all $i < 0$ and $i > \delta$, we have $\mathcal{A}_{(i,*)} = 0$. There is an isomorphism $\mathcal{A}_{(\delta,*)} \cong \mathbb{T}(-\beta)$ of graded \mathbb{T} -modules. For all $0 \leq i \leq \delta$, we have the equality of $\text{Sym}(I)$ -ideals

$$\mathcal{A}_{(\geq i,*)} = 0_{:\text{Sym}(I)} \mathfrak{m}^{\delta+1-i}.$$

(iv) \mathcal{A} is minimally generated in \mathbf{x} -degree at most $(g-2)D - d + 1$.

(v) Let $0 \leq i \leq \delta$. The natural multiplication map $\mu: \mathcal{A}_{(i,*)} \otimes_{\mathbb{T}} \text{Sym}(I)_{(\delta-i,*)} \rightarrow \mathcal{A}_{(\delta,*)}$, $\mathbf{a} \otimes \mathbf{b} \mapsto \mathbf{a} \cdot \mathbf{b}$ is a perfect pairing that induces the abstract isomorphism

$$\nu: \mathcal{A}_{(i,*)} \xrightarrow{\cong} \text{Hom}_{\mathbb{T}}(\text{Sym}(I)_{(\delta-i,*)}, \mathcal{A}_{(\delta,*)})$$

seen in part (ii).

(vi) For all $2 \leq i \leq d+1$, there is an isomorphism of bigraded B -modules

$$H_{\mathfrak{n}}^i(\text{Sym}(I)) \cong {}^*\text{Ext}_{\mathbb{R}}^{i-1}(\text{Sym}(I), \mathbb{R})(-(g-1)D, g-1).$$

A couple of words regarding the results of [Theorem 5.3](#) are in place. Part (i) simply means that I is an ideal of linear type on the punctured spectrum on R . Part (ii) comes by applying our generalization of Jouanolou duality and from the fact that we prove $\text{Sym}(I)$ to be d -weakly Gorenstein with respect to the ideal $\mathfrak{m}\text{Sym}(I)$. Part (iii) shows that the graded components of \mathcal{A} with respect to the \mathbf{x} -grading can be read from the natural filtration

$$\mathcal{A} = 0_{:\text{Sym}(I)} \mathfrak{m}^{\delta+1} \supset 0_{:\text{Sym}(I)} \mathfrak{m}^{\delta} \supset \cdots \supset 0_{:\text{Sym}(I)} \mathfrak{m}^2 \supset 0_{:\text{Sym}(I)} \mathfrak{m} = \mathcal{A}_{(\delta,*)}.$$

Part (iv) goes even further, it gives an upper bound for the \mathbf{x} -degree of the minimal generators of \mathcal{A} , and it can actually be sharp (see [Remark 6.12](#)). Part (v) implies that the abstract isomorphism

$$\mathcal{A} = H_{\mathfrak{m}}^0(\text{Sym}(I)) \cong {}^*\text{Hom}_{\mathbb{T}}(\text{Sym}(I), \mathbb{T})(-\delta, -\beta)$$

naturally comes from a multiplication map, and this becomes a fundamental fact in our approach to study blowup algebras. Part (vi) gives a generalized Jouanolou duality statement with \mathbb{R} taking the role of \mathbb{T} . To show part (vi), since $\text{Sym}(I)$ is only 0-weakly Gorenstein with respect to the ideal $\mathfrak{n}\text{Sym}(I)$ (see

[Remark 4.8](#)), we instead rely on the isomorphism of part (ii) and a duality result of Herzog and Rahimi ([24]).

In a similar vein, we relate the freeness of \mathcal{A} to the depth of $\mathcal{R}(I)$. Explicitly, we prove in [Proposition 5.6](#) that \mathcal{A} is a free T -module if and only if $\text{depth}(\mathcal{R}(I)) \geq d$.

A remarkable feature of the classical Jouanolou duality is that it can be made completely explicit in terms of *Morley forms*. However, the usual notion of Morley forms is not enough in our setting. For this reason, we devise a new reduction procedure that makes explicit the perfect pairing of part (v). We now briefly describe our generalization of the theory of Morley forms. Let \mathbb{D} be the kernel of the natural multiplication map $\text{Sym}(I) \otimes_T \text{Sym}(I) \rightarrow \text{Sym}(I)$. We choose a suitable element $\Delta \in \text{Sym}(I) \otimes_T \text{Sym}(I)$ in the annihilator of the diagonal ideal \mathbb{D} that is homogeneous of degree δ in the κ -grading. Our definition of Morley forms $\text{morl}_{(\delta-i,i)}$ comes by considering the graded components of this element Δ . By applying [Theorem 2.18](#), we show that there is an explicit and computable homogeneous T -homomorphism

$$\xi : \text{Hom}_T(\text{Sym}(I)_{(\delta-i,*)}, \mathcal{A}_{(\delta,*)}) \rightarrow \mathcal{A}_{(i,*)}, \quad u \mapsto \frac{\Phi_\Delta(o \circ u)}{\alpha}$$

that gives the inverse map of the isomorphism $\nu : \mathcal{A}_{(i,*)} \rightarrow \text{Hom}_T(\text{Sym}(I)_{(\delta-i,*)}, \mathcal{A}_{(\delta,*)})$. Here Φ_Δ is a map with target \mathcal{A} and $\alpha \in T$ is a non zero element, both determined by Δ . For more details on the notation, see [§2.1](#), [§2.2](#), and [§5.1](#).

Since $\text{Sym}(I)$ is assumed to be Cohen-Macaulay, it is of dimension $d+1$ and $\mathfrak{m}\text{Sym}(I)$ is a minimal prime of $\text{Sym}(I)$. When $\text{Sym}(I)$ is a complete intersection at the minimal prime $\mathfrak{m}\text{Sym}(I)$, we provide a simple and direct method to find the required element Δ (see [Remark 5.8](#)). Several classes of ideals satisfy the condition that $\text{Sym}(I)$ is a complete intersection at $\mathfrak{m}\text{Sym}(I)$ (see [Remark 6.4](#) and [Remark 6.9](#)). If $\text{Sym}(I)$ is a complete intersection, then Δ coincides with the element used by Jouanolou in the construction of the classical Morley forms.

The last part of the paper is dedicated to applications. We show that our results can deal with cases that were unreachable with previously existing methods. Our presentation here is divided in terms of two families of ideals that we treat.

Zero dimensional ideals. If $I \subset R$ is additionally an \mathfrak{m} -primary ideal, then we prove that $\mathcal{R}(I)$ can even be approximated by two better understood algebras. One is the usual symmetric algebra $\text{Sym}(I)$. And the other, less standard choice is the symmetric algebra $\text{Sym}(E)$, where E is the module defined by the Koszul syzygies. In this case, the ideal I satisfies all the conditions of [Theorem 5.3](#), and so we can use $\text{Sym}(I)$ and all the above results to approximate and to study $\mathcal{R}(I)$. On the other hand, in [Theorem 6.3](#), we also show that there is a short exact sequence $0 \rightarrow \mathcal{B} \rightarrow \text{Sym}(E) \rightarrow \mathcal{R}(I) \rightarrow 0$ with $\mathcal{B} = H_{\mathfrak{m}}^0(\text{Sym}(E))$, and that there is an isomorphism of bigraded B -modules

$$H_{\mathfrak{m}}^i(\text{Sym}(E)) \cong {}^* \text{Ext}_T^i(\text{Sym}(E), T)(-dD - d, -1)$$

for all $0 \leq i \leq d-1$. This opens new possibilities to study $\mathcal{R}(I)$ when I is assumed be a zero dimensional ideal. We plan to pursue this approach further in a subsequent paper.

Gorenstein ideals of codimension three. In addition, suppose that $I \subset R$ is a Gorenstein ideal of codimension three. This case is covered by [Theorem 5.3](#) and so one can use the above results to study $\mathcal{R}(I)$ (see [Theorem 6.6](#)). Let $h \geq 1$ be the degree of the homogeneous elements in a minimal alternating

presentation matrix $\varphi \in \mathbb{R}^{(d+1) \times (d+1)}$ of I . For ease of exposition, we restrict to the case $h = 2$ with the extra assumption that the monomial support of the entries of φ generates an almost complete intersection, and we leave the other cases to be studied in a subsequent paper. For this case, in [Theorem 6.10](#), we give a complete picture of the problem. We show that the ideal $\mathcal{J} \subset \mathbb{B}$ of defining equations of $\mathcal{R}(I)$ is minimally generated by the defining ideal of $\text{Sym}(I)$ and three forms of bidegrees $(0, 2d - 2)$, $(1, d - 1)$ and $(1, d - 1)$. In fact, we explicitly compute these minimal generators of \mathcal{J} in terms of the Jacobian dual of φ and Morley forms! In [Theorem 6.10](#), we also prove that $\deg(\mathcal{G}) = 2^{d-2}$, $\deg(Y) = 2d - 2$, $\text{depth}(\mathcal{R}(I)) \geq d$, and $\text{depth}(\text{gr}_1(\mathcal{R})) \geq d - 1$, where $\mathcal{G} : \mathbb{P}_{\mathbb{k}}^{d-1} \dashrightarrow \mathbb{P}_{\mathbb{k}}^d$ is the corresponding rational map and $Y \subset \mathbb{P}_{\mathbb{k}}^d$ is the closure of the image of \mathcal{G} .

Outline. The structure of the paper is as follows. In [Section 2](#), we provide an extension of Jouanolou duality and Morley forms. In [Section 3](#), we identify determinantal rings that satisfy the weakly Gorenstein condition. The weakly Gorenstein condition for symmetric algebras is studied in [Section 4](#). In [Section 5](#), we provide a general framework of duality statements that are relevant in the study of blowup algebras. Lastly, [Section 6](#) is dedicated to the study of specific families of ideals where we apply the methods developed in this paper.

2. AN EXTENSION OF JOUANOLOU DUALITY AND MORLEY FORMS

In this section, we provide a generalization of Jouanolou duality that can be applied to a plethora of situations. We assume the following setup throughout.

Setup 2.1. Let S be a positively graded Noetherian ring with S_0 a factor ring of a local Gorenstein ring T . Choose a positively graded polynomial ring $\mathbb{B} = T[x_1, \dots, x_d]$ such that we have a graded surjection $\mathbb{B} \twoheadrightarrow S$. Let $\mathfrak{m} = (x_1, \dots, x_d)$ be the graded irrelevant ideal of \mathbb{B} . Set $b := \deg(x_1) + \dots + \deg(x_d)$.

Remark 2.2. In this paper, we freely use basic properties of canonical modules, and our standard reference is [[3](#), Chapter 3]. Since T is assumed to be a local Gorenstein ring, the graded canonical module of \mathbb{B} is given by $\omega_{\mathbb{B}} = \mathbb{B}(-b)$ (see [[3](#), §3.6]). As a consequence the graded canonical module of S can be computed as $\omega_S = \text{Ext}_{\mathbb{B}}^c(S, \mathbb{B}(-b))$ where $c = \dim(\mathbb{B}) - \dim(S)$.

For a graded \mathbb{B} -module M , we denote the graded T -dual as

$${}^*\text{Hom}_T(M, T) := \bigoplus_{j \in \mathbb{Z}} \text{Hom}_T([M]_{-j}, T),$$

and the corresponding right derived functor as ${}^*\text{Ext}_T^i(M, T)$. Note that ${}^*\text{Ext}_T^i(M, T)$ is naturally a graded \mathbb{B} -module for all $i \geq 0$. For more details on the functors ${}^*\text{Ext}_T^i$ and their properties, the reader is referred to [[3](#), §1.5].

The following result yields a version of Jouanolou duality in terms of the canonical module of S , and it is applicable in great generality.

Theorem 2.3. *Assume [Setup 2.1](#). Suppose that S is perfect over \mathbb{B} and of codimension c . Then we have a graded isomorphism of S -modules*

$$H_{\mathfrak{m}}^i(\omega_S) \cong {}^*\text{Ext}_T^{i+c-d}(S, T)$$

for all $i \in \mathbb{Z}$.

First proof. Let $F_\bullet : 0 \rightarrow F_c \rightarrow \cdots \rightarrow F_1 \rightarrow F_0$ be a minimal graded B -resolution of S . Since S is perfect of codimension c , F_\bullet has length equal to c . We then have the following isomorphisms of graded B -modules

$$\begin{aligned}
 {}^*\mathrm{Ext}_T^{i+c-d}(S, T) &\cong H^{i+c-d}({}^*\mathrm{Hom}_T(F_\bullet, T)) \\
 &\cong H^{i+c-d}({}^*\mathrm{Hom}_T(F_\bullet \otimes_B B, T)) \\
 &\cong H^{i+c-d}(\mathrm{Hom}_B(F_\bullet, {}^*\mathrm{Hom}_T(B, T))) && \text{by Hom-tensor adjointness} \\
 &\cong H^{i+c-d}(\mathrm{Hom}_B(F_\bullet, B) \otimes_B {}^*\mathrm{Hom}_T(B, T)) \\
 &\cong H^{i+c-d}(\mathrm{Hom}_B(F_\bullet, B) \otimes_B H_m^d(B)(-b)) && \text{by graded local duality} \\
 &\cong H_{c-(i+c-d)}(\mathrm{Hom}_B(F_\bullet, B)[-c] \otimes_B H_m^d(B)(-b)) && \text{by a homological shift} \\
 &\cong \mathrm{Tor}_{d-i}^B(\omega_S(b), H_m^d(B)(-b)) = \mathrm{Tor}_{d-i}^B(\omega_S, H_m^d(B)).
 \end{aligned}$$

The last step in the above sequence of isomorphisms follows from the fact that $\mathrm{Hom}_B(F_\bullet, B(-b))[-c]$ is a minimal homogeneous B -resolution of ω_S , because by assumption S is perfect over B .

The Čech complex $C_m^\bullet : 0 \rightarrow B \rightarrow \bigoplus_{i=1}^d B_{x_i} \rightarrow \cdots \rightarrow B_{x_1 \cdots x_d} \rightarrow 0$ with respect to the sequence x_1, \dots, x_d is a complex of flat B -modules, $H^i(C_m^\bullet) = 0$ for $i < d$, and $H^d(C_m^\bullet) \cong H_m^d(B)$. By computing $\mathrm{Tor}_{d-i}^B(\omega_S, H_m^d(B))$ via this flat resolution of $H_m^d(B)$, we obtain that

$$\mathrm{Tor}_{d-i}^B(\omega_S, H_m^d(B)) \cong H^{d-(d-i)}(\omega_S \otimes_B C_m^\bullet) \cong H_m^i(\omega_S).$$

Finally, by combining all these isomorphisms, we obtain

$$H_m^i(\omega_S) \cong \mathrm{Tor}_{d-i}^B(\omega_S, H_m^d(B)) \cong {}^*\mathrm{Ext}_T^{i+c-d}(S, T). \quad \square$$

*Second proof.*¹ As before, let $F_\bullet : 0 \rightarrow F_c \rightarrow \cdots \rightarrow F_1 \rightarrow F_0$ be a minimal graded B -resolution of S . Since S is perfect by assumption, it follows that $G_\bullet := \mathrm{Hom}_B(F_\bullet, B(-b))[-c]$ yields a minimal graded B -resolution of ω_S . From the functorial isomorphism $H_m^d(B) \cong {}^*\mathrm{Hom}_T(\mathrm{Hom}_B(B, B(-b)), T)$, we obtain the following isomorphism of graded complexes

$$\begin{aligned}
 H_m^d(G_\bullet) &\cong {}^*\mathrm{Hom}_T(\mathrm{Hom}_B(G_\bullet, B(-b)), T) \\
 &\cong {}^*\mathrm{Hom}_T(\mathrm{Hom}_B(\mathrm{Hom}_B(F_\bullet, B(-b))[-c], B(-b)), T) \\
 &\cong {}^*\mathrm{Hom}_T(F_\bullet[c], T).
 \end{aligned}$$

The spectral sequences coming from the second quadrant double complex $G_\bullet \otimes_B C_m^\bullet$ converge in the second pages and yield the following isomorphism

$$H_m^i(\omega_S) \cong H_{d-i}(H_m^d(G_\bullet))$$

for all $i \geq 0$. After putting the above isomorphisms together, we obtain

$$\begin{aligned}
 H_m^i(\omega_S) &\cong H_{d-i}(H_m^d(G_\bullet)) \cong H_{d-i}({}^*\mathrm{Hom}_T(F_\bullet[c], T)) \\
 &\cong H^{i+c-d}({}^*\mathrm{Hom}_T(F_\bullet, T)) \\
 &\cong {}^*\mathrm{Ext}_T^{i+c-d}(S, T).
 \end{aligned}$$

¹This proof is *à la Jouanolou* in the sense that we utilize a spectral sequence argument that is present in several parts of Jouanolou's work.

This concludes the second proof of the theorem. \square

To obtain a “true” generalization of Jouanolou’s duality in terms of S , we need to relate the local cohomology modules of S and ω_S . For this purpose, we introduce the following general definition.

Definition 2.4. In addition to [Setup 2.1](#) suppose that S is Cohen-Macaulay. Let $\mathfrak{a} \subset S$ be a homogeneous ideal and $i \geq 0$ be an integer. We say that S is *i-weakly Gorenstein with respect to \mathfrak{a}* if there exists a homogeneous element $y \in \omega_S$ such that

$$\dim(\text{Supp}(\omega_S/Sy) \cap V(\mathfrak{a})) < \dim(S) - i$$

and $Sy \otimes_S S_{\mathfrak{p}} \cong \omega_{S_{\mathfrak{p}}}$ for all $\mathfrak{p} \in \text{Ass}(S)$. We will refer to y as a *weak generator* of ω_S .

An equivalent condition is that there exists a homogeneous element $y \in \omega_S$ that generates ω_S at the associated primes of S and at all the primes containing \mathfrak{a} with codimension $\leq i$; in particular, S becomes a Gorenstein ring after localizing at all these primes. However, the weakly Gorenstein property is considerably stronger due to the uniform choice of y that works for every prime in question. In fact, if S is 1-weakly Gorenstein with respect to a nilpotent ideal then S is already Gorenstein. In general, enlarging \mathfrak{a} weakens the i -weakly Gorenstein condition, whereas increasing i strengthens it. A simple reinterpretation of the condition is given in the following remark.

Remark 2.5. In addition to [Setup 2.1](#) suppose that S is Cohen-Macaulay. Then S is i -weakly Gorenstein with respect to a homogeneous ideal $\mathfrak{a} \subset S$ if and only if there is a homogeneous element $y \in \omega_S$ such that

- (i) $\text{ht}(\text{Ann}_S(\omega_S/Sy) + \mathfrak{a}) \geq i + 1$, and
- (ii) $\text{ht}(\text{Ann}_S(\omega_S/Sy)) \geq 1$.

The usefulness of this definition becomes apparent with the following lemma.

Lemma 2.6. *Assume that S is a positively graded Cohen-Macaulay ring. Suppose S is $(i + 1)$ -weakly Gorenstein with respect to a homogeneous ideal $\mathfrak{a} \subset S$. Let $y \in \omega_S$ be a weak generator of the canonical module. Then we have a graded isomorphism of S -modules*

$$H_{\mathfrak{a}}^i(\omega_S) \cong H_{\mathfrak{a}}^i(S)(-\deg(y)).$$

Proof. As the canonical module is faithful, for any $\mathfrak{p} \in \text{Ass}(S)$ we get $\text{Ann}_{S_{\mathfrak{p}}}(Sy \otimes_S S_{\mathfrak{p}}) = \text{Ann}_{S_{\mathfrak{p}}}(\omega_S \otimes_S S_{\mathfrak{p}}) = 0$, and so it follows that $\text{Ann}_S(y) = 0$. As a consequence, we have $Sy \cong S(-\deg(y))$. From the short exact sequence $0 \rightarrow Sy \rightarrow \omega_S \rightarrow \omega_S/Sy \rightarrow 0$, we obtain the exact sequence in cohomology

$$H_{\mathfrak{a}}^{i-1}(\omega_S/Sy) \rightarrow H_{\mathfrak{a}}^i(Sy) \rightarrow H_{\mathfrak{a}}^i(\omega_S) \rightarrow H_{\mathfrak{a}}^i(\omega_S/Sy).$$

Thus, to conclude the proof it suffices to show that $\text{grade}(\mathfrak{a}, \omega_S/Sy) \geq i + 1$. Equivalently, we need to prove that $\text{depth}((\omega_S/Sy)_{\mathfrak{p}}) \geq i + 1$ for all $\mathfrak{p} \in \text{Supp}(\omega_S/Sy) \cap V(\mathfrak{a})$. From the definition of weakly Gorenstein we get that $\text{depth}(S_{\mathfrak{p}}) = \dim(S_{\mathfrak{p}}) \geq i + 2$ for all $\mathfrak{p} \in \text{Supp}(\omega_S/Sy) \cap V(\mathfrak{a})$. It then follows from the short exact sequence above that $\text{depth}((\omega_S/Sy)_{\mathfrak{p}}) \geq i + 1$ for all $\mathfrak{p} \in \text{Supp}(\omega_S/Sy) \cap V(\mathfrak{a})$, as required. \square

Remark 2.7. If in addition to the assumption of [Lemma 2.6](#) one has $i \geq \text{grade}(\mathfrak{a})$, then $\deg(y)$ is independent of the choice of y .

