BOUNDS ON DEGREES OF VECTOR FIELDS

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1. Introduction

This paper is concerned with the structure of the module of derivations and its interplay with vector fields and singularities of varieties. Modules of derivations are not well understood – despite great advances on the Zariski-Lipman conjecture (see [20, 27, 45] for instance), there is still no complete characterization for when they are free. The paper focuses on Poincaré's problem on the degrees of vector fields.

In 1891, Poincaré asked the following question that became known as Poincaré's problem [37]: How can one decide whether a homogeneous differential equation given by a polynomial vector field $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ has a rational solution? This question has been rephrased as the problem to find *upper* bounds for the degree of any curve $\mathcal{C}$ to which $\mathcal{F}$ is tangent, possibly in terms of the degree of $\mathcal{F}$.

According to [18, p.57], 'This question is fundamental but difficult, and it has stimulated a lot of research for well over a century' (see [4–7, 10, 12, 15–18, 21, 33, 36, 41, 42]). It has often been addressed in greater generality, for curves and even varieties in $\mathbb{P}^n_{\mathbb{C}}$, and invariants other than the degrees of the variety and the vector field have been considered, because even for plane curves bounds only involving degrees are not always possible [5, 6, 33].

In this article, we study the generalized Poincaré problem from the opposite perspective, by establishing *lower* bounds on the degree of the vector field in terms of invariants of the variety, say X. This approach has the advantage that all the vector fields on $\mathbb{P}^n_k$ tangent to X, when restricted to X, can be encoded in a single module,

$$\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon\,,$$

see Proposition 2.4. Here R is the homogeneous coordinate ring of a subscheme $X \subset \mathbb{P}^n_k$, which, for the purpose of this introduction, is assumed to be reduced and irreducible over an algebraically closed field of characteristic zero; by $\mathfrak{m}$ we denote the homogeneous maximal ideal of R, by $\operatorname{Der}_k(R)$ the module of derivations, and by $\varepsilon \in \operatorname{Der}_k(R)$ the Euler derivation. If depth $R \geq 2$, then $\mathfrak{m}^{-1}\varepsilon = R\varepsilon$ and the module above essentially carries the same information as $\operatorname{Der}_k(R)$. The least degree of a vector field that leaves X invariant and does not vanish along X is 1 plus the initial degree of the

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module $\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon$, and our reformulation of Poincaré's problem becomes: *Find lower bounds for the initial degree* $\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon)$.

In the current paper we address this problem mainly for curves. We generalize bounds that were known for plane curves and we obtain new estimates as well. Our proofs are algebraic. In order to understand how tight the lower bounds for the initial degree of $\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon$ are, we also provide upper bounds, which sometimes lead to equalities. Our estimates use global invariants, such as the genus of the curve $\mathcal{C}$, the Castenuovo-Mumford regularity, or the a-invariant of the homogeneous coordinate ring R; invariants that can be considered global as well as local, like the singularity degree of $\mathcal{C}$, the total Tjurina number, or the multiplicity of R modulo the Jacobian ideal; and local information, such as the type of the singularities or a new invariant that we call *Loewy multiplicity*.

For smooth curves we prove that the initial degree satisfies the inequality $\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \geq a(R)+1$, which is an equality if $\mathcal{C}$ is arithmetically Gorenstein. In one of our main results, Theorem 4.10, we generalize this inequality to the case of curves with at most planar singularities. We show that

$$\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \geq a(R) + 1 + |\operatorname{Sing}(\mathcal{C})| - \operatorname{Lmult}(R/J_R)\,. \tag{1}$$

Here $a(R) = -\operatorname{indeg}(\omega_R)$ denotes the a-invariant of R, which is equal to $\operatorname{reg}\mathcal{C} - 3$ if $\mathcal{C}$ is arithmetically Cohen-Macaulay; and $\operatorname{Lmult}(R/J_R)$ denotes the Loewy multiplicity of R modulo the Jacobian ideal, which is bounded above by the sum of the local Tjurina numbers of $\mathcal{C}$ in this case. If $\mathcal{C}$ has only ordinary nodes as singularities, then $|\operatorname{Sing}(\mathcal{C})| - \operatorname{Lmult}(R/\operatorname{Jac}(R)) = 0$ and we obtain the inequality $\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \geq a(R)+1$, which is again an equality whenever $\mathcal{C}$ is arithmetically Gorenstein. The case of ordinary nodes had been treated before with the additional assumption that, first, $\mathcal{C}$ is a plane curve [7], then, $\mathcal{C}$ is a complete intersection [4], and, finally, $\mathcal{C}$ is arithmetically Cohen-Macaulay [16, 17].

The proof of inequality (1) has two main ingredients. Inspired by the use of general projections in [16], we prove more generally in Theorem 4.8 that if $A \subset R$ is a finite and birational extension of standard graded domains over a perfect field and A is Gorenstein of dimension at least two, then

$$\operatorname{indeg}(\operatorname{Der}_k(A)/A\varepsilon_A)+a(A)-a(R) \geq \operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) \geq \operatorname{indeg}(\operatorname{Der}_k(A)/A\varepsilon_A)-a(A)+a(R)\,.$$

In addition, for hypersurfaces of arbitrary dimension with only isolated singularities, we are able to bound $\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon)$ from below in terms of the a-invariant of R/J_R. Applying these two results to a curve $\mathcal{C}$ with only planar singularities, we prove inequality (1) by general projection to a plane curve. The general projection does not change the singularities of $\mathcal{C}$ and introduces only ordinary nodes as additional singularities, which guarantees that the difference $|\operatorname{Sing}(\mathcal{C})| - \operatorname{Lmult}(R/J_R)$ is unaltered.

Our other main results generalize, from plane curves to arbitrary curves, earlier work of du Plessis and Wall [12] and of Esteves and Kleiman [15] that uses the sum of local Tjurina numbers. For a curve $\mathcal{C} \subset \mathbb{P}^n_k$ of degree d with homogeneous coordinate ring R, we map a sufficiently general two-dimensional complete intersection S onto R, and write δ for the Castelnuovo-Mumford regularity of S and $J \subset R$ for the image of the Jacobian ideal of S. If $\mathcal{F}$ is a vector field that leaves $\mathcal{C}$ invariant and $0 \neq I_{\mathcal{F}} \subset R$ is an ideal defining the singular locus of $\mathcal{F}$, then $\mathcal{F}$ and J coincide up to a degree shift and this shift is closely related to the degree $\deg \mathcal{F}$, which we wish to control. The degree shift is reflected in the difference of multiplicities $e(R/I_{\mathcal{F}}) - e(R/J)$. Thus to estimate the degree shift, and hence $\deg \mathcal{F}$, from below it suffices to establish a lower bound for $e(R/I_{\mathcal{F}})$. To do so, we prove a non-vanishing result for maps between local cohomology modules of Koszul cycles that yields lower bounds for the regularity of the saturation $I_{\mathcal{F}}^{\mathrm{sat}}$ and hence for $e(R/I_{\mathcal{F}})$. The line of argument just described is inspired by the work of Esteves and Kleiman in [15]. Thus we obtain in Theorem 5.6 that

$$\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}) \geq \delta - 2 - \frac{e(R/J) - \delta}{d-1}\,,$$

unless $\mathcal{C}$ is a smooth complete intersection, in which case the initial degree is $a(R) + 1$.

This result is the starting point for various estimates in terms of the arithmetic genus p_a, the geometric genus p_g, and the sum τ of the local Tjurina numbers of the curve. Theorem 5.10 says that

$$\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \geq \frac{d}{d-1}\, a(R) - \frac{\tau - 2}{d-1}$$

if $\mathcal{C}$ is locally a complete intersection, and

$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \geq \frac{2p_a - \tau}{d-1}$$

if in addition $\mathcal{C}$ is arithmetically Cohen-Macaulay. For plane curves we prove in Corollary 5.7 that

$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \geq d - \frac{3}{2} - \sqrt{2\tau + 2p_g - d^2 + 3d - \frac{7}{4}}\,.$$

These bounds are sharp for plane curves of low genus and for other classes of curves, as illustrated in Proposition 5.12.

In Section 6 we turn to upper bounds for $\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon)$ in order to understand how sharp the lower bounds in Section 5 and Section 4 are. As a special case of Theorem 6.1, for instance, we prove that $a(R) + 1$ is an upper bound whenever $\mathcal{C}$ is arithmetically Gorenstein. In Theorem 6.5 we determine the minimal graded free resolution of $\operatorname{Der}_k(R)/R\varepsilon$ as a module over a polynomial ring if $\mathcal{C} \subset \mathbb{P}^3_k$ is smooth and arithmetically Cohen-Macaulay. From this we obtain the initial degree, the minimal number of generators, and the entire Hilbert series of $\operatorname{Der}_k(R)/R\varepsilon$. In particular, we see that the upper bound $a(R) + 1$ fails dramatically without the assumption of arithmetic Gorensteinness. Curiously, the work in Section 5 on local cohomology of Koszul cycles implies another result about the structure of the module of derivations – namely we prove in Proposition 7.1 that the Euler derivation cannot generate a free direct summand of $\operatorname{Der}_k(R)$ when $\mathcal{C}$ is arithmetically Gorenstein.