Proof. Write $t = \text{grade}(\mathfrak{a})$ and notice that $H_{\mathfrak{a}}^t(S) \neq 0$. Since $H_{\mathfrak{a}}^t(S)(-\text{deg}(\mathfrak{y})) \cong H_{\mathfrak{a}}^t(\omega_S)$, by [Lemma 2.6](#), it follows that $\text{deg}(\mathfrak{y})$ only depends on S . \square

Finally, we are ready for our promised generalization of Jouanolou duality.

Theorem 2.8 (Generalized Jouanolou duality). *Assume [Setup 2.1](#). Suppose that S is perfect over B of codimension c , and that S is $(i+1)$ -weakly Gorenstein with respect to \mathfrak{m}_S . Let $\mathfrak{y} \in \omega_S$ be a weak generator of the canonical module. Then we have a graded isomorphism of S -modules*

$$H_{\mathfrak{m}}^i(S) \cong {}^* \text{Ext}_{\mathbb{T}}^{i+c-d}(S, \mathbb{T})(\text{deg}(\mathfrak{y}))$$

for all $i \in \mathbb{Z}$.

Proof. This follows by combining [Theorem 2.3](#) and [Lemma 2.6](#). \square

2.1. Perfect pairing and Morley forms. In this subsection, our goal is to show that the isomorphism of [Theorem 2.8](#) for $i=0$ and $c=d$ arises from a perfect pairing given by multiplication. In addition, we want to make this isomorphism explicit via Morley forms.

Observation 2.9. Assume [Setup 2.1](#). The following statements hold:

- (i) The module ${}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T})$ is concentrated in nonpositive degrees.
- (ii) $H_{\mathfrak{m}}^0({}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T})) = {}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T})$.
- (iii) $[{}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T})]_0 \cong \mathbb{T}$.

Lemma 2.10. *If S is standard graded, then for every $i \geq 0$,*

$$0 : {}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T}) \mathfrak{m}^i = [{}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T})]_{\geq -i+1}.$$

Proof. Since the other inclusion is clear, we only need to show that

$$0 : {}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T}) \mathfrak{m}^i \subset [{}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T})]_{\geq -i+1}.$$

Let f be a homogeneous element of $0 : {}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T}) \mathfrak{m}^i$. Suppose that f has degree j with $j \leq -i$. It follows that the map f restricted to $S_{\leq i-1}$ is zero. On the other hand, $\mathfrak{m}^i f = 0$ implies that f restricted to $S_{\geq i}$ is also zero. We conclude that $f = 0$. \square

The following setup is used for the remainder of this subsection.

Setup 2.11. In addition to [Setup 2.1](#), assume that S is standard graded with $S_0 = \mathbb{T}$, perfect over B with $\dim(S) = \dim(\mathbb{T})$, and S is 1-weakly Gorenstein with respect to \mathfrak{m}_S . Write \mathcal{A} for $H_{\mathfrak{m}}^0(S)$. Let $\mathfrak{y} \in \omega_S$ be a weak generator of the canonical module of degree $-\delta$.

The next result describes the isomorphism of [Theorem 2.8](#) (with $i=0$ and $c=d$) in terms of a perfect pairing induced by the natural multiplication map. It also shows, in particular, that $\mathcal{A} \neq 0$ and $\delta \geq 0$.

Theorem 2.12. *Assume [Setup 2.11](#). The following statements hold:*

- (i) *The S -module $\mathcal{A} = H_{\mathfrak{m}}^0(S)$ is concentrated in degree at most δ .*
- (ii) *The \mathbb{T} -module \mathcal{A}_{δ} is free of rank one.*

(iii) For $0 \leq i \leq \delta$, the natural multiplication map

$$\mu : \mathcal{A}_i \otimes_{\mathbb{T}} \mathcal{S}_{\delta-i} \xrightarrow{\text{mult}} \mathcal{A}_\delta$$

is a perfect pairing that induces an isomorphism

$$\nu : \mathcal{A}_i \xrightarrow{\cong} \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathcal{A}_\delta).$$

In addition, if we write $\mathcal{A}_\delta = \mathbb{T}\mathfrak{s}$ and fix an isomorphism $\circ : \mathbb{T}\mathfrak{s} \xrightarrow{\cong} \mathbb{T}$ with $\mathfrak{s} \mapsto 1$, then the composition of μ and \circ ,

$$\mathcal{A}_i \otimes_{\mathbb{T}} \mathcal{S}_{\delta-i} \xrightarrow{\text{mult}} \mathcal{A}_\delta = \mathbb{T}\mathfrak{s} \xrightarrow{\circ} \mathbb{T}$$

is also a perfect pairing that induces an isomorphism

$$\nu' : \mathcal{A}_i \xrightarrow{\cong} \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathbb{T})$$

as in [Theorem 2.8](#).

(iv) $0 :_{\mathfrak{S}} \mathfrak{m} = \mathcal{A}_\delta = \mathbb{T}\mathfrak{s} \cong \mathbb{T}$.

Proof. According to [Theorem 2.8](#) we have a graded isomorphism of \mathbb{S} -modules

$$\mathcal{A} = \mathbf{H}_{\mathfrak{m}}^0(\mathbb{S}) \cong {}^*\text{Hom}_{\mathbb{T}}(\mathbb{S}, \mathbb{T})(-\delta).$$

Now part (i) and (ii) follow from [Observation 2.9](#). The same graded isomorphism of \mathbb{S} -modules identifies the map μ with the multiplication map

$$\text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathbb{T}) \otimes_{\mathbb{T}} \mathcal{S}_{\delta-i} \xrightarrow{\text{mult}} \text{Hom}_{\mathbb{T}}(\mathcal{S}_0, \mathbb{T}) = \mathbb{T},$$

which is a perfect pairing since it induces the identity map

$$\text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathbb{T}) \xrightarrow{\text{id}} \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathbb{T}). \quad \square$$

Finally, part (iv) follows from [Theorem 2.8](#), [Lemma 2.10](#), and parts (i) and (iii).

We will now construct explicit inverses of the maps ν using a generalization of Jouanolou's Morley forms. To any homogeneous element of degree δ in the annihilator of the diagonal ideal \mathbb{D} we associate forms that we call *Morley forms* in honor of Jouanolou. Recall that the *diagonal ideal* \mathbb{D} of the enveloping algebra $S^e := S \otimes_{\mathbb{T}} S$ is the kernel of the natural multiplication map $S^e \rightarrow S$. This ideal is generated by the elements $\overline{x_i} \otimes 1 - 1 \otimes \overline{x_i}$, where $\overline{x_i}$ denotes the image of x_i in S .

We think of S^e and of ${}^*\text{Hom}_{\mathbb{T}}({}^*\text{Hom}_{\mathbb{T}}(\mathbb{S}, \mathbb{T}), S)$ as $S - S$ -bimodules with S acting on the left and on the right. The largest submodules on which the left and right S -module structures coincide are $0 :_{S^e} \mathbb{D}$ and ${}^*\text{Hom}_S({}^*\text{Hom}_{\mathbb{T}}(\mathbb{S}, \mathbb{T}), S)$, respectively. Consider the homogeneous homomorphism of $S - S$ -bimodules

$$S^e \rightarrow {}^*\text{Hom}_{\mathbb{T}}({}^*\text{Hom}_{\mathbb{T}}(\mathbb{S}, \mathbb{T}), S)$$

given by $s_1 \otimes s_2 \mapsto (f \mapsto s_2 f(s_1))$. Restricting this map we obtain a homogeneous S -linear map

$$0 :_{S^e} \mathbb{D} \rightarrow {}^*\text{Hom}_S({}^*\text{Hom}_{\mathbb{T}}(\mathbb{S}, \mathbb{T}), S)$$

(see also [\[51, proof of Theorem 3.1\]](#), [\[40, Proposition F.9\]](#), [\[17, proof of Theorem A.1\]](#)). From [Observation 2.9\(ii\)](#) we see that the target of this map is ${}^*\text{Hom}_S({}^*\text{Hom}_{\mathbb{T}}(\mathbb{S}, \mathbb{T}), \mathbf{H}_{\mathfrak{m}}^0(\mathbb{S}))$. Hence we obtain a

homogeneous S -linear map

$$0 :_{S^e} \mathbb{D} \rightarrow {}^* \text{Hom}_S({}^* \text{Hom}_T(S, T), \mathcal{A}).$$

[Theorem 2.8](#) implies

$${}^* \text{Hom}_S({}^* \text{Hom}_T(S, T), \mathcal{A}) \cong {}^* \text{Hom}_S({}^* \text{Hom}_T(S, T), {}^* \text{Hom}_T(S, T))(-\delta),$$

and using Hom-Tensor adjointness we obtain

$${}^* \text{Hom}_S({}^* \text{Hom}_T(S, T), {}^* \text{Hom}_T(S, T)) \cong {}^* \text{Hom}_T(S \otimes_S {}^* \text{Hom}_T(S, T), T) \cong {}^* \text{Hom}_T({}^* \text{Hom}_T(S, T), T).$$

The last graded S -module is concentrated in degrees ≥ 0 and its degree 0 component is T . It follows that any homogeneous S -automorphism of ${}^* \text{Hom}_T(S, T)$ is the identity map times a unit in T . Hence all homogeneous S -isomorphisms of degree δ from ${}^* \text{Hom}_T(S, T)$ to \mathcal{A} are equal up to multiplication by a unit in T .

The above discussion yields the following remark.

Remark 2.13. Every homogeneous element $\Delta = \sum_i s_{i,1} \otimes s_{i,2}$ of degree δ in $0 :_{S^e} \mathbb{D}$ yields a homogeneous S -linear map

$$\Phi_\Delta : {}^* \text{Hom}_T(S, T) \rightarrow \mathcal{A}, \quad \mathbf{u} \mapsto \sum_i s_{i,2} \mathbf{u}(s_{i,1})$$

of degree δ . If Φ_Δ is a bijection, then Φ_Δ is independent of the choice of Δ , up to multiplication by a unit in T .

Let $\pi : S \rightarrow T$ be the homomorphism of T -algebras with $\pi(x_i) = 0$ for all i , and consider the map $\varepsilon = S \otimes_T \pi : S^e \rightarrow S$. Write $\eta : S^e \rightarrow S$ for the natural multiplication map and recall that $\eta(0 :_{S^e} \mathbb{D}) = \mathfrak{d}_N(S/T)$ is the *Noether different* of S over T , which defines the ramification locus of S over T . To identify elements Δ as in [Remark 2.13](#) that provide the desired isomorphisms, we need to understand the relationship between the three ideals $\mathfrak{d}_N(S/T)$, $\varepsilon(0 :_{S^e} \mathbb{D})$, and $0 :_S \mathfrak{m} = \mathcal{A}_\delta$ (the last equality follows from [Theorem 2.12\(iv\)](#)). Experimental evidence supports the following conjecture:

Conjecture 2.14. (i) $\varepsilon(0 :_{S^e} \mathbb{D}) = 0 :_S \mathfrak{m}$.
 (ii) If T is a domain, then $\mathfrak{d}_N(S/T) = (\text{rank}_T S) \cdot \varepsilon(0 :_{S^e} \mathbb{D})$.

The next two results provide further evidence for this conjecture and show, in particular, that it holds after tensoring with the total ring of quotients of T .

Remark 2.15. One has $\varepsilon(0 :_{S^e} \mathbb{D}) \subset 0 :_S \mathfrak{m}$.

Proof. This is clear since $\varepsilon(\mathbb{D}) = \mathfrak{m}$. □

Proposition 2.16. Assume [Setup 2.11](#). For L the total ring of quotients of T , we let S_L be the standard graded L -algebra $S \otimes_T L$, and S_L^e and \mathbb{D}_L be the corresponding enveloping algebra and diagonal ideal. The following statements hold:

- (i) $0 :_{S_L^e} \mathbb{D}_L \cong S_L(-\delta)$.
- (ii) For every associated prime \mathfrak{p} of $\mathfrak{m}S$, we have $[0 :_{S_L^e} \mathbb{D}_L]_\delta \not\subset (1 \otimes \mathfrak{p}) S_L^e$. In particular,

$$[0 :_{S^e} \mathbb{D}]_\delta \not\subset (1 \otimes \mathfrak{p}) S^e.$$

- (iii) For every associated prime \mathfrak{q} of T , $\varepsilon([0 :_{S^e} \mathbb{D}]_\delta) \not\subseteq \mathfrak{q}S$. In fact, if $\Delta \in [0 :_{S^e} \mathbb{D}]_\delta$ with $\Delta \notin (1 \otimes \mathfrak{p})S^e$ for every associated prime \mathfrak{p} of mS , then $\varepsilon(\Delta) \notin \mathfrak{q}S$ for every associated prime \mathfrak{q} of T .
- (iv) $\varepsilon(0 :_{S_L^e} \mathbb{D}_L) = \varepsilon([0 :_{S_L^e} \mathbb{D}_L]_\delta) = 0 :_{S_L} m_L$; in particular, $\varepsilon(0 :_{S^e} \mathbb{D})_{\mathfrak{q}} = \varepsilon([0 :_{S^e} \mathbb{D}]_\delta)_{\mathfrak{q}} = (0 :_S m)_{\mathfrak{q}}$ for every associated prime \mathfrak{q} of S .
- (v) If L is a field, then $\mathfrak{d}_N(S_L/L) = (\dim_L(S_L)) \cdot \varepsilon(0 :_{S_L^e} \mathbb{D}_L)$.

Proof. The ring L is a finite product of Artinian local Gorenstein rings. Since $\dim(S) = \dim(T)$, the B -module S is perfect of codimension d . Therefore S_L is a perfect module of codimension d over $B_L = B \otimes_T L = L[x_1, \dots, x_d]$. Thus S_L is a finite module over L of finite projective dimension. Now the Auslander-Buchsbaum formula, applied to the factors of L , shows that S_L is flat over L .

By [Setup 2.11](#), $\omega_{S_L} \cong S_L \cdot \mathfrak{y} \cong S_L(\delta)$. Since S_L is flat over L , it follows that $\omega_{S_L^e} \cong S_L^e(2\delta)$, as can be seen from a homogeneous free resolution of S_L over B_L . In particular S_L is Gorenstein.

Now part (i) follows because

$$0 :_{S_L^e} \mathbb{D}_L \cong \text{Hom}_{S_L^e}(S_L^e/\mathbb{D}_L, S_L^e) \cong \text{Hom}_{S_L^e}(S_L, \omega_{S_L^e}(-2\delta)) \cong \omega_{S_L}(-2\delta) \cong S_L(-\delta).$$

As to part (ii), since $[0 :_{S_L^e} \mathbb{D}_L]_\delta$ generates the ideal $0 :_{S_L^e} \mathbb{D}_L$ by part (i), it suffices to prove $0 :_{S_L^e} \mathbb{D}_L \not\subseteq (1 \otimes \mathfrak{p})S_L^e$. Suppose the contrary. Since S_L^e is an Artinian Gorenstein ring, it follows that

$$\mathbb{D}_L = 0 :_{S_L^e} (0 :_{S_L^e} \mathbb{D}_L) \supset 0 :_{S_L^e} ((1 \otimes \mathfrak{p})S_L^e) \supset (1 \otimes (0 :_{S_L} \mathfrak{p}))S_L^e.$$

Applying the multiplication map $S_L^e \rightarrow S_L$ that has \mathbb{D}_L as its kernel, one sees that $0 :_{S_L} \mathfrak{p} = 0$. This is impossible because S_L is Artinian and $\mathfrak{p}S_L \neq S_L$ as $\mathfrak{p} = \mathfrak{q} + mS$ for some associated prime \mathfrak{q} of T .

If \mathfrak{q} is an associated prime of T , then $\mathfrak{p} = \mathfrak{q} + mS$ is an associated prime of mS , and $\varepsilon^{-1}(\mathfrak{q}S) = (1 \otimes \mathfrak{p})S^e$. Hence we can use part (ii) to establish part (iii).

By [Remark 2.15](#), part (iv) follows once we show that the inclusion $\varepsilon([0 :_{S_L^e} \mathbb{D}_L]_\delta) \subset 0 :_{S_L} m_L$ is an equality. [Theorem 2.12\(iv\)](#) implies that $0 :_{S_L} m_L = L\mathfrak{s}$, hence $\varepsilon([0 :_{S_L^e} \mathbb{D}_L]_\delta) = K\mathfrak{s}$ for some ideal K of L . If $K \neq L$, then K is contained in a prime ideal of L , contradicting part (iii).

Let \mathfrak{q} be an associated, hence minimal, prime of S . If \mathfrak{q} contains mS , then \mathfrak{q} contracts to a minimal prime of T and the asserted equality locally at \mathfrak{q} follows from the one just proved. If on the other hand \mathfrak{q} does not contain mS , then $(0 :_S m)_{\mathfrak{q}} = 0$ and we are done by [Remark 2.15](#).

Part (v) is a consequence of part (iv) and the equality $\mathfrak{d}_N(S_L/L) = (\dim_L(S_L)) \cdot (0 :_{S_L} m_L)$. This equality follows from [\[17, Theorems A.1 and A.5\]](#) and the fact that if $\dim_L(S_L)$ is a multiple of the characteristic then the trace map $\text{Tr} := \text{Tr}_{S_L/L}$ is zero; indeed $\text{Tr}(L) = 0$ by the assumption on the characteristic and $\text{Tr}(mS_L) = 0$ because the elements of mS_L are nilpotent. \square

We choose an element $\Delta \in [0 :_{S^e} \mathbb{D}]_\delta$ with $\Delta \notin (1 \otimes \mathfrak{p})S^e$ for every associated prime \mathfrak{p} of mS . If the residue field of T is infinite, such an element exists by [Proposition 2.16\(ii\)](#) and any general element in $[0 :_{S^e} \mathbb{D}]_\delta$ will do. We consider S^e as a standard bigraded T -algebra with

$$\text{bideg}(x_i \otimes 1) = (1, 0) \quad \text{and} \quad \text{bideg}(1 \otimes x_i) = (0, 1).$$

Thus we obtain the decomposition

$$\Delta = \sum_{i=0}^{\delta} \text{morl}_{(\delta-i, i)} \quad \text{where} \quad \text{morl}_{(\delta-i, i)} \in [S^e]_{(\delta-i, i)}.$$

We say that $\text{morl}_{(\delta-i,i)}$ is the i -th Morley form associated to Δ . We now list some basic properties of these Morley forms. Recall that by [Theorem 2.12\(iv\)](#),

$$0 :_{\mathcal{S}} \mathfrak{m} = \mathcal{A}_\delta = \mathbb{T}\mathfrak{s} \cong \mathbb{T}.$$

Lemma 2.17. *The following statements hold:*

- (i) $\text{morl}_{(\delta,0)} = \alpha \cdot \mathfrak{s} \otimes 1 \in \mathcal{S}_\delta \otimes_{\mathbb{T}} \mathcal{S}_0$ for some non zerodivisor α in \mathbb{T} .
- (ii) For any $\mathfrak{b} \in \mathcal{S}_l$ with $l \leq i$, we have the equality

$$(\mathfrak{b} \otimes 1) \cdot \text{morl}_{(\delta-i,i)} = (1 \otimes \mathfrak{b}) \cdot \text{morl}_{(\delta-i+l,i-l)} \in \mathcal{S}_{\delta-i+l} \otimes_{\mathbb{T}} \mathcal{S}_i.$$

Proof. As for part (i), notice that $\text{morl}_{(\delta,0)} = \varepsilon(\Delta) \otimes 1$. [Remark 2.15](#) and [Theorem 2.12\(iv\)](#) show that $\varepsilon(\Delta) = \alpha \cdot \mathfrak{s}$ for some $\alpha \in \mathbb{T}$, and α is a non zerodivisor by [Proposition 2.16\(iii\)](#). Part (ii) is obvious since $\Delta \in 0 :_{\mathcal{S}^e} \mathbb{D}$. \square

In the following theorem by utilizing our generalized Morley forms, we obtain an explicit inverse of the isomorphism $\nu : \mathcal{A}_i \rightarrow \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathcal{A}_\delta)$ from [Theorem 2.12\(iii\)](#). For the statement of [Theorem 2.18](#), we observe that \mathcal{A}_i is \mathbb{T} -torsionfree owing to the isomorphism $\mathcal{A}_i \cong \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathbb{T})$ and that the element α of [Lemma 2.17\(i\)](#) is a non zerodivisor in \mathbb{T} .

Theorem 2.18. *Assume [Setup 2.11](#), let \mathfrak{s} and \mathfrak{o} be as in [Theorem 2.12\(iii\)](#), let α be as in [Lemma 2.17\(ii\)](#), and let Φ_Δ be as in [Remark 2.13](#). Let $0 \leq i \leq \delta$. The following statements hold:*

- (i) For any $\mathfrak{u} \in \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathcal{A}_\delta)$, we have that $\Phi_\Delta(\mathfrak{o} \circ \mathfrak{u}) \in \alpha \cdot \mathcal{A}_i$. We have a \mathbb{T} -homomorphism

$$\xi : \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathcal{A}_\delta) \rightarrow \mathcal{A}_i, \quad \mathfrak{u} \mapsto \frac{\Phi_\Delta(\mathfrak{o} \circ \mathfrak{u})}{\alpha}.$$

- (ii) The \mathbb{T} -homomorphisms

$$\nu : \mathcal{A}_i \rightarrow \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathcal{A}_\delta) \quad \text{and} \quad \xi : \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathcal{A}_\delta) \rightarrow \mathcal{A}_i$$

are inverse to each other.

Proof. For any $\mathfrak{a} \in \mathcal{A}_i$ and the corresponding multiplication map

$$\nu(\mathfrak{a}) \in \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathcal{A}_\delta),$$

we have the equalities

$$\begin{aligned} \Phi_\Delta(\mathfrak{o} \circ \nu(\mathfrak{a})) &= (\mathfrak{o} \otimes \text{id}_{\mathcal{S}})((\mathfrak{a} \otimes 1) \cdot \text{morl}_{(\delta-i,i)}) \\ &= (\mathfrak{o} \otimes \text{id}_{\mathcal{S}})((1 \otimes \mathfrak{a}) \cdot \text{morl}_{(\delta,0)}) && \text{by [Lemma 2.17\(ii\)](#)} \\ &= (\mathfrak{o} \otimes \text{id}_{\mathcal{S}})((1 \otimes \mathfrak{a}) \cdot (\alpha \cdot \mathfrak{s} \otimes 1)) && \text{by [Lemma 2.17\(i\)](#)} \\ &= \alpha \cdot \mathfrak{a} \in \alpha \cdot \mathcal{A}_i. \end{aligned}$$

Since the map $\nu : \mathcal{A}_i \rightarrow \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathcal{A}_\delta)$ was shown to be an isomorphism in [Theorem 2.12\(iii\)](#), we obtain that

$$\Phi_\Delta(\mathfrak{o} \circ \mathfrak{u}) \in \alpha \cdot \mathcal{A}_i \quad \text{for all} \quad \mathfrak{u} \in \text{Hom}_{\mathbb{T}}(\mathcal{S}_{\delta-i}, \mathcal{A}_\delta).$$

As \mathcal{A}_i is \mathbb{T} -torsionfree and α is a non zerodivisor on \mathbb{T} , division by α in $\alpha \cdot \mathcal{A}_i$ is well-defined. This completes the proof of part (i).

Notice that the above computation already shows that $\xi \circ \nu = \text{id}_{\mathcal{A}_i}$. Since ν is an isomorphism by [Theorem 2.12\(iii\)](#), ξ is necessarily the inverse map of ν . \square

The map ξ in [Theorem 2.18\(i\)](#) can be made even more explicit by using Morley forms and multiplication of polynomials and inverse polynomials: There is a natural embedding followed by an isomorphism of graded B -modules

$$\iota : {}^* \text{Hom}_{\mathbb{T}}(S, T) \hookrightarrow {}^* \text{Hom}_{\mathbb{T}}(B, T) \cong T[x_1^{-1}, \dots, x_d^{-1}].$$

Clearly, the S -module and the B -module structure of $\text{Im}(\iota)$ coincide. So if $s \in S$ and $h \in \text{Im}(\iota)$, then $s \cdot h = \tilde{s}h$, where $\tilde{s} \in B$ is any preimage of s and the multiplication on the right hand side is simply multiplication of a polynomial and an inverse polynomial via the B -module structure of $T[x_1^{-1}, \dots, x_d^{-1}]$. Tensoring with S one obtains a homomorphism of graded $B \otimes_{\mathbb{T}} S$ -modules

$$\psi : {}^* \text{Hom}_{\mathbb{T}}(S, T) \otimes_{\mathbb{T}} S \longrightarrow T[x_1^{-1}, \dots, x_d^{-1}] \otimes_{\mathbb{T}} S,$$

and again the S^e -module and the $B \otimes_{\mathbb{T}} S$ -module structure of $\text{Im}(\psi)$ coincide. In concrete terms, if $\beta \in S^e$ and $H \in \text{Im}(\psi)$, then

$$\beta \cdot H = \tilde{\beta} H,$$

where $\tilde{\beta} \in B \otimes_{\mathbb{T}} S$ is any preimage of β .

With these identifications, the map Φ_{Δ} of [Remark 2.13](#) becomes

$$\Phi_{\Delta} : {}^* \text{Hom}_{\mathbb{T}}(S, T) \longrightarrow T \otimes_{\mathbb{T}} \mathcal{A}, \quad w \mapsto \Delta \cdot \psi(w \otimes 1),$$

and with notation as in [Theorem 2.18\(i\)](#) we obtain

$$\Phi_{\Delta}(o \circ u) = \Delta \cdot \psi((o \circ u) \otimes 1).$$

In this equality, we can replace Δ by $\text{morl}_{(\delta-i, i)}$ because $o \circ u$ is homogeneous of degree $i - \delta$. Thus we have proved the following corollary, which is needed in [§6.2](#):

Corollary 2.19. *With the assumptions of [Theorem 2.18](#) and notation as in the discussion above, we have*

$$T \otimes_{\mathbb{T}} \mathcal{A}_i = \frac{1}{\alpha} \cdot \text{morl}_{(\delta-i, i)} \cdot \psi(\text{Hom}_{\mathbb{T}}(S_{\delta-i}, T) \otimes 1).$$

2.2. The case where the coefficient ring T is graded. In this short subsection, we deal with the case where T is a graded ring. This case is of particular importance due to its applicability in the study of blowup algebras that we are going to pursue later. The proofs are exactly the same and one only needs to indicate the necessary shifts in bidegree. [Definition 2.4](#) of the weak Gorenstein property can be easily adapted to the bigraded setting.

Setup 2.20. Let T be a positively graded Gorenstein ring with T_0 a local ring and let $\omega_T \cong T(\alpha)$ with $\alpha \in \mathbb{Z}$ be its canonical module. Let $B = T[x_1, \dots, x_d]$ be a bigraded polynomial ring such that $\text{bideg}(t) = (0, \deg(t))$ for all homogeneous $t \in T$ and $\text{bideg}(x_i) = (b_i, 0)$ with $b_i > 0$ a positive integer for all $1 \leq i \leq d$. Let $S = B/\mathcal{J}$, where \mathcal{J} is a bihomogenous ideal. Let $\mathfrak{m} \subset B$ be the ideal $\mathfrak{m} = (x_1, \dots, x_d)$. Set $\mathfrak{b} := \mathfrak{b}_1 + \dots + \mathfrak{b}_d$.

We now restate our generalization of Jouanolou duality in the current bigraded setting.

Theorem 2.21. *Assume [Setup 2.20](#). Suppose that S is perfect over B and of codimension c . Then we have a bigraded isomorphism of S -modules*

$$H_m^i(\omega_S) \cong {}^* \text{Ext}_T^{i+c-d}(S, T)(0, \alpha)$$

for all $i \in \mathbb{Z}$.

Proof. Either of the proofs of [Theorem 2.3](#) adapt to this case directly. One only needs to notice that the bigraded canonical module of B is $\omega_B = B(-b, \alpha)$, and so $\omega_S \cong \text{Ext}_B^c(S, B(-b, \alpha))$. \square

Theorem 2.22 (Generalized Jouanolou duality). *Assume [Setup 2.20](#). Suppose that S is perfect over B of codimension c , and that S , as a bigraded ring, is $(i+1)$ -weakly Gorenstein with respect to m_S . If $y \in \omega_S$ is a bihomogeneous weak generator of ω_S , then there is a bigraded isomorphism of S -modules*

$$H_m^i(S) \cong {}^* \text{Ext}_T^{i+c-d}(S, T)(\text{bideg}(y) + (0, \alpha)).$$

Proof. Since [Lemma 2.6](#) also holds in a bigraded setting, the result follows from [Theorem 2.21](#). \square

For the rest of this subsection we assume, in addition to [Setup 2.20](#), that $b_1 = \cdots = b_d = 1$, S is perfect over B with $\dim(S) = \dim(T)$, and S is 1-weakly Gorenstein with respect to m_S . Let $y \in \omega_S$ be a bihomogeneous weak generator of the canonical module of bidegree $(-\delta, -\gamma)$.

In this case the element \mathfrak{s} of [Theorem 2.12](#) can be chosen to be bihomogeneous of bidegree $(\delta, \gamma - \alpha)$, the map ν of [Theorem 2.12\(iii\)](#) is homogeneous, and the map ν' of [Theorem 2.12\(iii\)](#) is homogeneous of degree $\alpha - \gamma$. If the element Δ of [Remark 2.13](#) is bihomogeneous of bidegree (δ, ρ) , then so is the induced map Φ_Δ .

We consider the enveloping algebra $B \otimes_T B$ as a trigraded ring with

$$\text{trideg}(x_i \otimes 1) = (1, 0, 0), \text{trideg}(1 \otimes x_i) = (0, 1, 0), \text{trideg}(t) = (0, 0, \deg(t)) \text{ for homogeneous } t \in T.$$

This induces a triple grading on S^e . If the element $\Delta \in [0 :_{S^e} \mathbb{D}]_\delta$ is bihomogeneous of bidegree (δ, ρ) , we obtain the decomposition

$$\Delta = \sum_{i=0}^{\delta} \text{morl}_{(\delta-i, i)} \quad \text{where} \quad \text{morl}_{(\delta-i, i)} \in [S^e]_{(\delta-i, i, \rho)}.$$

With this grading the element α of [Lemma 2.17](#) and the isomorphism ξ of [Theorem 2.18](#) are homogeneous of degrees α and $\gamma - \alpha$, respectively.

3. WEAKLY GORENSTEIN DETERMINANTAL RINGS

In this section, we study the weakly Gorenstein condition for determinantal rings. Our findings show that under reasonable assumptions the weakly Gorenstein property holds. These results combined with our generalization of Jouanolou duality ([Theorem 2.22](#)) yield new tools to study local cohomology modules of determinantal rings.

Setup 3.1. Let B be a Noetherian ring and $\varphi \in B^{m \times n}$ be an $m \times n$ matrix with $m \leq n$. Let $1 \leq t \leq m$ and S be the determinantal ring $S := B/I_t(\varphi)$. We consider the following submatrices of φ

$$\left[\begin{array}{c|c} \delta & * \\ \hline * & * \end{array} \right], \quad \left[\begin{array}{c|c} & \\ \hline \varphi' & * \end{array} \right], \quad \left[\begin{array}{c} \varphi'' \\ \hline * \end{array} \right]$$

where $\delta \in B^{(t-1) \times (t-1)}$, $\varphi' \in B^{m \times (t-1)}$, and $\varphi'' \in B^{(t-1) \times n}$. Let $\Delta := \det(\delta) \in B$.

We first prove an elementary lemma.

Lemma 3.2. *We have the inclusion $I_{t-1}(\varphi') \cdot I_{t-1}(\varphi'')S \subset \Delta S$ in S .*

Proof. We proceed by induction on t . The case $t = 1$ is vacuous. As we have $\varphi_{i,1}\varphi_{1,j} - \varphi_{1,1}\varphi_{i,j} \in I_2(\varphi)$ for all i, j , the claim follows for the case $t = 2$.

Suppose that $t \geq 3$. Since containments are preserved under extensions, we may assume that $B = \mathbb{Z}[x_{i,j}]$ where $x_{i,j}$ are independent variables and $\varphi = (x_{i,j})$ is the generic $m \times n$ matrix. Since S is a Cohen-Macaulay domain (see [25, Corollary 4]) and $\Delta S \neq 0$, all the associated primes of ΔS have height one.

Let $P \in \text{Spec}(S)$ be a minimal prime of $(x_{1,1}, x_{1,2})S$. Set $\mathfrak{p} = P \cap \mathbb{Z}$, $\bar{B} = B \otimes_{\mathbb{Z}} \text{Quot}(\mathbb{Z}/\mathfrak{p})$, and $\bar{S} = S \otimes_{\mathbb{Z}} \text{Quot}(\mathbb{Z}/\mathfrak{p})$. Notice that $\text{ht}(P) \geq \text{ht}(P\bar{S})$. Since $t \geq 3$, the minimal monomial generators of the initial ideal of $I_t(\varphi)\bar{B}$ with any antidiagonal term order do not involve the variables $x_{1,1}$ and $x_{1,2}$ (see, e.g., [46, Theorem 16.28]), and this implies that $x_{1,1}, x_{1,2}$ form a regular sequence on \bar{S} (see, e.g., [16, Proposition 15.15]). Thus $\text{ht}(P\bar{S}) \geq 2$, and so $\text{ht}(P) \geq 2$. Since this holds for every P , it follows that $\text{ht}((x_{1,1}, x_{1,2})S) = 2$. Hence, there is an element $b \in B$ such that the image of $x_{1,1} + bx_{1,2}$ in S is in no associated prime of ΔS . After elementary column operations, we obtain that $\varphi_{1,1}$ is not contained in any of the associated primes of ΔS , and so it suffices to show the containment after localizing at the element $\varphi_{1,1}$. Once $\varphi_{1,1}$ is invertible, we can apply standard arguments to reduce to the case $t - 1$. \square

The following result provides families of determinantal rings that have the weakly Gorenstein property.

Theorem 3.3. *In addition to Setup 3.1, assume that B is normal, Gorenstein, and local with infinite residue field. Suppose that*

$$\text{ht}(I_t(\varphi)) = (m - t + 1)(n - t + 1) \quad \text{and} \quad \text{ht}(I_{t-1}(\varphi)) > (m - t + 1)(n - t + 1).$$

Let $\mathfrak{a} \subset S$ be an ideal.

After elementary row operations, we may assume that $\text{ht}(I_{t-1}(\varphi'')S) > 0$. Then S is i -weakly Gorenstein with respect to \mathfrak{a} , for $i = \text{ht}(I_{t-1}(\varphi'')S + \mathfrak{a}) - 1$.

After elementary column operations, we may further assume that $\text{ht}(I_{t-1}(\varphi')S) > 0$. In this case $I_{t-1}(\varphi')^{n-m}S$ is the canonical module of S , and the image of Δ^{n-m} in this ideal is a weak generator of the canonical module.

Proof. Since the row and column spaces of $\varphi \otimes_B S$ have rank $t - 1$, we may assume that $\text{ht}(I_{t-1}(\varphi'')S) \geq 1$ after elementary row operations and $\text{ht}(I_{t-1}(\varphi')S) \geq 1$ after elementary column operations. Let

$B' = B[X]$ where $X = (x_{i,j})$ is a generic $m \times n$ matrix with $x_{i,j}$ independent variables, and set $S' = B'/I_t(X)$. From [2, Theorem] we know the canonical module of S' satisfies the isomorphism $\omega_{S'} \cong I_{t-1}(X')^{n-m}S'$, where X' is the submatrix given by the first $t-1$ columns of X (see also [4, Chapter 8]). The ring S' is Cohen-Macaulay, $S \cong S'/I_1(X-\varphi)$, and the entries of the matrix $X-\varphi$ form a regular sequence on S' (notice that $\text{ht}(I_t(X)) = \text{ht}(I_t(\varphi))$ by assumption, and thus $\dim(S) = \dim(S') - mn$). It follows that

$$\omega_S \cong \omega_{S'} \otimes_{S'} S \cong I_{t-1}(X')^{n-m}S' \otimes_{S'} S \twoheadrightarrow I_{t-1}(\varphi')^{n-m}S.$$

Thus we obtain a surjection

$$\omega_S \twoheadrightarrow I_{t-1}(\varphi')^{n-m}S.$$

As $\text{ht}(I_{t-1}(\varphi)) > (m-t+1)(n-t+1)$, S is generically a complete intersection on the Gorenstein ring B , hence generically Gorenstein. It follows that ω_S has rank 1. On the other hand, $I_{t-1}(\varphi')^{n-m}S$ is an ideal of positive grade and hence again a module of rank 1. It follows that the kernel of the above surjection is a torsion submodule of ω_S , and hence zero because ω_S is a maximal Cohen-Macaulay module.

Let \mathfrak{y} be the image of Δ^{n-m} in S . Notice that $\mathfrak{y} \in \omega_S$ under the isomorphism $\omega_S \cong I_{t-1}(\varphi')^{n-m}S$. From Lemma 3.2 it follows that $\text{Ann}_S(\omega_S/S\mathfrak{y}) \supset I_{t-1}(\varphi'')^{n-m}S$, and the latter ideal has positive height. Thus, by Remark 2.5, S is i -weakly Gorenstein with respect to \mathfrak{a} when $i = \text{ht}(I_{t-1}(\varphi'')S + \mathfrak{a}) - 1$. \square

We believe that simple combinations of Theorem 2.22 and Theorem 3.3 can lead to a better understanding of local cohomology modules of determinantal rings in several cases of interest. To be more precise, we provide a sample that will be useful later (see Theorem 6.3).

Proposition 3.4. *Let $R = \mathbb{k}[x_1, \dots, x_d]$ and $T = \mathbb{k}[y_1, \dots, y_n]$ be standard graded polynomial rings over a field \mathbb{k} . Let $\mathfrak{m} = (x_1, \dots, x_d) \subset R$ be the irrelevant ideal. Let B be the standard bigraded polynomial ring $R \otimes_{\mathbb{k}} T$. Let f_1, \dots, f_n be a sequence of forms in R of the same degree $D \geq 1$. Let φ be the $2 \times n$ matrix given by*

$$\varphi = \begin{pmatrix} y_1 & \cdots & y_n \\ f_1 & \cdots & f_n \end{pmatrix},$$

and set $S = B/I_2(\varphi)$. If $\text{ht}(I_2(\varphi)) = n-1$, then there is a bigraded isomorphism of S -modules

$$H_{\mathfrak{m}}^i(S) \cong {}^* \text{Ext}_T^{i+n-1-d}(S, T)(d - (n-1)D, -1)$$

for all $0 \leq i \leq d-1$.

Proof. We first show the result when \mathbb{k} is infinite.

Claim 3.4.1. *The claimed isomorphisms exist when \mathbb{k} is infinite.*

To prove the claim notice that $\text{ht}(I_1(\varphi'')) \geq n$ and $\text{ht}(I_1(\varphi'')S + \mathfrak{m}S) = \text{ht}((y_1, \dots, y_n)S + \mathfrak{m}S) = \dim(S) = d+1$. Thus Theorem 3.3 implies that S is d -weakly Gorenstein with respect to $\mathfrak{m}S$.

After elementary column operations over the infinite field \mathbb{k} , we may assume that y_1 is a non zerodivisor on S . Notice that $I_2(\varphi)$ is a geometric $(n-1)$ -residual intersection of the ideal (y_1, f_1) because

$$I_2(\varphi) = (\Delta_{1,2}, \dots, \Delta_{1,n}) : (f_1, y_1),$$

$\text{ht}(I_2(\varphi)) = n - 1$ and $\text{ht}(I_2(\varphi) + (f_1, y_1)) \geq n$, where $\Delta_{1,i}$ denotes the 2×2 minor corresponding to columns 1 and i (see [30]). From the formulas for the canonical module of residual intersections ([31, Proposition 2.3], [45]), we obtain that $\omega_S \cong (f_1, y_1)^{n-2} S(-d + (n-1)D, -1)$. Let $y \in \omega_S$ be the class of y_1^{n-2} , which by Theorem 3.3 can be chosen as the element that makes S d -weakly Gorenstein with respect to m_S . Since $\text{bideg}(y) = (0, n-2) - (-d + (n-1)D, -1) = (d - (n-1)D, n-1)$, Theorem 2.22 yields

$$H_m^i(S) \cong {}^* \text{Ext}_T^{i+n-1-d}(S, T)(d - (n-1)D, -1)$$

for all $0 \leq i \leq d-1$, as required.