2. Preliminary results and a translation from geometry to algebra

Let R be a standard graded algebra over a field with homogeneous maximal ideal $\mathfrak{m}$. Let Ω be the module of differentials of R and let ε denote the *Euler derivation*. In this section we prove that the vector fields studied in [4–6, 15–18, 41, 42] correspond to elements in the R-module $\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon$, see Proposition 2.4.

We begin by reviewing basic definitions related to vector fields; we basically follow the definitions from the excellent reference [16, p. 4-5].

We adopt the following setting:

Setting 2.1. Let $n \geq 2$, and $S = k[x_1, \ldots, x_n]$ the homogeneous coordinate ring of $\mathbb{P}_k^{n-1}$, with maximal homogeneous ideal $\mathfrak{m}_S$. Let $I \subset S$ be a saturated homogeneous ideal and $R = S/I$ be the homogeneous coordinate ring of the corresponding projective scheme $X \subset \mathbb{P}_k^{n-1}$. Let $\mathfrak{m}$ be the maximal homogeneous ideal of R and ε the Euler derivation.

The Euler sequence

$$0 \longrightarrow Z \longrightarrow \Omega_k(S) = \oplus_{i=1}^n S dx_i \cong S^n(-1) \xrightarrow{[x_1 \ \ldots \ x_n]} \mathfrak{m}_S \longrightarrow 0$$

defines the cotangent sheaf $\Omega_{\mathbb{P}_k^{n-1}}$ as $\widetilde{Z}$. Notice that Z is the first syzygy module in the Koszul complex of $x_1, \ldots, x_n$.

A *vector field* on $\mathbb{P}_k^{n-1}$ of degree m is a homogeneous map of degree $m-1$

$$\eta : Z \longrightarrow S\,.$$

As $\mathrm{Ext}_S^1(\mathfrak{m}_S, S) = 0$, any such map is the restriction of a map

$$\xi : \Omega_k(S) \cong S^n(-1) \xrightarrow{[a_1 \ \ldots \ a_n]} S\,,$$

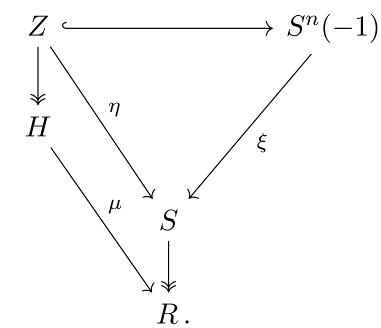
where the a_i are forms of degree m.

There is a commutative diagram with exact rows

$$(2) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & \Omega_k(S) \cong S^n(-1) & \longrightarrow & \mathfrak{m}_S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varrho} & & \\ 0 & \longrightarrow & L & \longrightarrow & \Omega_k(R) & \longrightarrow & \mathfrak{m} & \longrightarrow & 0 \end{array}\,.$$

We write $H = \mathrm{im}\,\varphi$ and notice that this is the image of the second differential in the Koszul complex built on the R-linear map $\Omega_k(R) \longrightarrow R$ corresponding to the Euler derivation (by the universal property). In particular, L/H is the first homology of this Koszul complex and hence it is annihilated by $\mathfrak{m}$. Moreover, $H = L$ if I is generated by forms whose degrees are not multiples of the characteristic. Indeed, in this case the Euler relations shows that $\ker \psi$ maps onto $\ker \varrho$, hence φ is surjective by the Snake Lemma.

One says that the vector field η *leaves* X *invariant* or that X is an *integral subscheme* of η (if X is a curve we say that X is a *leaf* of η) if η induces a map $\mu : H \longrightarrow R$, necessarily linear and homogeneous of degree $m-1$,