We now settle the proposition for an arbitrary field \mathbb{k} . Let \mathbb{k}' be an infinite field containing \mathbb{k} , and set $S' := S \otimes_{\mathbb{k}} \mathbb{k}'$. The computation above of the canonical module shows that $\dim_{\mathbb{k}'}([\omega_{S'}]_{(d-(n-1)D, n-1)}) = 1$. This implies that $\dim_{\mathbb{k}}([\omega_S]_{(d-(n-1)D, n-1)}) = 1$. Therefore, take a non-zero element $0 \neq y \in [\omega_S]_{(d-(n-1)D, n-1)}$. It necessarily follows that, up to multiplication by a unit in \mathbb{k}' , the extension of y into $\omega_{S'} \cong \omega_S \otimes_{\mathbb{k}} \mathbb{k}'$ equals the element chosen in Claim 3.4.1 to make S' weakly Gorenstein over $m_{S'}$. This completes the proof for the general case. \square

4. WEAKLY GORENSTEIN SYMMETRIC ALGEBRAS

The principal motivation of this section is to find a comprehensive family of ideals whose symmetric algebra has the weakly Gorenstein property. This first goal will allow us to apply our generalized Jouanolou duality (Theorem 2.22) in several new cases. However, along the way, we achieve more:

- (i) we compute explicitly the canonical module of a large family of symmetric algebras,
- (ii) when the approximation complex \mathcal{Z}_{\bullet} is acyclic, we define a new complex that also resolves the symmetric algebra but has the advantage of having free modules in the last $g-1$ positions, where g is the grade of the ideal,
- (iii) when the ideal is an almost complete intersection or perfect of deviation two, we give an explicit free resolution for the symmetric algebra, provided \mathcal{Z}_{\bullet} is acyclic.

Modules of Koszul cycles were used before in [21] to obtain information about the free resolution of the symmetric algebra of an ideal $I \subset R$ and about $\omega_{\text{Sym}(I)} \otimes_{\text{Sym}(I)} R$.

Throughout this section we assume the following setup.

Setup 4.1. Let (R, m) be a d -dimensional Cohen-Macaulay local ring. Let $I = (f_1, \dots, f_n) \subsetneq R$ be an ideal minimally generated by n elements, and set $g = \text{ht}(I) \geq 1$ to be the height of I . Let $B = R[y_1, \dots, y_n]$ be a standard graded polynomial ring. There is a surjection of standard graded R -algebras $B \twoheadrightarrow \text{Sym}(I)$ mapping y_i to $f_i \in [\text{Sym}(I)]_1$. Let K_{\bullet} be the Koszul complex associated to the sequence f_1, \dots, f_n . Let Z_{\bullet} and H_{\bullet} be the cycles and homologies of K_{\bullet} , respectively.

Definition 4.2. Let $k \geq 0$ be an integer.

- (G) I satisfies the *condition* G_k if $\mu(I_p) \leq \text{ht}(p)$ for all $p \in V(I) \subset \text{Spec}(R)$ with $\text{ht}(p) \leq k-1$.
- (F) I satisfies the *condition* F_0 if $\mu(I_p) \leq \text{ht}(p) + 1$ for all $p \in \text{Spec}(R)$.
- (SD) I has the *sliding depth condition* SD_k if $\text{depth}(H_i) \geq \min\{d-g, d-n+i+k\}$ for all i .
- (SCM) I is *strongly Cohen-Macaulay* if H_i is a Cohen-Macaulay module for all i .

One can check that the conditions strongly Cohen-Macaulay and SD_k yield the following inequalities

$$\text{depth}(Z_i) \geq \begin{cases} \min\{d, d - g + 2\} & \text{if } I \text{ is strongly Cohen-Macaulay} \\ \min\{d, d - n + i + k + 1\} & \text{if } I \text{ satisfies } SD_k \end{cases}$$

for all $i \geq 0$ (see [23, p.161]). Recall that height 2 perfect ideals and height 3 Gorenstein ideals (perfect ideals with last Betti number 1) are strongly Cohen-Macaulay (see [30, Proposition 0.3], [1, 18, 49], [57]).

Let Z_\bullet be the approximation complex

$$Z_\bullet: 0 \rightarrow Z_{n-1} \otimes_{\mathbb{R}} B(-n+1) \rightarrow \cdots \rightarrow Z_i \otimes_{\mathbb{R}} B(-i) \rightarrow \cdots \rightarrow Z_1 \otimes_{\mathbb{R}} B(-1) \rightarrow Z_0 \otimes_{\mathbb{R}} B$$

corresponding to the sequence f_1, \dots, f_n , which is a complex of graded B -modules. For more details regarding approximation complexes the reader is referred to [20]. For the sake of completeness, we include a couple of well-known results regarding symmetric algebras.

Lemma 4.3. *If $I \subset R$ has F_0 on the punctured spectrum of R , then $\dim(\text{Sym}(I)) = \max\{n, d + 1\}$.*

Proof. By the Huneke-Rossi formula (see [28, Theorem 2.6], [56, Theorem 1.2.1]), we have

$$\dim(\text{Sym}(I)) = \sup\{\mu(I_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(R)\}.$$

Since I has F_0 on the punctured spectrum of R , we deduce $\dim(\text{Sym}(I)) = \max\{\mu(I), \dim(R) + 1\} = \max\{n, d + 1\}$. \square

Lemma 4.4. *If $I \subset R$ satisfies the conditions SD_0 and F_0 , then the following statements hold:*

- (i) $\text{Sym}(I)$ is Cohen-Macaulay.
- (ii) The approximation complex Z_\bullet is acyclic.

Proof. See for instance [20, Theorem 5.4, Theorem 10.1]. \square

We now construct a new complex that will allow us to compute the canonical module of $\text{Sym}(I)$ in many cases. Apparently, the existence of this quite useful complex has been unnoticed. Its construction was inspired to us by [21, Theorem 5.8].

Proposition 4.5. *Let $\mathcal{D}_{p,q}$ be the double complex*

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \wedge^n \mathbb{R}^n \otimes_{\mathbb{R}} B(-n+g-1) \\ & & & & & & \downarrow \partial_f \\ \vdots & & \vdots & & \vdots & & \vdots \\ & & & & & & \downarrow \partial_f \\ 0 & & \wedge^n \mathbb{R}^n \otimes_{\mathbb{R}} B(-n+2) & \xrightarrow{\partial_y} & \cdots & \xrightarrow{\partial_y} & \wedge^{n-g+3} \mathbb{R}^n \otimes_{\mathbb{R}} B(-n+g-1) \\ & & \downarrow \partial_f & & & & \downarrow \partial_f \\ \wedge^n \mathbb{R}^n \otimes_{\mathbb{R}} B(-n+1) & \xrightarrow{\partial_y} & \wedge^{n-1} \mathbb{R}^n \otimes_{\mathbb{R}} B(-n+2) & \xrightarrow{\partial_y} & \cdots & \xrightarrow{\partial_y} & \wedge^{n-g+2} \mathbb{R}^n \otimes_{\mathbb{R}} B(-n+g-1) \end{array}$$

where $\mathcal{D}_{p,q} = \wedge^{n-g+2+p+q} \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{B}(-n+g-1-p)$, and ∂_f and ∂_y are the differentials of the Koszul complexes $K(f_1, \dots, f_n; \mathbb{B})$ and $K(y_1, \dots, y_n; \mathbb{B})$, respectively. Let $\mathfrak{T}_{\bullet} := \text{Tot}(\mathcal{D}_{\bullet, \bullet})$ be the total complex

$$\mathfrak{T}_{\bullet}: 0 \rightarrow \bigoplus_{j=1}^{g-1} (\wedge^n \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{B}(-n+g-j)) \rightarrow \cdots \rightarrow \wedge^{n-g+2} \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{B}(-n+g-1) \rightarrow 0$$

of $\mathcal{D}_{\bullet, \bullet}$. We can splice together \mathfrak{T}_{\bullet} and the truncated complex $0 \rightarrow \mathcal{Z}_{n-g} \rightarrow \cdots \rightarrow \mathcal{Z}_0 \rightarrow 0$ to obtain a complex of graded \mathbb{B} -modules

$$\mathfrak{L}_{\bullet}: 0 \rightarrow \mathfrak{T}_{g-2} \rightarrow \cdots \rightarrow \mathfrak{T}_1 \rightarrow \mathfrak{T}_0 \rightarrow \mathcal{Z}_{n-g} \rightarrow \cdots \rightarrow \mathcal{Z}_0 \rightarrow 0.$$

The complex \mathfrak{L}_{\bullet} is acyclic if and only if \mathcal{Z}_{\bullet} is.

Proof. Since the Koszul complex is depth sensitive and $g = \text{grade}(I)$, it follows that $H_i = 0$ for all $i \geq n-g+1$. Hence, after computing homology by columns in the double complex $\mathcal{D}_{\bullet, \bullet}$ we get that the only non-zero row is the bottom one, which is then given by

$$0 \rightarrow \mathcal{Z}_{n-1} \rightarrow \cdots \rightarrow \mathcal{Z}_{n-g+2} \xrightarrow{\Psi} \mathcal{Z}_{n-g+1} \rightarrow 0.$$

This implies that $H_i(\mathfrak{T}_{\bullet}) = H_{n-g+1+i}(\mathcal{Z}_{\bullet})$ for all $i \geq 1$ and that $H_0(\mathfrak{T}_{\bullet}) = \text{Coker}(\Psi)$. Since \mathcal{Z}_{\bullet} is a complex, there is a natural map from $\text{Coker}(\Psi)$ to $\text{Ker}(\Phi)$, where $\mathcal{Z}_{n-g} \xrightarrow{\Phi} \mathcal{Z}_{n-g-1} \rightarrow \cdots \rightarrow \mathcal{Z}_0$. Therefore, we can always make the claimed splicing, and \mathfrak{L}_{\bullet} is acyclic if and only if \mathcal{Z}_{\bullet} is. \square

The following theorem contains the main results of this section.

Theorem 4.6. *Assume Setup 4.1 where \mathbb{R} is Gorenstein with infinite residue field. Suppose that the ideal $I \subset \mathbb{R}$ has height $g = \text{ht}(I) \geq 2$ and satisfies the conditions F_0 and SD_1 . After changing the generators f_1, \dots, f_n of I the element y_1 is a non zerodivisor on $\text{Sym}(I)$, and the following statements hold:*

- (i) $\omega_{\text{Sym}(I)} \cong (f_1, y_1)^{g-2} \text{Sym}(I)(-1)$.
- (ii) $\text{Sym}(I)$ is d -weakly Gorenstein with respect to the ideal $\mathfrak{m} \text{Sym}(I)$, and the image of y_1^{g-2} in $\omega_{\text{Sym}(I)}$ is a weak generator of the canonical module of degree $g-1$.

Proof. The condition F_0 implies, in particular, that $n \leq d+1$. From Lemma 4.3 and Lemma 4.4 we have that $\text{Sym}(I)$ is a Cohen-Macaulay ring of dimension $d+1$ and that the approximation complex \mathcal{Z}_{\bullet} is acyclic. As $\dim(S/(y_1, \dots, y_n)S) = \dim(\mathbb{R}) = d$, it then follows that $\text{grade}((y_1, \dots, y_n)S) \geq 1$. So, after possibly changing the generators of I , we can assume that y_1 is a non zerodivisor on S , as asserted.

The condition SD_1 yields the following lower bounds for the depth of the Koszul cycles

$$\text{depth}(\mathcal{Z}_i) \geq \min\{d, d-n+i+2\} \quad \text{for all } i.$$

This implies that $\text{depth}(\mathcal{Z}_i) \geq \min\{d+n, d+i+2\}$. We consider the complex of graded \mathbb{B} -modules

$$\mathfrak{L}_{\bullet}: 0 \rightarrow \mathfrak{T}_{g-2} \rightarrow \cdots \rightarrow \mathfrak{T}_1 \rightarrow \mathfrak{T}_0 \rightarrow \mathcal{Z}_{n-g} \rightarrow \cdots \rightarrow \mathcal{Z}_0 \rightarrow 0$$

constructed in Proposition 4.5, which is acyclic by the current assumptions. To simplify notation, set $S := \text{Sym}(I)$.

Denote by $(\mathfrak{L}_{\bullet}, \partial_{\bullet})$ the complex \mathfrak{L}_{\bullet} with its differential. Let $U_i = \text{Coker}(\partial_{i+1})$. For all $0 \leq i \leq n-g$, we have the short exact sequence

$$0 \rightarrow U_{i+1} \rightarrow \mathcal{Z}_i \rightarrow U_i \rightarrow 0$$

that gives the induced exact sequence

$$\mathrm{Ext}_{\mathbb{B}}^{n-1-i-1}(\mathbb{U}_{i+1}, \omega_{\mathbb{B}}) \rightarrow \mathrm{Ext}_{\mathbb{B}}^{n-1-i}(\mathbb{U}_i, \omega_{\mathbb{B}}) \rightarrow \mathrm{Ext}_{\mathbb{B}}^{n-1-i}(\mathcal{Z}_i, \omega_{\mathbb{B}}).$$

For $0 \leq i \leq n-g$, we have $\mathrm{Ext}_{\mathbb{B}}^{n-1-i}(\mathcal{Z}_i, \omega_{\mathbb{B}}) = 0$ since $n-i-1 > \max\{0, n-i-2\} \geq d+n-\mathrm{depth}(\mathcal{Z}_i)$. Thus we obtain the surjection

$$\mathrm{Ext}_{\mathbb{B}}^{n-1-i-1}(\mathbb{U}_{i+1}, \omega_{\mathbb{B}}) \twoheadrightarrow \mathrm{Ext}_{\mathbb{B}}^{n-1-i}(\mathbb{U}_i, \omega_{\mathbb{B}})$$

for all $0 \leq i \leq n-g$. By composing these surjections, we obtain a surjective homomorphism of graded \mathbb{B} -modules

$$\mathrm{Ext}_{\mathbb{B}}^{g-2}(\mathbb{U}_{n-g+1}, \omega_{\mathbb{B}}) \twoheadrightarrow \mathrm{Ext}_{\mathbb{B}}^{n-1}(\mathbb{U}_0, \omega_{\mathbb{B}}) = \mathrm{Ext}_{\mathbb{B}}^{n-1}(S, \omega_{\mathbb{B}}) \cong \omega_S.$$

The complex $0 \rightarrow \mathfrak{T}_{g-2} \rightarrow \cdots \rightarrow \mathfrak{T}_1 \rightarrow \mathfrak{T}_0 \rightarrow 0$ gives a homogeneous free \mathbb{B} -resolution of \mathbb{U}_{n-g+1} .

If $g = 2$, then \mathbb{U}_{n-g+1} is equal to the free \mathbb{B} -module $\mathbb{B}(-n+1)$. In this case we have a surjection $\mathrm{Hom}_{\mathbb{B}}(\mathbb{U}_{n-g+1}, \omega_{\mathbb{B}}) \twoheadrightarrow \omega_S$, hence ω_S is generated by one homogeneous element of degree 1; this means that $\omega_S \cong S(-1)$ and, in particular, S is Gorenstein. Hence the proof is complete for the case $g = 2$.

Assume that $g \geq 3$. We are going to show that $\mathrm{Ext}_{\mathbb{B}}^{g-2}(\mathbb{U}_{n-g+1}, \omega_{\mathbb{B}})$, after tensoring with S and factoring out S -torsion, maps isomorphically to both ω_S and $(f_1, y_1)^{g-2}S(-1)$, which will finish the proof of (i).

The module $\mathrm{Ext}_{\mathbb{B}}^{g-2}(\mathbb{U}_{n-g+1}, \omega_{\mathbb{B}})$ is the cokernel of the map

$$\mathrm{Hom}_{\mathbb{B}}(\mathfrak{T}_{g-3}, \omega_{\mathbb{B}}) \rightarrow \mathrm{Hom}_{\mathbb{B}}(\mathfrak{T}_{g-2}, \omega_{\mathbb{B}}).$$

The map $\mathfrak{T}_{g-2} \rightarrow \mathfrak{T}_{g-3}$ is given explicitly as

$$\bigoplus_{j=1}^{g-1} \mathbb{B}(-n+g-j) \rightarrow \bigoplus_{j=1}^{g-2} \mathbb{B}^n(-n+g-j),$$

and its matrix representation with respect to the standard bases is

$$\mathfrak{A} = \begin{bmatrix} f_1 & y_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{n-1}f_n & (-1)^{n-1}y_n & 0 & \cdots & 0 & 0 \\ \hline 0 & -f_1 & y_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^n f_n & (-1)^{n-1} y_n & \cdots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \cdots & (-1)^{g-3} f_1 & y_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^{n+g-4} f_n & (-1)^{n-1} y_n \end{bmatrix}.$$

Let $\mathfrak{C} = \mathfrak{A}^T$ and

$$\Omega := \mathrm{Coker} \left(\bigoplus_{j=1}^{g-2} \omega_{\mathbb{B}}^n(n-g+j) \xrightarrow{\mathfrak{C}} \bigoplus_{j=1}^{g-1} \omega_{\mathbb{B}}(n-g+j) \right).$$

Denote by w_j the generator of Ω corresponding to the j -th column of \mathfrak{C} , and write $\bar{}$ for images in S . Since in S we have the relations $\overline{f_i y_j} = \overline{f_j y_i}$, we obtain a surjective homomorphism of graded B -modules

$$\Omega \twoheadrightarrow (f_1, y_1)^{g-2} S(-1), \quad \text{where } w_j \mapsto (-1)^{\binom{j}{2}} \overline{f_1^{j-1} y_1^{g-j-1}}.$$

Thus we have surjective homomorphisms of graded S -modules

$$\Omega \otimes_B S \twoheadrightarrow (f_1, y_1)^{g-2} S(-1) \quad \text{and} \quad \Omega \otimes_B S \twoheadrightarrow \omega_S.$$

The matrix \mathfrak{A} has size $(g-2)n \times (g-1)$, and $g \geq 3$. So by the pigeonhole principle, every $(g-1) \times (g-1)$ submatrix of \mathfrak{A} contains, after possibly multiplying one column by -1 , a $2 \times (g-1)$ submatrix of the form

$$\begin{pmatrix} 0 & \cdots & 0 & (-1)^{i_1-1} f_{i_1} & (-1)^{i_1-1} y_{i_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (-1)^{i_2-1} f_{i_2} & (-1)^{i_2-1} y_{i_2} & 0 & \cdots & 0 \end{pmatrix}.$$

Accordingly, $I_{g-1}(\mathfrak{C}) \subset \left(\{f_i y_j - f_j y_i\}_{i,j} \right) \subset B$, hence $I_{g-1}(\mathfrak{C} \otimes_B S) = 0$. Notice that

$$\overline{y_1^{g-2}} \in I_{g-2}(\mathfrak{C} \otimes_B S)$$

and $\overline{y_1^{g-2}}$ is a non zerodivisor on S , therefore $\text{rank}_S(\mathfrak{C} \otimes_B S) = g-2$ and $\text{rank}_S(\Omega \otimes_B S) = 1$. The canonical module ω_S is faithful and the epimorphic image of a module of rank 1, hence ω_S is an S -module of rank 1. The ideal $(f_1, y_1)^{g-2} S$ contains the non zerodivisor $\overline{y_1^{g-2}}$, and so it is a torsion-free S -module of rank 1. Finally, since ω_S is always torsion-free, we obtain the isomorphisms

$$\omega_S \cong \frac{\Omega \otimes_B S}{\text{tor}(\Omega \otimes_B S)} \cong (f_1, y_1)^{g-2} S(-1).$$

This concludes the proof of part (i) of the theorem.

We now concentrate on part (ii) of the theorem, which becomes straightforward after having computed the canonical module ω_S .

Since $(\overline{y_1}, \dots, \overline{y_n}) \cdot \overline{f_1} \subset (\overline{y_1})$ in S , it follows that

$$(\overline{y_1}, \dots, \overline{y_n})^{g-2} \subset \text{Ann}_S \left((\overline{f_1}, \overline{y_1})^{g-2} / (\overline{y_1})^{g-2} \right) = \text{Ann}_S(\omega_S / S\mathfrak{u}_1),$$

where \mathfrak{u}_1 denotes the image in ω_S of the element $w_1 \in \Omega$. As a consequence, we get

$$\text{ht}(\text{Ann}_S(\omega_S / S\mathfrak{u}_1)) \geq 1 \quad \text{and} \quad \text{ht}(\text{Ann}_S(\omega_S / S\mathfrak{u}_1) + \mathfrak{m}S) \geq d+1.$$

Therefore, S is d -weakly Gorenstein with respect to $\mathfrak{m}S$, and this establishes the remaining part (ii). \square

Remark 4.7. It should be mentioned that the proof of [Theorem 4.6](#) works under the weaker assumption that R is a Cohen-Macaulay ring with canonical module ω_R . In that case, we have the formula

$$\omega_{\text{Sym}(I)} \cong \omega_R (f_1, y_1)^{g-2} \text{Sym}(I)(-1)$$

when $g = \text{ht}(I) \geq 2$ and $I \subset R$ satisfies F_0 and SD_1 .

Remark 4.8. In addition to the hypotheses of [Theorem 4.6](#) assume that $\mu(I) = d+1$. An interesting question is whether $\text{Sym}(I)$ is weakly Gorenstein with respect to \mathfrak{n} , where $\mathfrak{n} = (y_1, \dots, y_n) \subset B$. The natural choice for making $S := \text{Sym}(I)$ weakly Gorenstein with respect to \mathfrak{n} is to choose the element f_1^{g-2} instead of $\overline{y_1^{g-2}}$ in the proof of [Theorem 4.6](#). However, this choice does not work. Indeed, notice that $\text{ht}(\mathfrak{m}S) = 0$ and $\overline{y_1^{g-2}}$ is a non zerodivisor on S , so the image of f_1^{g-2} cannot generate $(f_1, y_1)^{g-2} S_{\mathfrak{p}} = S_{\mathfrak{p}}$

for all $\mathfrak{p} \in \text{Ass}(S)$. The same argument shows that no element in $\mathfrak{m}S$ would work. On the other hand, in [Theorem 5.3](#) we do consider the local cohomology modules $H_n^i(S)$ for $i \geq 2$ by using further duality results.

Remark 4.9. Our computation of the canonical module of $\text{Sym}(I)$ extends or complements known results in the literature. In [\[56, Theorem 5.7.8\]](#), by utilizing computations with divisor class groups, the formula of [Theorem 4.6\(i\)](#) was obtained. However, the result of [\[56, Theorem 5.7.8\]](#) further requires $\text{Sym}(I)$ to be a normal domain (which is a strong condition; in fact, in our applications $\text{Sym}(I)$ will typically not be a domain). In [\[22, Corollaries 2.5 and 2.9\]](#), a formula for the canonical module $\omega_{\mathcal{R}(I)}$ of $\mathcal{R}(I)$ is obtained assuming that I is generically a complete intersection, $\mathcal{R}(I)$ is Cohen-Macaulay, and $\text{gr}_I(\mathcal{R})$ is Gorenstein; that formula has the same form as our formula for $\omega_{\text{Sym}(I)}$. We exploit this connection between $\omega_{\text{Sym}(I)}$ and $\omega_{\mathcal{R}(I)}$ in the next corollary to deduce that I is of linear type. Recall that an ideal I is said to be *of linear type* if the natural map $\text{Sym}(I) \rightarrow \mathcal{R}(I)$ is an isomorphism. A version of [Corollary 4.10](#) is proved in [\[52, Theorem 3.8\]](#), with different methods and the hypothesis that I be strongly Cohen-Macaulay as opposed to F_0 and SD_1 .

Corollary 4.10. *In addition to [Setup 4.1](#) assume that:*

- (a) $I \subset R$ satisfies the conditions F_0 and SD_1 , and I is generically a complete intersection (a complete intersection locally at each of its associated primes) with $g = \text{ht}(I) \geq 2$.
- (b) $\text{gr}_I(\mathcal{R})$ is a Gorenstein ring.

Then I is of linear type and strongly Cohen-Macaulay.

Proof. Notice that R is Gorenstein because $\text{gr}_I(\mathcal{R})$ is (see, e.g., [\[19, proof of Proposition 11.16\]](#)). We first prove that I is of linear type. By [Lemma 4.3](#), we have $\dim(\text{Sym}(I)) = d + 1 = \dim(\mathcal{R}(I))$. After dualizing, the surjection $\text{Sym}(I) \rightarrow \mathcal{R}(I)$ yields the natural inclusion

$$\varphi : \omega_{\mathcal{R}(I)} \hookrightarrow \omega_{\text{Sym}(I)}.$$

Since I is generically a complete intersection with $g \geq 1$ and $\text{gr}_I(\mathcal{R})$ is Gorenstein, [\[52, Theorem 2.1\]](#) shows that $\mathcal{R}(I)$ is Cohen-Macaulay. With these hypotheses it follows from [\[22, Corollaries 2.5 and 2.9\]](#) that $\omega_{\mathcal{R}(I)} \cong (f_1, y_1)^{g-2} \mathcal{R}(I)(-1)$. Combining this with [Theorem 4.6\(i\)](#) we obtain a natural homomorphism of graded $\text{Sym}(I)$ -modules

$$\psi : \omega_{\text{Sym}(I)} \cong (f_1, y_1)^{g-2} \text{Sym}(I)(-1) \rightarrow (f_1, y_1)^{g-2} \mathcal{R}(I)(-1) \cong \omega_{\mathcal{R}(I)}.$$

Since $g \geq 2$, the ideal I is of linear type locally in codimension 1, and so both φ and ψ are isomorphisms locally at every prime ideal of R of height 1.

We claim that the composition $\psi \circ \varphi$ is an isomorphism. Since $\mathcal{R}(I)$ is Cohen-Macaulay, $\text{End}(\omega_{\mathcal{R}(I)})$ is naturally isomorphic to $\mathcal{R}(I)$. As $\psi \circ \varphi \in [\text{End}(\omega_{\mathcal{R}(I)})]_0$, it follows that $\psi \circ \varphi$ is multiplication by an element $\alpha \in [\mathcal{R}(I)]_0 = R$. But $\psi \circ \varphi$ is an isomorphism locally at every prime ideal of R of height 1, so the element α is a unit in R locally at every such prime. This can only happen if α is a unit, which shows that $\psi \circ \varphi$ is an isomorphism.

Thus $\omega_{\mathcal{R}(I)}$ is isomorphic to a direct summand of $\omega_{\text{Sym}(I)}$. Recall that $\text{Sym}(I)$ is Cohen-Macaulay by [Lemma 4.4\(i\)](#). Therefore $\text{Spec}(\text{Sym}(I))$ is connected in codimension one by Hartshorne's connectedness theorem (see, e.g., [\[16, Theorem 18.12\]](#)), and so $\omega_{\text{Sym}(I)}$ is indecomposable (see, e.g., [\[26, Theorem](#)

3.6]). Thus the $\text{Sym}(I)$ -modules $\omega_{\text{Sym}(I)}$ and $\omega_{\mathcal{R}(I)}$ are isomorphic and hence have the same annihilators. As canonical modules of unmixed rings are faithful, we conclude that the natural surjection $\text{Sym}(I) \rightarrow \mathcal{R}(I)$ is an isomorphism, as asserted.

Now I satisfies SD_0 , I is of linear type and generically a complete intersection, and $\text{gr}_I(\mathcal{R})$ is Gorenstein. With these hypotheses [52, Corollary 3.11] implies that I is also strongly Cohen-Macaulay. \square

Finally, we give an explicit free resolution for the symmetric algebra when the ideal is an almost complete intersection or a perfect ideal of deviation two, both under the assumption that \mathcal{Z}_\bullet is acyclic. The *deviation* of $I \subset \mathcal{R}$ is defined as $d(I) := \mu(I) - \text{ht}(I) = n - g$; one says that I is an almost complete intersection when $d(I) \leq 1$.

Theorem 4.11. *Assume Setup 4.1. Suppose that the approximation complex \mathcal{Z}_\bullet is acyclic and let*

$$F_\bullet : \cdots \rightarrow F_2 \rightarrow \mathcal{R}^n \xrightarrow{(f_1, \dots, f_n)} \mathcal{R} \rightarrow 0$$

be a free \mathcal{R} -resolution of \mathcal{R}/I . Let \mathcal{T}_\bullet be the acyclic complex of free \mathcal{B} -module defined in Proposition 4.5. The following statements hold:

- (i) *If $I \subset \mathcal{R}$ is an almost complete intersection, then a homogeneous free \mathcal{B} -resolution of $\text{Sym}(I)$ is given by*

$$\mathfrak{F}_\bullet : \cdots \rightarrow \mathfrak{F}_i \rightarrow \cdots \rightarrow \mathfrak{F}_1 \rightarrow \mathfrak{F}_0 \rightarrow 0$$

where $\mathfrak{F}_0 = \mathcal{B}$, $\mathfrak{F}_1 = F_2 \otimes_{\mathcal{R}} \mathcal{B}(-1)$, and $\mathfrak{F}_i = \mathcal{T}_{i-2} \oplus (F_{i+1} \otimes_{\mathcal{R}} \mathcal{B}(-1))$ for $i \geq 2$.

- (ii) *If $I \subset \mathcal{R}$ is a perfect ideal of deviation two and $F_\bullet : 0 \rightarrow F_g \rightarrow \cdots \rightarrow F_2 \rightarrow \mathcal{R}^n \xrightarrow{(f_1, \dots, f_n)} \mathcal{R} \rightarrow 0$ is a free \mathcal{R} -resolution of \mathcal{R}/I , then a homogeneous free \mathcal{B} -resolution of $\text{Sym}(I)$ is given by*

$$\mathfrak{F}_\bullet : \cdots \rightarrow \mathfrak{F}_i \rightarrow \cdots \rightarrow \mathfrak{F}_1 \rightarrow \mathfrak{F}_0 \rightarrow 0$$

where $\mathfrak{F}_0 = \mathcal{B}$, $\mathfrak{F}_1 = F_2 \otimes_{\mathcal{R}} \mathcal{B}(-1)$, and $\mathfrak{F}_i = \mathcal{T}_{i-3} \oplus ((F_{i+1} \oplus K_{i+1} \oplus F_{g-i+2}^*) \otimes_{\mathcal{R}} \mathcal{B}(-1))$ for $i \geq 2$.

Proof. Let \mathcal{L} be the kernel of the natural surjection $\mathcal{B} \rightarrow \text{Sym}(I)$.

- (i) From Proposition 4.5 we have the acyclic complex of graded \mathcal{B} -modules

$$\mathcal{L}_\bullet : 0 \rightarrow \mathcal{T}_{g-2} \rightarrow \cdots \rightarrow \mathcal{T}_1 \xrightarrow{\Psi} \mathcal{T}_0 \rightarrow \mathcal{Z}_1 \rightarrow \mathcal{B} \rightarrow 0$$

that resolves $\text{Sym}(I)$. The map $\text{Coker}(\Psi) \rightarrow \mathcal{Z}_1$ induces a morphism of complexes of graded \mathcal{B} -modules $u_\bullet : \mathcal{T}_\bullet \rightarrow \mathcal{G}_\bullet$, where

$$\mathcal{G}_\bullet : \cdots \rightarrow F_3 \otimes_{\mathcal{R}} \mathcal{B}(-1) \rightarrow F_2 \otimes_{\mathcal{R}} \mathcal{B}(-1) \rightarrow 0$$

is the homogeneous resolution of $\mathcal{Z}_1 = \mathcal{Z}_1 \otimes_{\mathcal{R}} \mathcal{B}(-1)$ obtained by truncating $F_\bullet \otimes_{\mathcal{R}} \mathcal{B}(-1)$. Since we have a short exact sequence $0 \rightarrow \text{Coker}(\Psi) \rightarrow \mathcal{Z}_1 \rightarrow \mathcal{L} \rightarrow 0$, the mapping cone $C(u_\bullet)$ yields a homogeneous free \mathcal{B} -resolution of \mathcal{L} . So, the proof of this part is complete.

- (ii) The complex of Proposition 4.5 is now given by

$$\mathcal{L}_\bullet : 0 \rightarrow \mathcal{T}_{g-2} \rightarrow \cdots \rightarrow \mathcal{T}_1 \xrightarrow{\Psi} \mathcal{T}_0 \rightarrow \mathcal{Z}_2 \xrightarrow{\Phi} \mathcal{Z}_1 \rightarrow \mathcal{B} \rightarrow 0.$$

Notice that there is a short exact sequence $0 \rightarrow \mathcal{B}_2(K_\bullet) \rightarrow \mathcal{Z}_2(K_\bullet) \rightarrow \mathcal{H}_2(K_\bullet) \rightarrow 0$, the truncated Koszul complex $0 \rightarrow K_n \rightarrow \cdots \rightarrow K_3 \rightarrow 0$ is a free \mathcal{R} -resolution of $\mathcal{B}_2(K_\bullet)$, and $\text{Hom}_{\mathcal{R}}(F_\bullet, \mathcal{R})[-g]$ gives a free \mathcal{R} -resolution of $\mathcal{H}_2(K_\bullet) \cong \omega_{\mathcal{R}/I}$ as \mathcal{R}/I is a perfect \mathcal{R} -module. By the Horseshoe lemma, a homogeneous

free B -resolution of $Z_2 = Z_2 \otimes_R B(-1)$ is given by a complex P_\bullet with $P_i = (K_{i+3} \oplus F_{g-i}^*) \otimes_R B(-1)$. As in part (i), a mapping cone construction along the short exact sequence $0 \rightarrow \text{Coker}(\Psi) \rightarrow Z_2 \rightarrow \text{Im}(\Phi) \rightarrow 0$ yields a homogeneous free B -resolution Q_\bullet of $\text{Im}(\Phi)$ with $Q_i = P_i \oplus \mathfrak{F}_{i-1}$. Recall that the complex G_\bullet of part (i) is a homogeneous free B -resolution of $Z_1 = Z_1 \otimes_R B(-1)$. Therefore, another mapping cone construction along the short exact sequence $0 \rightarrow \text{Im}(\Phi) \rightarrow Z_1 \rightarrow \mathcal{L} \rightarrow 0$ gives a homogeneous free B -resolution L_\bullet of \mathcal{L} with $L_0 = F_2 \otimes_R B(-1)$ and

$$L_i = Q_{i-1} \oplus (F_{i+2} \otimes_R B(-1)) = \mathfrak{F}_{i-2} \oplus ((F_{i+2} \oplus K_{i+2} \oplus F_{g-i+1}^*) \otimes_R B(-1))$$

for all $i \geq 1$. The resolution \mathfrak{F}_\bullet is now given by setting $\mathfrak{F}_0 = B$ and $\mathfrak{F}_i = L_{i-1}$ for all $i \geq 1$. \square

Remark 4.12. The proof above also gives a description of the differentials in the resolutions \mathfrak{F}_\bullet of [Theorem 4.11](#).

5. A GENERAL FRAMEWORK OF DUALITIES TO STUDY BLOWUP ALGEBRAS

In this section, we apply our generalization of Jouanolou duality to study the defining equations of several interesting classes of Rees algebras. Determining the defining equations of Rees algebras is a problem of utmost importance with applications in Algebraic Geometry, Commutative Algebra and applied areas like Geometric Modeling (see [\[6, 11–14, 27, 39, 41–44, 47, 48, 55\]](#)).

Setup 5.1. Let k be a field, $R = k[x_1, \dots, x_d]$ be a standard graded polynomial ring and $\mathfrak{m} = (x_1, \dots, x_d) \subset R$ be the graded irrelevant ideal. Let $I \subset R$ be an ideal minimally generated by n forms f_1, \dots, f_n of the same degree $D \geq 1$. Let $T = k[y_1, \dots, y_n]$ be a standard graded polynomial ring and $\mathfrak{n} = (y_1, \dots, y_n) \subset T$ be the graded irrelevant ideal. Let B be the standard bigraded polynomial ring $B = R \otimes_k T$ (i.e., $\text{bideg}(x_i) = (1, 0)$ and $\text{bideg}(y_i) = (0, 1)$). Since we are primarily interested in the \mathbf{x} -grading, for any bigraded B -module M , we denote by M_i the graded T -module $M_i := \bigoplus_{j \in \mathbb{Z}} [M]_{(i,j)}$.

As customary, we consider the k -algebra homomorphism

$$\Phi : B \rightarrow \mathcal{R}(I) = R[It] = \bigoplus_{j=0}^{\infty} I^j t^j \subset R[t], \quad x_i \mapsto x_i \text{ and } y_i \mapsto f_i t.$$

Our goal is to determine the defining ideal $\mathcal{J} := \text{Ker}(\Phi) \subset B$ of the Rees algebra $\mathcal{R}(I)$. Let $\mathcal{G} : \mathbb{P}_k^{d-1} \dashrightarrow \mathbb{P}_k^{n-1}$ be the rational map

$$(x_1 : \dots : x_d) \mapsto (f_1(x_1, \dots, x_d) : \dots : f_n(x_1, \dots, x_d))$$

determined by the forms f_1, \dots, f_n generating I . The Rees algebra $\mathcal{R}(I)$ provides the bihomogeneous coordinate ring of the closure of the graph of \mathcal{G} , and this reinforces the interest in finding the defining equations of $\mathcal{R}(I)$.

Typically, a good way to study the Rees algebra is to approximate it by the symmetric algebra, which is much better understood, at least as far as defining equations are concerned. If

$$F_1 \xrightarrow{\varphi} F_0 \rightarrow I \rightarrow 0$$

is a homogeneous minimal free presentation of I , where the i -th basis element of F_0 maps to f_i , then the defining ideal of $\text{Sym}(I)$ is the kernel \mathcal{L} of the induced map $B = \text{Sym}(F_0) \rightarrow \text{Sym}(I)$. This ideal can be

described explicitly as

$$\mathcal{L} = (g_1, \dots, g_u) = I_1([y_1, \dots, y_n] \cdot \varphi)$$

with $u = \text{rank}(F_1) = \mu(\text{syz}(I))$. There is a natural exact sequence of bigraded B -modules

$$0 \rightarrow \mathcal{A} \rightarrow \text{Sym}(I) \rightarrow \mathcal{R}(I) \rightarrow 0,$$

where $\mathcal{A} = \mathcal{J}/\mathcal{L}$ coincides with the R -torsion of the symmetric algebra. Therefore, one can study \mathcal{A} to determine (or to obtain information about) the defining ideal \mathcal{J} of the Rees algebra $\mathcal{R}(I)$. Our main contribution in this direction is [Theorem 5.3](#) below. To prove it, we need the following adaptation of [Theorem 4.6\(ii\)](#) to our bigraded setting:

Proposition 5.2. *Assume [Setup 5.1](#). Suppose that the ideal $I \subset R$ has height $g = \text{ht}(I) \geq 2$ and satisfies the conditions F_0 and SD_1 . Then $\text{Sym}(I)$ is d -weakly Gorenstein with respect to $\mathfrak{m}\text{Sym}(I)$, and the canonical module has a weak generator that is bihomogeneous of bidegree $(d - (g - 1)D, g - 1)$.*

Proof. We first prove the claim for the case when the field \mathbb{k} is infinite. [Theorem 4.6](#) implies that S is d -weakly Gorenstein with respect to $\mathfrak{m}S$, and a weak generator u_1 of the canonical module is identified in the proof of the same theorem. To show that u_1 is bihomogeneous with

$$\text{bideg}(u_1) = (d - (g - 1)D, g - 1),$$

we note that the complex

$$\mathfrak{L}_\bullet : 0 \rightarrow \mathfrak{T}_{g-2} \rightarrow \cdots \rightarrow \mathfrak{T}_1 \rightarrow \mathfrak{T}_0 \rightarrow \mathfrak{Z}_{n-g} \rightarrow \cdots \rightarrow \mathfrak{Z}_0 \rightarrow 0$$

of [Proposition 4.5](#) can be made bihomogeneous. Thus we have

$$\mathfrak{T}_{g-2} = \bigoplus_{j=1}^{g-1} B(-(g-j)D, -n + g - j).$$

On the other hand, the surjection

$$\text{Hom}_B(\mathfrak{T}_{g-2}, \omega_B) \otimes_B S = \bigoplus_{j=1}^{g-1} S((g-j)D - d, -g + j) \twoheadrightarrow \omega_S$$

introduced in the proof of [Theorem 4.6](#) is already bihomogeneous. As u_1 is the image of the first standard basis element of the module on the left, it follows that u_1 is indeed bihomogeneous with $\text{bideg}(u_1) = (d - (g - 1)D, g - 1)$. This completes the proof of the claim when \mathbb{k} is infinite.

Next we treat the case of an arbitrary ground field \mathbb{k} . We proceed as in the proof of [Proposition 3.4](#). Let \mathbb{k}' be an infinite field containing \mathbb{k} . Write $R' := R \otimes_{\mathbb{k}} \mathbb{k}'$ and $S' := S \otimes_{\mathbb{k}} \mathbb{k}'$, and let u'_1 be the weak generator of $\omega_{S'} \cong \omega_S \otimes_{\mathbb{k}} \mathbb{k}'$ considered in the previous paragraph. As in the proof of [Proposition 3.4](#), it suffices to show that, up to multiplication by a unit in \mathbb{k}' , the element u'_1 is extended from a bihomogeneous element of ω_S , and this in turn follows once we have proved that the bigraded Hilbert function of $\omega_{S'}$ has value 1 when evaluated at $\text{bideg}(u'_1)$. [Theorem 4.6](#) gives an isomorphism $\omega_{S'} \cong (f_1, y_1)^{g-2} S'$. The proof of the same theorem shows that, up to sign, this isomorphism maps u_1 to the image $\overline{y_1^{g-2}}$ of y_1^{g-2} in S' . That proof and the argument in the previous paragraph also show that the isomorphism $\omega_{S'} \cong (f_1, y_1)^{g-2} S'$ is bihomogeneous, though not necessarily of bidegree $(0, 0)$. So it suffices to prove that the bigraded Hilbert function of $(f_1, y_1)^{g-2} S'$ has value 1 when evaluated at $\text{bideg}(\overline{y_1^{g-2}}) = (0, g - 2)$, which is obvious. \square

Theorem 5.3. *Assume [Setup 5.1](#). Suppose that $I = (f_1, \dots, f_n) \subset R$ has height $g = \text{ht}(I) \geq 2$, that I satisfies the conditions G_d and SD_1 , and that $n = d + 1$. Let $\delta := (g - 1)D - d$ and $\beta := d - g + 2$. Then the following statements hold:*

(i) $\mathcal{A} = H_m^0(\text{Sym}(I))$.

(ii) *For all $0 \leq i \leq d - 1$, there is an isomorphism of bigraded B -modules*

$$H_m^i(\text{Sym}(I)) \cong {}^* \text{Ext}_T^i(\text{Sym}(I), T)(-\delta, -\beta).$$

In particular, $\mathcal{A} = H_m^0(\text{Sym}(I)) \cong {}^ \text{Hom}_T(\text{Sym}(I), T)(-\delta, -\beta)$.*

(iii) *For all $i < 0$ and $i > \delta$, we have $\mathcal{A}_i = 0$. There is an isomorphism $\mathcal{A}_\delta \cong T(-\beta)$ of graded T -modules. For all $0 \leq i \leq \delta$, we have the equality of $\text{Sym}(I)$ -ideals*

$$\mathcal{A}_{\geq i} = 0 :_{\text{Sym}(I)} m^{\delta+1-i}.$$

(iv) *If $g \geq 3$ and I satisfies SD_2 , then \mathcal{A} is minimally generated in \mathbf{x} -degrees at most $(g - 2)D - d + 1$.*

(v) *Let $0 \leq i \leq \delta$. The natural multiplication map $\mu : \mathcal{A}_i \otimes_T \text{Sym}(I)_{\delta-i} \rightarrow \mathcal{A}_\delta$, $a \otimes b \mapsto a \cdot b$ is a perfect pairing that induces the abstract isomorphism of graded T -modules*

$$\nu : \mathcal{A}_i \xrightarrow{\cong} \text{Hom}_T(\text{Sym}(I)_{\delta-i}, \mathcal{A}_\delta)$$

seen in part (ii).

(vi) *For all $2 \leq i \leq d + 1$, there is an isomorphism of bigraded B -modules*

$$H_n^i(\text{Sym}(I)) \cong {}^* \text{Ext}_R^{i-1}(\text{Sym}(I), R)(-(g - 1)D, g - 1).$$

Proof. To simplify notation, set $S := \text{Sym}(I)$.

(i) From the assumed conditions it follows that I is of linear type on the punctured spectrum of R (see [\[20, Theorem 5.1, Corollary 4.8\]](#)). This implies that $\mathcal{A} = H_m^0(S)$.

(ii) One uses [Proposition 5.2](#) and applies [Theorem 2.22](#) with $c = n - 1 = d$ and $a = -n = -d - 1$.

(iii) The isomorphism $\mathcal{A} \cong {}^* \text{Hom}_T(S, T)(-\delta, -\beta)$ implies that $\mathcal{A}_i = 0$ when $i < 0$ and $i > \delta$, and it gives the isomorphism $\mathcal{A}_\delta \cong T(-\beta)$ of graded T -modules. The same isomorphism for \mathcal{A} together with [Lemma 2.10](#) shows the equality $\mathcal{A}_{\geq i} = 0 :_S m^{\delta+1-i}$.

(iv) Since the ideal $I \subset R$ satisfies SD_2 , the graded strands of bidegree $(*, k)$ of the complex \mathcal{L}_\bullet of [Proposition 4.5](#) are acyclic complexes of R -modules that satisfy the assumptions of [\[44, Theorem 4.3\]](#) with $i := d$ and $t := 1$. As $g \geq 3$, the $(d - 1)$ -st module in each of these complexes is

$$[\mathcal{F}_{g-3}]_{(*, k)} = \left[\bigoplus_{j=1}^{g-2} B^n(-(g-1-j)D, -n+g-j) \right]_{(*, k)},$$

so it follows that these R -modules are generated in degrees at most $(g - 2)D$. Thus according to [\[44, Theorem 4.3\]](#), the \mathbf{x} -degrees of the minimal generators of \mathcal{A} are at most $(g - 2)D - d + 1$.

(v) This follows from the analogue of [Theorem 2.12\(iii\)](#) in the bigraded setting, which can be proved using part (ii).

(vi) Recall that S is a Cohen-Macaulay ring of dimension $d + 1$ (see [Lemma 4.3](#) and [Lemma 4.4](#)). We apply a duality result of Herzog and Rahimi [[24](#), Theorem, Corollary 1.6] that is being reproduced in [Theorem 5.4](#) below, where we also provide a short direct proof for the reader's convenience. For $2 \leq i \leq d + 1$, we have the following isomorphisms of bigraded B -modules, where $-\vee^B$ denotes bigraded \mathbb{k} -duals,

$$\begin{aligned} H_n^i(S) &\cong (H_m^{d+1-i}(\omega_S))^{\vee^B} && \text{by [Theorem 5.4](#)} \\ &\cong (H_m^{d+1-i}(S))^{\vee^B}(d - (g-1)D, g-1) && \text{by [Lemma 2.6](#) and [Proposition 5.2](#)} \\ &\cong H_n^i(\omega_S)(d - (g-1)D, g-1) && \text{by [Theorem 5.4](#).} \end{aligned}$$

By exchanging the roles of R and T in [Theorem 2.21](#), we obtain the isomorphism

$$H_n^i(\omega_S) \cong {}^* \text{Ext}_R^{i-1}(S, R)(-d, 0)$$

for every $i \geq 0$. Combining the above isomorphisms we conclude that

$$H_n^i(S) \cong {}^* \text{Ext}_R^{i-1}(S, R)(-(g-1)D, g-1),$$

as required. \square

If B is a standard bigraded polynomial ring over a field \mathbb{k} as in [Setup 5.1](#) and M is a bigraded B -module, then by

$$(M)^{\vee^B} := \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathbb{k}}([M]_{(-i,-j)}, \mathbb{k})$$

we denote bigraded \mathbb{k} -dual of M . The next result provides a short proof of Herzog-Rahimi duality. This duality was used already in the proof of the theorem above.

Theorem 5.4 (Herzog-Rahimi duality [[24](#), Theorem, Corollary 1.6]). *Let B, m, n be as in [Setup 5.1](#) and let M be a finitely generated bigraded B -module. Then there are two converging spectral sequences of bigraded B -modules*

$$H_n^p(\text{Ext}_B^q(M, \omega_B)) \implies (H_m^{d+n-p-q}(M))^{\vee^B}$$

and

$$H_m^p(\text{Ext}_B^q(M, \omega_B)) \implies (H_n^{d+n-p-q}(M))^{\vee^B}.$$

In particular, if M is Cohen-Macaulay, then there are two isomorphisms of bigraded B -modules

$$H_n^i(\omega_M) \cong (H_m^{\dim(M)-i}(M))^{\vee^B} \quad \text{and} \quad H_m^i(\omega_M) \cong (H_n^{\dim(M)-i}(M))^{\vee^B}$$

for every $i \in \mathbb{Z}$.

Proof. Let $F_{\bullet} : \cdots \rightarrow F_i \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ be a minimal bigraded free B -resolution of M . As in the second proof of [Theorem 2.3](#), via the spectral sequences coming from the second quadrant double complex $F_{\bullet} \otimes_R C_m^{\bullet}$ we obtain the isomorphisms $H_m^i(M) \cong H_{d-i}(H_m^d(F_{\bullet}))$. Dualizing with the exact functor $(\bullet)^{\vee^B}$, we now get

$$(H_m^i(M))^{\vee^B} \cong H^{d-i} \left((H_m^d(F_{\bullet}))^{\vee^B} \right) \cong H^{d-i} \left(H_n^n(\text{Hom}_B(F_{\bullet}, \omega_B)) \right)$$

where the last isomorphism follows from the functorial isomorphism $(H_m^d(B))^{\vee_B} \cong H_n^n(\text{Hom}_B(B, \omega_B))$. Let $G^\bullet := \text{Hom}_B(F_\bullet, \omega_B)$ and consider the first quadrant double complex $G^\bullet \otimes_T C_n^\bullet$; the corresponding spectral sequences are given by

$$I E_2^{p,q} = \begin{cases} H^p(H_n^n(G^\bullet)) \cong (H_m^{d-p}(M))^{\vee_B} & \text{if } q = n \\ 0 & \text{otherwise} \end{cases}$$

and

$$II E_2^{p,q} = H_n^p(H^q(G^\bullet)) \cong H_n^p(\text{Ext}_B^q(M, \omega_B)).$$

Therefore, we obtain the convergent spectral sequence

$$H_n^p(\text{Ext}_B^q(M, \omega_B)) \implies (H_m^{d+n-p-q}(M))^{\vee_B}.$$

The other spectral sequence is obtained by a completely symmetric argument. The two claimed isomorphisms follow immediately from the spectral sequences. \square

Remark 5.5. The isomorphisms of [Theorem 5.3](#) are probably best seen in the form of a diagram. Assume [Setup 5.1](#) with all the conditions and notations of [Theorem 5.3](#). We have the following diagram of bihomogeneous B -isomorphisms, where a label on an arrow specifies the range of i where the isomorphism is valid:

$$\begin{array}{ccc} (*\text{Ext}_R^{d-i}(\text{Sym}(I), R))^{\vee_B}(d, 0) & & \\ \downarrow \text{by Theorem 2.21} & & \\ (H_n^{d+1-i}(\omega_{\text{Sym}(I)}))^{\vee_B} & \xleftrightarrow{\text{by Theorem 5.4}} & H_m^i(\text{Sym}(I)) \\ & & \downarrow \text{by Lemma 2.6 and Proposition 5.2 } 0 \leq i \leq d-1 \\ (H_n^{d+1-i}(\text{Sym}(I)))^{\vee_B}(-\delta, -\beta + n) & \xleftrightarrow{\text{by Theorem 5.4}} & H_m^i(\omega_{\text{Sym}(I)})(-\delta, -\beta + n) \\ & & \downarrow \text{by Theorem 2.21} \\ & & *\text{Ext}_T^i(\text{Sym}(I), T)(-\delta, -\beta) \end{array}$$

In particular, when $0 \leq i \leq d-1$, we obtain the surprising fact that the above six B -modules are related by bigraded B -isomorphisms.

We end this subsection with a simple condition for \mathcal{A} to be free as a T -module.

Proposition 5.6. *Assume [Setup 5.1](#). Suppose that $\text{Sym}(I)$ is Cohen-Macaulay and I is of linear type on the punctured spectrum of R . Then $\text{depth}(\mathcal{R}(I)) \geq d$ if and only if \mathcal{A} is a free T -module.*

Proof. Again, we write $S := \text{Sym}(I)$. Since S is Cohen-Macaulay and $\dim(S) \geq \dim(\mathcal{R}(I))$, the assertion is obvious if $\mathcal{A} = 0$. So we may assume that $\mathcal{A} \neq 0$. In particular, $I \neq 0$ and so $\dim(S) \geq d+1$. As S is

Cohen-Macaulay and I is not of linear type, but is of linear type on the punctured spectrum, it follows that $n = d + 1$ (see [28, Theorem 2.6] and [20, Proposition 2.4]). Therefore $\dim(T) = d + 1$.

We have the equality $\mathcal{A} = H_m^0(\text{Sym}(I))$ because I is of linear type on the punctured spectrum. Thus there is a positive integer $k > 0$ such that \mathcal{A} is a module over $B/m^k B$. From the finite homomorphism $T \rightarrow B/m^k B$, we see that \mathcal{A} is a finitely generated T -module and that

$$\text{depth}_T(\mathcal{A}) = \text{depth}_{B/m^k B}(\mathcal{A}) = \text{depth}_B(\mathcal{A}).$$

Since $\text{depth}(S) \geq d + 1$, the short exact sequence of B -modules $0 \rightarrow \mathcal{A} \rightarrow S \rightarrow \mathcal{R}(I) \rightarrow 0$ shows that $\text{depth}(\mathcal{R}(I)) \geq d$ if and only if $\text{depth}_B(\mathcal{A}) \geq d + 1$ or, equivalently, $\text{depth}_T(\mathcal{A}) \geq d + 1$. The last inequality holds if and only if \mathcal{A} is a free T -module because $d + 1 = \dim(T)$. \square

5.1. Explicit equations via Morley forms. We are now going to apply the theory of Morley forms developed in §2.1 to symmetric algebras of ideals. Unlike in the classical case (where $\text{Sym}(I)$ is a complete intersection), here the Morley forms *do not* give a perfect pairing. We instead need to introduce a division/reduction to make explicit the perfect pairing seen in Theorem 5.3(v). Throughout this subsection, we assume the hypotheses of Theorem 5.3.

Setup 5.7. Assume Setup 5.1. Suppose that $I = (f_1, \dots, f_n) \subset R$ has height $g = \text{ht}(I) \geq 2$, that I satisfies the conditions G_d and SD_1 , and that $n = d + 1$. In particular, Theorem 5.3 yields the isomorphism

$$\mathcal{A} = H_m^0(\text{Sym}(I)) \cong {}^* \text{Hom}_T(\text{Sym}(I), T)(-\delta, -\beta)$$

where $\delta := (g - 1)D - d$ and $\beta := d - g + 2$. Assume that the defining ideal \mathcal{L} of $\text{Sym}(I)$ contains a bihomogeneous regular sequence ℓ_1, \dots, ℓ_d that generates \mathcal{L} at $\mathfrak{m}B \in \text{Min}(\mathcal{L})$ and satisfies the conditions $\sum_{i=1}^d \deg_x(\ell_i) = \delta + d$ and $\deg_y(\ell_i) = 1$ (recall that $\text{ht}(\mathcal{L}) = d = \text{ht}(\mathfrak{m}B)$).

Owing to the above bigraded isomorphism for \mathcal{A} , we have $\mathcal{A}_{(\delta, \star)} \cong T(-\delta, -\beta)$, and the current definition of δ is consistent with the one in Remark 2.13 and the discussion following Theorem 2.22 (see also Proposition 5.2).

We write

$$[\ell_1, \dots, \ell_d] = [x_1, \dots, x_d] \cdot G$$

where $G \in B^{d \times d}$ is a $d \times d$ matrix whose entries are bihomogeneous with constant bidegrees along the columns. Multiplying with the adjoint of G yields

$$(x_1, \dots, x_d) \cdot \det(G) \subset (\ell_1, \dots, \ell_d) \subset \mathcal{L}.$$

Let $\text{syl} \in \text{Sym}(I)$ be the image of $\det(G)$ in $\text{Sym}(I)$. Notice that $\text{syl} \in 0 :_{\text{Sym}(I)} \mathfrak{m} \subset \mathcal{A}$. Moreover, syl is bihomogeneous with $\text{bideg}(\text{syl}) = (\sum_{i=1}^d \deg_x(\ell_i) - d, d) = (\delta, d)$, so

$$\text{syl} \in \mathcal{A}_{(\delta, d)}.$$

As $(\ell_1, \dots, \ell_d) \subset \mathfrak{m}B$ is an inclusion of complete intersection ideals of height d , we have that

$$(\ell_1, \dots, \ell_d) :_B \mathfrak{m}B = (\ell_1, \dots, \ell_d, \det(G))$$

(see [58], [36, Proposition 3.8.1.6]). Thus, up to multiplication by a unit in \mathbb{k} , the element $\det(G)$ modulo (ℓ_1, \dots, ℓ_d) only depends on the ideal (ℓ_1, \dots, ℓ_d) . Hence, up to multiplication by a unit in \mathbb{k} , the element

syl is uniquely determined by (ℓ_1, \dots, ℓ_d) . Following standard notation in the literature, we call syl the *Sylvester form of ℓ_1, \dots, ℓ_d with respect to x_1, \dots, x_d* (or the *determinant of the Jacobian dual*).

The element $\text{syl} \in \mathcal{A}_{(\delta, d)}$ is non-zero. Indeed, since ℓ_1, \dots, ℓ_d generate \mathcal{L} locally at the minimal prime $\mathfrak{mB} \in \text{Min}(\mathcal{L})$, it follows that

$$(\ell_1, \dots, \ell_d) :_{\mathbb{B}} \mathcal{L} \not\subset \mathfrak{mB}.$$

Since \mathfrak{mB} is a prime ideal, we obtain $\mathcal{L} \not\supset (\ell_1, \dots, \ell_d) :_{\mathbb{B}} \mathfrak{mB} = (\ell_1, \dots, \ell_d, \det(G))$. Thus $\det(G) \notin \mathcal{L}$, as asserted.

We choose a generator $\text{rsyl} \in \mathcal{A}_{(\delta, \beta)} \cong \mathbb{k}(-\delta, -\beta)$ and call it a *reduced Sylvester form*. It coincides with the element \mathfrak{s} in [Theorem 2.12](#) and the discussion following [Theorem 2.22](#). As $\text{syl} \in \mathcal{A}_{(\delta, d)}$ and $\mathcal{A}_{(\delta, \star)} \cong \mathbb{T}(-\delta, -\beta)$, there exists a unique element $\alpha \in \mathbb{T}_{d-\beta}$ so that

$$\text{syl} = \alpha \cdot \text{rsyl}.$$

In other words, $\text{rsyl} = \frac{\text{syl}}{\alpha}$.

We now construct an explicit element $\Delta \in [0 :_{\text{Sym}(I)^e} \mathbb{D}]_{(\delta, \star)}$ that can be used in the definition of Morley forms, as described in [§2.1](#) and the discussion following [Theorem 2.22](#). As each $\ell_i \otimes 1 - 1 \otimes \ell_i$ is in the kernel of the multiplication map $\mathbb{B}^e \rightarrow \mathbb{B}$, we can write

$$[\ell_1 \otimes 1 - 1 \otimes \ell_1, \dots, \ell_d \otimes 1 - 1 \otimes \ell_d] = [x_1 \otimes 1 - 1 \otimes x_1, \dots, x_d \otimes 1 - 1 \otimes x_d] \cdot H,$$

where $H \in (\mathbb{B}^e)^{d \times d}$ is a $d \times d$ matrix whose entries are bihomogeneous with constant bidegrees along the columns. Let Δ be the image of $\det(H)$ in $\text{Sym}(I)^e$. Notice that $\Delta \in 0 :_{\text{Sym}(I)^e} \mathbb{D}$ and that Δ is bihomogeneous of bidegree (δ, d) , in other words

$$\Delta \in [0 :_{\text{Sym}(I)^e} \mathbb{D}]_{(\delta, d)}.$$

Let $\Pi : \mathbb{B} \rightarrow \mathbb{T}$ be the homomorphism of \mathbb{T} -algebras with $\Pi(x_i) = 0$ for all i , and consider the map $\varepsilon = \mathbb{B} \otimes_{\mathbb{T}} \Pi : \mathbb{B}^e \rightarrow \mathbb{B}$. Applying ε to the entries of H we obtain a matrix G as above. Since syl is well defined up to multiplication by a unit in \mathbb{k} , we may assume that

$$\varepsilon(\Delta) = \text{syl},$$

where $\varepsilon : S^e \rightarrow S$ is defined as in [§2.1](#). Recall that $\text{syl} = \alpha \cdot \text{rsyl}$ and $\text{rsyl} = \mathfrak{s}$.

On the other hand, $\varepsilon(\Delta) = \varepsilon(\text{morl}_{(\delta, 0)})$ and $\text{morl}_{(\delta, 0)} = \alpha \cdot \mathfrak{s} \otimes 1$ with α defined as in [Lemma 2.17\(i\)](#). Comparing the two expressions for $\varepsilon(\Delta)$ we see that the current definition of α coincides with the one in [§2.1](#).

Since $\varepsilon(\Delta) = \text{syl}$ and $\text{syl} \neq 0$ by the above, we also see that $\Delta \notin \ker \varepsilon = (1 \otimes \mathfrak{m}) \text{Sym}(I)^e$, where \mathfrak{m} generates the unique associated prime of the ideal $\mathfrak{m} \text{Sym}(I)$. Thus the hypotheses of [Theorem 2.18](#) are satisfied and we obtain:

Remark 5.8. Assume [Setup 5.7](#). The element $\Delta = \overline{\det(H)} \in \text{Sym}(I)^e$ can be used to define Morley forms as in [§2.1](#). Thus we obtain the decomposition

$$\Delta = \overline{\det(H)} = \sum_{i=0}^{\delta} \text{morl}_{(\delta-i, i)} \quad \text{where} \quad \text{morl}_{(\delta-i, i)} \in [\text{Sym}(I)^e]_{(\delta-i, i, d)}.$$

Applying [Theorem 2.18](#) with $\mathfrak{s} := \text{rsyl}$ and α defined by the identity $\text{syl} = \alpha \cdot \text{rsyl}$, we obtain explicit isomorphisms between the \mathbb{T} -modules $\text{Hom}_{\mathbb{T}}(\text{Sym}(I)_{\delta-i}, \mathcal{A}_{\delta})$ and \mathcal{A}_i .

The above requirement that $\text{Sym}(I)$ is a complete intersection at the minimal prime \mathfrak{m} of $\text{Sym}(I)$ is satisfied in several situations of interest (see [Remark 6.4](#) and [Remark 6.9](#)).

6. APPLICATIONS TO CERTAIN FAMILIES OF IDEALS

In this section, we apply the results and techniques developed in previous sections. For organizational purposes, we divide the section into two subsections. Each subsection covers a family of ideals where our results are applied.

6.1. Zero dimensional ideals. In this subsection, we study Rees algebras of zero dimensional ideals. In this case, we are able to approximate the Rees algebra with two algebras: one is the symmetric algebra of the ideal, and the other is the symmetric algebra of the module whose syzygies are the Koszul syzygies of the ideal. This provides two methods for studying Rees algebras. We will use the following setup.

Setup 6.1. Assume [Setup 5.1](#) with $I \subset R$ an \mathfrak{m} -primary ideal, and suppose that $n = d + 1$. Let $(K_\bullet, \partial_\bullet)$ be the Koszul complex of the sequence f_1, \dots, f_n , and H_\bullet be its homology. Let $E := \text{Coker}(\partial_2)$ be the module defined by the Koszul syzygies.

The next lemma collects some basic facts about the relation between the module E and the ideal I .

Lemma 6.2. *The following statements hold:*

(i) $\text{Sym}(E) \cong B/\mathcal{K}$ where \mathcal{K} is the determinantal ideal

$$\mathcal{K} = I_2 \begin{pmatrix} y_1 & y_2 & \cdots & y_{d+1} \\ f_1 & f_2 & \cdots & f_{d+1} \end{pmatrix}.$$

(ii) $\mathcal{R}(E) \cong \mathcal{R}(I)$, and there is a natural exact sequence $0 \rightarrow H_{\mathfrak{m}}^0(\text{Sym}(E)) \rightarrow \text{Sym}(E) \rightarrow \mathcal{R}(I) \rightarrow 0$.

(iii) $\dim(\text{Sym}(E)) = \dim(\text{Sym}(I)) = d + 1$.

Proof. (i) By construction, a presentation of E is given by $K_2 \xrightarrow{\partial_2} K_1 \rightarrow E \rightarrow 0$. Let $\{e_1, \dots, e_{d+1}\}$ be a basis of K_1 with $\partial_1(e_i) = f_i$ for all i . Then $\{e_i \wedge e_j\}_{1 \leq i < j \leq d+1}$ is a basis of K_2 , and the map ∂_2 is defined by $\partial_2(e_i \wedge e_j) = f_i e_j - f_j e_i$. Let $[\partial_2]$ be the matrix representation of ∂_2 with respect to the chosen bases of K_1 and K_2 . Then the defining ideal of $\text{Sym}(E)$ is

$$\mathcal{K} = I_1([\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{d+1}] \cdot [\partial_2]) = \left(\{f_i y_j - f_j y_i\}_{1 \leq i < j \leq d+1} \right) = I_2 \begin{pmatrix} y_1 & y_2 & \cdots & y_{d+1} \\ f_1 & f_2 & \cdots & f_{d+1} \end{pmatrix}.$$

(ii) The existence of the exact sequence follows from the fact that $E_{\mathfrak{p}} \cong I_{\mathfrak{p}} = R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$. The exact sequence in turn shows that the kernel of the natural surjection $\text{Sym}(E) \rightarrow \mathcal{R}(I)$ is the R -torsion of $\text{Sym}(E)$.

(iii) This part follows from the Huneke-Rossi formula (see [\[28\]](#), [\[56, Theorem 1.2.1\]](#)). □

As a consequence of the lemma above, we have the following two short exact sequences

$$0 \rightarrow \mathcal{A} \rightarrow \text{Sym}(I) \rightarrow \mathcal{R}(I) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{B} \rightarrow \text{Sym}(E) \rightarrow \mathcal{R}(I) \rightarrow 0$$

where $\mathcal{A} = H_{\mathfrak{m}}^0(\text{Sym}(I))$ and $\mathcal{B} = H_{\mathfrak{m}}^0(\text{Sym}(E))$. The next theorem makes $\text{Sym}(I)$ and $\text{Sym}(E)$ good candidates to approximate $\mathcal{R}(I)$ when $I \subset R$ is \mathfrak{m} -primary.

Theorem 6.3. *Assume [Setup 6.1](#). Then the following statements hold:*

(i) For all $0 \leq i \leq d-1$, there are isomorphisms of bigraded B -modules

$$H_m^i(\text{Sym}(I)) \cong {}^* \text{Ext}_T^i(\text{Sym}(I), T)(-dD - d - D, -2).$$

In particular, $\mathcal{A} = H_m^0(\text{Sym}(I)) \cong {}^* \text{Hom}_T(\text{Sym}(I), T)(-dD - d - D, -2)$.

(ii) For all $0 \leq i \leq d-1$, there are isomorphisms of bigraded B -modules

$$H_m^i(\text{Sym}(E)) \cong {}^* \text{Ext}_T^i(\text{Sym}(E), T)(-dD + d, -1).$$

In particular, $\mathcal{B} = H_m^0(\text{Sym}(E)) \cong {}^* \text{Hom}_T(\text{Sym}(E), T)(-dD + d, -1)$.

Proof. Part (i) follows from [Theorem 5.3\(ii\)](#) (notice that $d \geq 2$), and part (ii) is a consequence of [Proposition 3.4](#). \square

Remark 6.4. After having the abstract duality statements of part (ii) in the above theorem, all the statements of [Theorem 5.3](#) hold for $\text{Sym}(E)$ with bigraded shifts given by $\delta := dD - d$ and $\beta := 1$. Also, at least when \mathbb{k} is infinite, the defining ideal of $\text{Sym}(E)$ contains a regular sequence as in [Setup 5.7](#), and so the technique of generalized Morley forms developed in [§5.1](#), and in particular [Remark 5.8](#), apply to $\text{Sym}(E)$. We are not pursuing this approach in the present paper due to length constraints. In a subsequent paper, we plan to study Rees algebras of \mathfrak{m} -primary ideal; from a geometric point of view this is relevant, as it entails studying the graph of morphisms $\mathbb{P}_{\mathbb{k}}^{d-1} \rightarrow \mathbb{P}_{\mathbb{k}}^d$ parametrizing a hypersurface.

6.2. Gorenstein ideals of height three. In this subsection, we concentrate on Gorenstein ideals of height 3 and we determine the defining equations of the Rees algebra for a particular family.

Setup 6.5. In addition to [Setup 5.1](#), assume that $I = (f_1, \dots, f_n) \subset R$ is a Gorenstein ideal of height 3 with $\mu(I) = n = d+1$ and that I satisfies G_d . Let $\varphi \in \mathbb{R}^{(d+1) \times (d+1)}$ be an alternating presentation matrix of I whose non-zero entries are homogeneous of degree $h \geq 1$ (consequently, $\deg(f_i) = D = \frac{n-1}{2}h = \frac{d}{2}h$).

Theorem 6.6. Assume [Setup 6.5](#). Then we have bigraded isomorphisms of B -modules

$$H_m^i(\text{Sym}(I)) \cong {}^* \text{Ext}_T^i(\text{Sym}(I), T)(-d(h-1), -d+1)$$

for all $0 \leq i \leq d-1$.

Proof. The statement is a particular case of [Theorem 5.3\(ii\)](#). \square

We notice that with the assumptions above the ideal I is of linear type locally on the punctured spectrum (by [Theorem 5.3\(i\)](#), for instance) and that $I_1(\varphi)$ is an \mathfrak{m} -primary ideal (because I satisfies G_d and $\mu(I) \geq d$). If $I_1(\varphi)$ is a complete intersection, the defining ideal of $\mathcal{R}(I)$ has been determined explicitly in [[43](#), Theorem 9.1 and Remark 9.2], without any restriction on the number of generators of I (see also [[32](#), 2.10] for the case $n = d+1$). Thus we may assume that $I_1(\varphi)$ is not a complete intersection. In this subsection we are going to treat this case under the following hypotheses:

Setup 6.7. In addition to [Setup 6.5](#), assume that $I_1(\varphi)$ is an almost complete intersection, but not a complete intersection. Let c_1, \dots, c_{d+1} be homogeneous generators of $I_1(\varphi)$, necessarily of degree h . The defining ideal of $\text{Sym}(I)$ is generated by the entries of the row vector $[y_1, \dots, y_{d+1}] \cdot \varphi$. This vector can be rewritten as

$$[y_1, \dots, y_{d+1}] \cdot \varphi = [c_1, \dots, c_{d+1}] \cdot A,$$

where $A \in T^{(d+1) \times (d+1)}$ is a $(d+1) \times (d+1)$ matrix with linear entries in T . The matrix A is referred to as the *Jacobian dual* of φ in the literature. Let $\Delta_{i,j}$ be the minor of A obtained by deleting the i -th row and the j -th column multiplied by $(-1)^{i+j}$.

Lemma 6.8. *Assume Setup 6.7. The matrix A has the following properties:*

- (i) $\text{rank}(A) = d$.
- (ii) $\text{Ker}(A)$ is a free T -module generated by the vector $[y_1, \dots, y_{d+1}]^t$.
- (iii) The adjoint of A^t is of the form

$$\text{adj}(A^t) = \begin{bmatrix} y_1 \delta_1 & y_2 \delta_1 & \cdots & y_{d+1} \delta_1 \\ y_1 \delta_2 & y_2 \delta_2 & \cdots & y_{d+1} \delta_2 \\ \vdots & \vdots & \cdots & \vdots \\ y_1 \delta_{d+1} & y_2 \delta_{d+1} & \cdots & y_{d+1} \delta_{d+1} \end{bmatrix}$$

where $\delta_i = \Delta_{i,j}/y_j \in T$.

- (iv) The ideal $I_d(A) \subset T$ satisfies $\text{ht}(I_d(A)) \geq 2$.
- (v) $\text{Ker}(A^t)$ is a free T -module generated by the vector $[\delta_1, \dots, \delta_{d+1}]^t$.

Proof. (i) We first show that $\text{rank}(A) \geq d$. Let Q be the quotient field of T . Notice that $\mathcal{L}' := \mathcal{L}(\mathbb{R} \otimes_{\mathbb{k}} Q)$ is a proper ideal of the polynomial ring $\mathbb{R} \otimes_{\mathbb{k}} Q$ and that $\text{ht}(\mathcal{L}') \geq \text{ht}(\mathcal{L}) = n - 1 = d$. Thus \mathcal{L}' requires at least d generators, showing that $\text{rank}(A) \geq d$.

Since φ is alternating, it follows that

$$[m, x_1^h, \dots, x_d^h] \cdot A \cdot [y_1, \dots, y_{d+1}]^t = [y_1, \dots, y_{d+1}] \cdot \varphi \cdot [y_1, \dots, y_{d+1}]^t = 0,$$

and so $A \cdot [y_1, \dots, y_{d+1}]^t = 0$. This shows that $\text{rank}(A) \leq d$, which settles part (i).

(ii) By the above paragraph, we have a complex $0 \rightarrow T \xrightarrow{[y_1, \dots, y_{d+1}]^t} T^{d+1} \xrightarrow{A} T^{d+1}$. This complex is exact by the Buchsbaum-Eisenbud acyclicity criterion since $\text{ht}(y_1, \dots, y_{d+1}) \geq 2$ and $\text{rank}(A) = d$. This shows part (ii).

(iii) As $A \cdot [y_1, \dots, y_{d+1}]^t = 0$ and $A \cdot \text{adj}(A) = 0$, part (ii) implies that each column of $\text{adj}(A)$ is a multiple of $[y_1, \dots, y_{d+1}]^t$. This establishes part (iii).

(iv) Suppose that $\text{ht}(I_d(A)) \leq 1$. In this case, $I_d(A) \subset \mathfrak{p}$ for some homogeneous $\mathfrak{p} \in \text{Spec}(T)$ of height one.

The only minimal primes of the defining ideal \mathcal{L} of $\text{Sym}(I)$ are \mathfrak{m}_B and \mathfrak{J} , the defining ideal of $\mathcal{R}(I)$. Indeed, let \mathfrak{P} be a minimal prime of \mathcal{L} ; if $\mathfrak{P} \cap \mathbb{R} = \mathfrak{m}$ then $\mathfrak{P} \supset \mathfrak{m}_B$ and so $\mathfrak{P} = \mathfrak{m}_B$; if $\mathfrak{P} \cap \mathbb{R} = \mathfrak{q} \subsetneq \mathfrak{m}$ then $\mathcal{L}_{\mathfrak{P}} = \mathfrak{J}_{\mathfrak{P}}$, because I is of linear type on the punctured spectrum of \mathbb{R} , and so $\mathfrak{P} = \mathfrak{J}$.

We now show that $\mathfrak{p} \not\subset \mathfrak{J}$. To see this we determine the analytic spread $\ell := \ell(I)$ of I , which is the dimension of the special fiber ring $\mathcal{F} := \mathcal{R}(I) \otimes_{\mathbb{R}} \mathbb{k} = \mathcal{R}(I)_{(0,*)}$. For this we may assume that \mathbb{k} is infinite, in which case I is integral over a homogeneous ideal J generated by ℓ elements. Since I is of linear type on the punctured spectrum, it follows that $\text{ht}(J : I) \geq d$. Thus $\mu(J) \geq d$ by [30, Theorem 3.1(i)] because I is strongly Cohen-Macaulay and satisfies G_d . Therefore $\ell \geq d$. On the other hand, one always has $\ell \leq d$, hence $\ell = d$. This shows that \mathcal{F} is a domain of embedding codimension 1, and so $\mathcal{F} \cong T/fT$, where f is homogeneous with $\deg(f) - 1 = r$, the reduction number of I .

By part (iii), $\delta_i y_j \in \mathfrak{p}$ for all i, j and so $\delta_i \in \mathfrak{p}$ for all i . Since at least one $\delta_i \neq 0$ by part (i), it follows that \mathfrak{p} contains a non-zero homogeneous polynomial in T of degree $d - 1$. If $\mathfrak{p} \subset \mathcal{J}$, then $\mathcal{J} \cap T = fT$ contains a non-zero homogeneous element of degree $d - 1$, which shows that $r = \deg(f) - 1 \leq d - 2$. By [50, Theorem 3.1] (see also [53, Corollary 5.6]) the inequality $r \leq d - 2 = \ell - 2$ implies that $\mu(I_1(\varphi)) \leq d$. So $I_1(\varphi)$ is a complete intersection, which is ruled out by Setup 6.7. This proves that $\mathfrak{p} \not\subset \mathcal{J}$.

As $\mathfrak{p} \subset T$ we trivially have that $\mathfrak{p} \not\subset \mathfrak{m}_B$, hence \mathfrak{p} is not contained in any minimal prime of \mathcal{L} . Thus $\text{ht}(\mathcal{L} + \mathfrak{p}B) \geq \text{ht}(\mathcal{L}) + 1 = d + 1$, and so $\text{ht}((\mathcal{L} + \mathfrak{p}B)/\mathfrak{p}B) \geq d$. The rest of the argument is similar to the proof of part (i), but now we take Q to be the quotient field of T/\mathfrak{p} and $\mathcal{L}' := \mathcal{L}(\mathbb{R} \otimes_{\mathbb{k}} Q)$. As the ideal \mathcal{L}' is proper and has height at least d , it follows that \mathcal{L}' requires at least d generators. Therefore the image of A in Q has rank at least d , which shows that $I_d(A) \not\subset \mathfrak{p}$, contrary to our assumption. This proves that indeed $\text{ht}(I_d(A)) \geq 2$.

(v) Part (iii) gives the complex

$$0 \rightarrow T \xrightarrow{[\delta_1, \dots, \delta_{d+1}]^t} T^{d+1} \xrightarrow{A^t} T^{d+1}$$

and the inclusion $(\delta_1, \dots, \delta_{d+1}) \supset I_d(A)$. Since $\text{ht}(I_d(A)) \geq 2$ by (iv), the Buchsbaum-Eisenbud acyclicity criterion shows that this complex is exact. \square

Remark 6.9. Assume that \mathbb{k} is an infinite field. Then there exists a bihomogeneous regular sequence ℓ_1, \dots, ℓ_d belonging to \mathcal{L} that generates \mathcal{L} at $\mathfrak{m}_B \in \text{Min}(\mathcal{L})$, and satisfies $\text{bideg}(\ell_i) = (h, 1)$ (as required in Setup 5.7).

Proof. Since $\text{rank}(A) = d$ (see Lemma 6.8(i)), $\mu(\mathcal{L} \otimes_B B_{\mathfrak{m}_B}) \leq d$. Now the claim follows because \mathcal{L} is generated in bidegree $(h, 1)$. \square

In the next theorem, we determine explicitly the defining equations of the Rees algebra of a Gorenstein ideal as in Setup 6.7 with $h = 2$ and $I_1(\varphi)$ a monomial ideal.

Theorem 6.10. *In addition to Setup 6.7 assume that $h = 2$ and $I_1(\varphi)$ is a monomial ideal. Without loss of generality we may assume that $I_1(\varphi) = (x_1 x_2, x_1^2, \dots, x_d^2)$. Then the following statements hold:*

(i) *The natural isomorphism of bigraded B -modules ${}^* \text{Hom}_T(B, T) \cong T[x_1^{-1}, \dots, x_d^{-1}]$ restricts to an isomorphism*

$$\begin{aligned} {}^* \text{Hom}_T(\text{Sym}(I), T) &\cong B \cdot (\delta_1 x_1^{-1} x_2^{-1} + \delta_2 x_1^{-2} + \dots + \delta_{d+1} x_d^{-2}) x_3^{-1} \dots x_d^{-1} \\ &\quad + B \cdot x_1^{-1} x_3^{-1} \dots x_d^{-1} \\ &\quad + B \cdot x_2^{-1} x_3^{-1} \dots x_d^{-1} \subset T[x_1^{-1}, \dots, x_d^{-1}]. \end{aligned}$$

In addition, $\mathcal{A} = H_m^0(\text{Sym}(I)) \cong {}^ \text{Hom}_T(\text{Sym}(I), T)(-d, -d + 1)$ is a free T -module that is minimally generated as a B -module by three elements of bidegrees $(0, 2d - 2)$, $(1, d - 1)$, and $(1, d - 1)$.*

(ii) *We use the notation of §2.2 and in particular Corollary 2.19. Define elements $U_i \in \text{Sym}(I)$ by the equations*

$$\begin{aligned} 1 \otimes_T U_0 &= \frac{1}{\alpha} \cdot \text{morl}_{(d,0)} \cdot ((\delta_1 x_1^{-1} x_2^{-1} + \delta_2 x_1^{-2} + \dots + \delta_{d+1} x_d^{-2}) x_3^{-1} \dots x_d^{-1} \otimes_T 1) \\ 1 \otimes_T U_1 &= \frac{1}{\alpha} \cdot \text{morl}_{(d-1,1)} \cdot (x_1^{-1} x_3^{-1} \dots x_d^{-1} \otimes_T 1) \end{aligned}$$

$$1 \otimes_{\mathbb{T}} \mathcal{U}_2 = \frac{1}{\alpha} \cdot \text{morl}_{(d-1,1)} \cdot (x_2^{-1} x_3^{-1} \cdots x_d^{-1} \otimes_{\mathbb{T}} 1).$$

These elements have bidegrees $(0, 2d-2)$, $(1, d-1)$, $(1, d-1)$, respectively, and they generate the ideal $\mathcal{A} \subset \text{Sym}(\mathbb{I})$ minimally. In particular, the defining ideal $\mathcal{J} \subset \mathbb{B}$ of $\mathcal{R}(\mathbb{I})$ is minimally generated by \mathcal{L} and lifts to \mathbb{B} of the elements $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2$.

Moreover by [Remark 6.9](#), if \mathbb{k} is an infinite field, then there exists a \mathbb{B} -regular sequence ℓ_1, \dots, ℓ_d in \mathcal{L} that satisfies the conditions of [Setup 5.7](#); therefore the discussion following [Setup 5.7](#) and in particular [Remark 5.8](#) provide explicit Morley forms and the linear polynomial $\alpha \in \mathbb{T}$.

- (iii) Let $\mathcal{G} : \mathbb{P}_{\mathbb{k}}^{d-1} \dashrightarrow \mathbb{P}_{\mathbb{k}}^d$ be the rational map determined by the forms f_1, \dots, f_n generating \mathbb{I} . Let $Y \subset \mathbb{P}_{\mathbb{k}}^d$ be the closure of the image of \mathcal{G} . The degree of \mathcal{G} is equal to $\deg(\mathcal{G}) = 2^{d-2}$ and $\deg(Y) = 2d-2$.
- (iv) $\text{depth}(\mathcal{R}(\mathbb{I})) = d$ and $\text{depth}(\text{gr}_{\mathbb{T}}(\mathbb{R})) = d-1$.

Proof. We write $S := \text{Sym}(\mathbb{I})$. From [Theorem 6.6](#), we have an isomorphism of bigraded S -modules $\mathcal{A} \cong {}^* \text{Hom}_{\mathbb{T}}(S, \mathbb{T})(-d, -d+1)$. In order to determine each of the \mathbb{T} -module $\text{Hom}_{\mathbb{T}}(S_i, \mathbb{T})$ we are going to construct a (not necessarily minimal) homogeneous free presentation

$$P_i \xrightarrow{\psi_i} Q_i \rightarrow S_i \rightarrow 0$$

of S_i .

Denote by M_i the set of monomials in R_i that are not in the ideal $I_1(\varphi)$. Let $C = \{x_1 x_2, x_1^2, \dots, x_d^2\}$ be the minimal monomial generating set of $I_1(\varphi)$ and let C^j be the minimal monomial generating set of $(I_1(\varphi))^j$. For $i \geq 2$ and $\alpha = (\alpha_3, \dots, \alpha_d) \in \{0, 1\}^{d-2}$, we define the set

$$D_{i,\alpha} := \{x_1^{b_1} x_2^{b_2} x_3^{\alpha_3} \cdots x_d^{\alpha_d} c \in R_{i-2} \mid c \in C^j \text{ and } b_1 + b_2 \leq 1\}.$$

Write $D_{i,\alpha} \cdot C$ for the set of products of elements in $D_{i,\alpha}$ and in C . The particular shape of C easily shows that the sets $D_{i,\alpha}$ give a partition of the minimal monomial generating set of $\mathfrak{m}^{i-2} \subset R$, and the sets $D_{i,\alpha} \cdot C$ give a partition of the minimal monomial generating set of $\mathfrak{m}^{i-2} I_1(\varphi) \subset R$.

We consider the (not necessarily minimal) free presentation of the graded \mathbb{T} -module S_i

$$P_i \xrightarrow{\psi_i} Q_i \rightarrow S_i \rightarrow 0,$$

where

$$Q_i := \bigoplus_{\alpha} \underbrace{\bigoplus_{u \in D_{i,\alpha}} \bigoplus_{c \in C} T \cdot q_{u,c}}_{Q_{i,\alpha}} \oplus \underbrace{\bigoplus_{v \in M_i} T \cdot q_v}_{N_i}$$

and

$$P_i := \bigoplus_{\alpha} \underbrace{\left(\bigoplus_{u \in D_{i,\alpha}} \bigoplus_{l=1}^{d+1} T \cdot p_{u,l} \oplus \bigoplus_{r \in F_{i,\alpha}} T \cdot p_r \right)}_{P_{i,\alpha}};$$

here $q_{u,c}$, q_v , $p_{u,l}$ and p_r are the chosen basis elements for the free \mathbb{T} -modules Q_i and P_i . The free \mathbb{T} -module B_i has the natural description $B_i = \bigoplus_{|\gamma|=i} \mathbb{T} \cdot \mathbf{x}^\gamma \cong \mathbb{T}^{\binom{i+d-1}{d-1}}$. In the free \mathbb{T} -module Q_i , $q_{u,c}$ corresponds to the pair $(u \in D_{i,\alpha}, c \in C)$ and q_v corresponds to the monomial $v \in M_i$. The map

$Q_i \rightarrow S_i$ is defined by sending $q_{u,c} \mapsto \overline{uc} \in S_i$ and $q_v \mapsto \overline{v} \in S_i$. The sets $D_{i,\alpha} \cdot C$ together with M_i give a partition of the monomials of degree i in R . Hence we have a natural surjection $Q_i \twoheadrightarrow S_i$.

Notice that there can exist $(u \in D_{i,\alpha}, c \in C)$ and $(u' \in D_{i,\alpha}, c' \in C)$ such that $uc = u'c'$, and as a consequence we need a relation to identify $q_{u,c}$ and $q_{u',c'}$. The fact that the sets $D_{i,\alpha} \cdot C$ are disjoint implies that there are no further relations among the chosen basis elements of Q_i . Denote by $F_{i,\alpha}$ a set indexing all the needed homogeneous relations (columns in ψ_i) among the $q_{u,c}$'s with $u \in D_{i,\alpha}$ and $c \in C$. Let $p_r \in P_i$ be the basis element that gives the relation $r \in F_{i,\alpha}$. For any $r \in F_{i,\alpha}$, we have that

$$\psi_i(p_r) \in \bigoplus_{u \in D_{i,\alpha}} \bigoplus_{c \in C} T \cdot q_{u,c} \subset Q_{i,\alpha}.$$

It now follows that $B_i \cong Q_i / (\sum_{r \in F_{i,\alpha}} T \cdot \psi_i(p_r))$. The relations of S_i come from a bihomogeneous generating set of the ideal $\mathfrak{m}^{i-2} \cdot \mathcal{L}$, which in turn can be chosen to be the union of bihomogeneous generating sets of $D_{i,\alpha} \cdot \mathcal{L}$. In the free T -module P_i , $p_{u,l}$ corresponds to the pair $(u \in D_{i,\alpha}, g_l)$. Since $g_l = [x_1 x_2, x_1^2, \dots, x_d^2] \cdot [a_{1,l}, a_{2,l}, \dots, a_{d,l}]^t$, for any $u \in D_{i,\alpha}$, we have that

$$[\psi_i(p_{u,1}), \psi_i(p_{u,2}), \dots, \psi_i(p_{u,d+1})] = [q_{u,x_1 x_2}, q_{u,x_1^2}, \dots, q_{u,x_d^2}] \cdot A.$$

By combining these facts, it follows that ψ_i is a presentation matrix of S_i . See [Example 6.11](#) for an illustration of this presentation.

We have $\text{Hom}_T(S_i, T) = \text{Ker}(\psi_i^t) \subset \text{Hom}_T(Q_i, T)$, and we denote the dual basis elements of $\text{Hom}_T(Q_i, T)$ by $q_{u,c}^*$ and q_v^* . The isomorphism $\mathcal{A} \cong {}^* \text{Hom}_T(S, T)(-d, -d+1)$ shows that $\text{Ker}(\psi_i^t) = \text{Hom}_T(S_i, T) = 0$ whenever $i > d$.

We notice that $\psi_i(P_{i,\alpha}) \subset Q_{i,\alpha}$, and we write $\psi_{i,\alpha} : P_{i,\alpha} \rightarrow Q_{i,\alpha}$ for the restriction map. Thus we obtain a direct sum decomposition of maps

$$\psi_i = \bigoplus_{\alpha} \psi_{i,\alpha}.$$

As a consequence,

$$\text{Hom}_T(S_i, T) \cong \text{Ker}(\psi_i^t) = \bigoplus_{\alpha} \text{Ker}(\psi_{i,\alpha}^t) \oplus \bigoplus_{v \in M_i} T \cdot q_v^*.$$

If $|\alpha| := \sum \alpha_i = i-2$, then $j = 0$ and $D_{i,\alpha} = \{z_\alpha\}$ where $z_\alpha := x_3^{\alpha_3} \cdots x_d^{\alpha_d}$, and so there are no relations among the $q_{u,c}$'s indexed by the pairs $(u \in D_{i,\alpha}, c \in C)$. Hence, when $|\alpha| = i-2$, [Lemma 6.8\(v\)](#) shows that $\text{Ker}(\psi_{i,\alpha}^t)$ is the free T -module given by

$$\text{Ker}(\psi_{i,\alpha}^t) = T \cdot \underbrace{(\delta_1 q_{z_\alpha, x_1 x_2}^* + \delta_2 q_{z_\alpha, x_1^2}^* + \cdots + \delta_{d+1} q_{z_\alpha, x_d^2}^*)}_{Z_{i,\alpha}}.$$

We obtain that

$$\begin{aligned} \text{Ker}(\psi_i^t) &= \bigoplus_{|\alpha| < i-2} \text{Ker}(\psi_{i,\alpha}^t) \oplus \bigoplus_{|\alpha| = i-2} \text{Ker}(\psi_{i,\alpha}^t) \oplus \bigoplus_{v \in M_i} T \cdot q_v^* \\ &= \bigoplus_{|\alpha| < i-2} \text{Ker}(\psi_{i,\alpha}^t) \oplus \bigoplus_{|\alpha| = i-2} T \cdot Z_{i,\alpha} \oplus \bigoplus_{v \in M_i} T \cdot q_v^*. \end{aligned}$$

We now consider the case $i = d$. Notice that $M_d = \emptyset$. Let $\sigma := (1, \dots, 1) \in \mathbb{N}^{d-2}$. The T -module \mathcal{A}_0 is free of rank at most 1; indeed, $\mathcal{A}_0 \subset T$ is the defining ideal of the special fiber ring of I , which is a principal ideal (see the proof of [Lemma 6.8](#)). The isomorphism $\mathcal{A}_0 \cong \text{Hom}_T(S_d, T)(-d+1)$ then shows that $\text{Hom}_T(S_d, T) = \text{Ker}(\psi_d^t)$ is also free of rank at most 1, hence indecomposable. As moreover $Z_{d,\sigma} \neq 0$, we conclude that

$$\text{Ker}(\psi_d^t) = \text{Ker}(\psi_{d,\sigma}^t) = T \cdot Z_{d,\sigma}$$

and $\text{Ker}(\psi_{d,\alpha}^t) = 0$ for all $\alpha \neq \sigma$.

For a given α set $\eta := x_3^{1-\alpha_3} \dots x_d^{1-\alpha_d}$. There is a commutative diagram of T -linear maps

$$\begin{array}{ccc} P_{i,\alpha} & \xrightarrow{\psi_{i,\alpha}} & Q_{i,\alpha} \\ \cdot\eta \Big\| & & \Big\| \cdot\eta \\ P_{i+d-2-|\alpha|,\sigma} & \xrightarrow{\psi_{i+d-2-|\alpha|,\sigma}} & Q_{i+d-2-|\alpha|,\sigma} \end{array}$$

where the vertical maps are isomorphisms. Thus $\text{Ker}(\psi_{i,\alpha}^t) \cong \text{Ker}(\psi_{i+d-2-|\alpha|,\sigma}^t)$. If $|\alpha| < i-2$, then $i+d-2-|\alpha| > d$, and therefore $\text{Ker}(\psi_{i+d-2-|\alpha|,\sigma}^t) = 0$. So it follows that $\text{Ker}(\psi_{i,\alpha}^t) = 0$.

In summary, we have now proved that for all $0 \leq i \leq d$,

$$\text{Hom}_T(S_i, T) = \text{Ker}(\psi_i^t) = \bigoplus_{|\alpha|=i-2} T \cdot Z_{i,\alpha} \oplus \bigoplus_{v \in M_i} T \cdot q_v^*$$

and that this is a free T -module. Using the natural embedding $\text{Hom}_T(S_i, T) \subset \text{Hom}_T(B_i, T)$, we can rewrite $\text{Hom}_T(S_i, T)$ as

$$\text{Hom}_T(S_i, T) = \bigoplus_{|\alpha|=i-2} T \cdot (\delta_1(x_1 x_2 z_\alpha)^* + \dots + \delta_{d+1}(x_d^2 z_\alpha)^*) \oplus \bigoplus_{v \in M_i} T \cdot v^* \subset \text{Hom}_T(B_i, T),$$

where $(x_1 x_2 z_\alpha)^*, \dots, (x_d z_\alpha)^*, v^*$ denote dual basis elements of $\text{Hom}_T(B_i, T)$.

Moreover, using the identification ${}^* \text{Hom}_T(B, T) \cong T[x_1^{-1}, \dots, x_d^{-1}]$, we obtain an isomorphism of bi-graded B -modules

$$\text{Hom}_T(S_i, T) \cong \bigoplus_{|\alpha|=i-2} T \cdot (\delta_1(x_1 x_2 z_\alpha)^{-1} + \dots + \delta_{d+1}(x_d^2 z_\alpha)^{-1}) \oplus \bigoplus_{v \in M_i} T \cdot v^{-1} \subset T[x_1^{-1}, \dots, x_d^{-1}].$$

Since $M_{d-1} = \{x_1 x_3 \dots x_d, x_2 x_3 \dots x_d\}$, we conclude that

$$\begin{aligned} {}^* \text{Hom}_T(S, T) &\cong B \cdot (\delta_1 x_1^{-1} x_2^{-1} + \delta_2 x_1^{-2} + \dots + \delta_{d+1} x_d^{-2}) x_3^{-1} \dots x_d^{-1} \\ &\quad + B \cdot x_1^{-1} x_3^{-1} \dots x_d^{-1} \\ &\quad + B \cdot x_2^{-1} x_3^{-1} \dots x_d^{-1} \subset T[x_1^{-1}, \dots, x_d^{-1}]. \end{aligned}$$

The three generators above are minimal for bidegree reasons. The bihomogeneous isomorphism

$$\mathcal{A} = H_m^0(S) \cong {}^* \text{Hom}_T(S, T)(-d, -d+1)$$

shows that \mathcal{A} is minimally generated as an S -module by three bihomogeneous elements of bidegrees $(0, 2d-2)$, $(1, d-1)$, $(1, d-1)$. This completes the proof of part (i).

Part (ii) is a direct consequence of part (i) and [Corollary 2.19](#).

[[10](#), Theorems 5.4 and 5.8(iii)] and [[9](#), Theorem 2.4(iii)] show that $\deg(\mathcal{G}) \cdot \deg(Y) = (d - 1)2^{d-1}$. Since $\deg(Y) = 2(d - 1)$ by part (i), it follows that $\deg(\mathcal{G}) = 2^{d-2}$. Thus part (iii) is proven.

Part (iv) is a consequence of part (i), [Proposition 5.6](#), [[29](#), proof of Proposition 1.1], and [[50](#), Theorem 3.1]. □

Example 6.11. Let $d = 4$, $R = \mathbb{k}[x_1, \dots, x_4]$, $T = \mathbb{k}[y_1, \dots, y_5]$ and $h = 2$. In this case, the T -module S_3 has (a non-minimal) presentation given by $T^{22} \xrightarrow{\psi_3} T^{22} \rightarrow S_3 \rightarrow 0$, where

$$\psi_3 = \begin{matrix} & x_1g_1 & x_1g_2 & x_1g_3 & x_1g_4 & x_1g_5 & x_2g_1 & x_2g_2 & x_2g_3 & x_2g_4 & x_2g_5 & & x_3g_1 & x_3g_2 & x_3g_3 & x_3g_4 & x_3g_5 & x_4g_1 & x_4g_2 & x_4g_3 & x_4g_4 & x_4g_5 \\ \begin{matrix} x_1 \cdot x_1x_2 \\ x_1 \cdot x_1^2 \\ x_1 \cdot x_2^2 \\ x_1 \cdot x_3^2 \\ x_1 \cdot x_4^2 \\ x_2 \cdot x_1x_2 \\ x_2 \cdot x_1^2 \\ x_2 \cdot x_2^2 \\ x_2 \cdot x_3^2 \\ x_2 \cdot x_4^2 \\ x_3 \cdot x_1x_2 \\ x_3 \cdot x_2^2 \\ x_3 \cdot x_3^2 \\ x_3 \cdot x_4^2 \\ x_4 \cdot x_1x_2 \\ x_4 \cdot x_1^2 \\ x_4 \cdot x_2^2 \\ x_4 \cdot x_3^2 \\ x_4 \cdot x_4^2 \\ x_1x_3x_4 \\ x_2x_3x_4 \end{matrix} & a_1 & b_1 & c_1 & d_1 & e_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & a_2 & b_2 & c_2 & d_2 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & a_3 & b_3 & c_3 & d_3 & e_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & a_4 & b_4 & c_4 & d_4 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & a_5 & b_5 & c_5 & d_5 & e_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & c_1 & d_1 & e_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & c_2 & d_2 & e_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & a_3 & b_3 & c_3 & d_3 & e_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & a_4 & b_4 & c_4 & d_4 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & a_5 & b_5 & c_5 & d_5 & e_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & c_1 & d_1 & e_1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & c_2 & d_2 & e_2 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & b_3 & c_3 & d_3 & e_3 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & b_4 & c_4 & d_4 & e_4 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & b_5 & c_5 & d_5 & e_5 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & c_1 & d_1 & e_1 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & c_2 & d_2 & e_2 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & b_3 & c_3 & d_3 & e_3 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & b_4 & c_4 & d_4 & e_4 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & b_5 & c_5 & d_5 & e_5 \\ & 0 \\ & 0 \end{matrix} .$$

The first 10 rows correspond to the basis elements $q_{u,c}$ where $u \in D_{3,(0,0)}$. The next 5 rows correspond to the basis elements $q_{u,c}$ where $u \in D_{3,(1,0)}$. The next 5 rows correspond to the basis elements $q_{u,c}$ where $u \in D_{3,(0,1)}$. The last two rows correspond to the two basis elements $q_{x_1x_3x_4}$ and $q_{x_2x_3x_4}$. The columns indexed by the $x_i g_j$'s correspond to the basis vectors $p_{u,l}$. The matrix also shows the direct sum decomposition of $\psi_3 = \psi_{3,\{0,0\}} \oplus \psi_{3,\{1,0\}} \oplus \psi_{3,\{0,1\}}$.

Remark 6.12. The fact that \mathcal{A} is generated in \mathbf{x} -degrees at most one, proved in [Theorem 6.10](#), was already predicted by [Theorem 5.3\(iv\)](#). This also shows that the bound in [Theorem 5.3\(iv\)](#) is sharp. The fact that the symmetric algebra and the Rees algebra of I first differ in \mathbf{y} -degree $d - 1$ was already proved in [[54](#), Theorem 2.5]. There it is also shown that $\mathcal{A}_{(\star, d-1)}$ is isomorphic to a shift of $\text{Ext}_{\mathbb{R}}^d(\mathbb{R}/I_1(\varphi), \mathbb{R})$. In the setting of [Theorem 6.10](#), this already implies that \mathcal{A} has exactly two minimal generators in \mathbf{y} -degree $d - 1$.

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