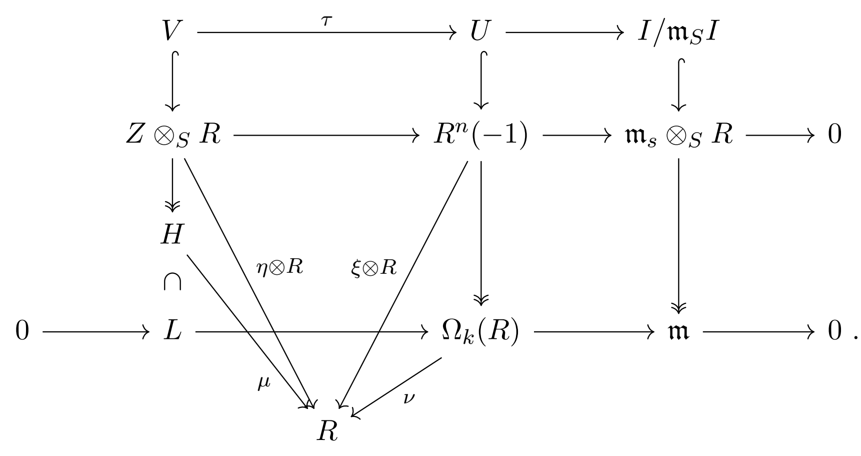


Notice that a map $H \longrightarrow R$ corresponds to a unique map $L \longrightarrow R$ if I is generated by forms whose degrees are not multiples of the characteristic or if depth $R \geq 2$.

Summarizing, every vector fields η of degree $m-1$ that leaves X invariant induces a unique homogeneous R-linear map $\mu : H \longrightarrow R$ of degree $m-1$.

Proposition 2.2. *Adopt Setting 2.1. A homogeneous R-linear map $\mu : H \longrightarrow R$ is induced by a vector field that leaves X invariant if and only if μ can be extended to a homogeneous R-linear map $\nu : \Omega_k(R) \longrightarrow R$.*

Proof. Given a vector field $\eta : Z \longrightarrow S$, the map $\mu : H \longrightarrow R$ is induced by η if and only if μ is induced by $\eta \otimes_S R$. Consider the commutative diagram with exact rows and columns



If the map μ is induced by η, hence by $\eta \otimes R$, then $(\eta \otimes R)(V) = 0$ which implies $(\xi \otimes R)(\operatorname{im} \tau) = 0$. By the above diagram, $\operatorname{coker} \tau \hookrightarrow I/\mathfrak{m}_S I$ and therefore $\mathfrak{m} \cdot U \subset \operatorname{im} \tau$. Thus $\mathfrak{m} \cdot (\xi \otimes R)(U) = 0$, which implies that $(\xi \otimes R)(U) = 0$ since depth $R > 0$. It follows that $\xi \otimes R$ induces a homogeneous R-linear map $\Omega_k(R) \longrightarrow R$, which gives μ when restricted to H.

Conversely, let $\nu : \Omega_k(R) \longrightarrow R$ be a homogeneous R-linear map. It can be lifted to a homogeneous S-linear map $\xi : S^n(-1) \longrightarrow S$ because $S^n(-1)$ is free. Set $\eta = \xi_{|Z}$. Since $(\xi \otimes R)(U) = 0$, we have $(\eta \otimes R)(V) = 0$ and so η induces a homogeneous R-linear map $\mu : H \longrightarrow R$, which is also the restriction of ν to H. $\square$

We write $-^* = \operatorname{Hom}_R(-, R)$. In light of Proposition 2.2 we are interested in the image of $\Omega_k(R)^* = \operatorname{Der}_k(R)$ in the module H^*. In the next proposition, we identify this image. We use the fact that there is a natural embedding $\operatorname{Der}_k(R) \hookrightarrow Q \otimes_R \operatorname{Der}_k(R)$ where Q is the total ring of fractions of R. For an ideal $\mathfrak{a}$ of R, we denote its inverse ideal by $\mathfrak{a}^{-1} := R :_Q \mathfrak{a}$. Notice that if $\operatorname{depth} R \geq 2$ then $\mathfrak{m}^{-1} = R$.

Example 2.3. Let $\mathcal{C}$ be the rational quartic curve given by the parametrization $\mathbb{P}^1_k \overset{(s^4:s^3t:st^3:t^4)}{\dashrightarrow} \mathbb{P}^3_k$. In this case $\mathfrak{m}^{-1}$ is $\overline{R}$, the integral closure of $R = k[s^4, s^3t, st^3, t^4]$, and $\overline{R} = R[s^2t^2] \subset Q$.

Proposition 2.4. *Adopt Setting 2.1. There are homogeneous exact sequences*

$$0 \longrightarrow \operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon \longrightarrow L^* \longrightarrow \operatorname{Ext}^2_R(k, R)\,,$$

$$0 \longrightarrow L^* \longrightarrow H^* \longrightarrow \operatorname{Ext}^1_R(C, R)\,,$$

where C is an R-module annihilated by $\mathfrak{m}$.

Therefore, there are natural homogeneous embeddings

$$\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon \hookrightarrow L^* \hookrightarrow H^*,$$

where the first embedding is an isomorphism if $\operatorname{depth} R \geq 3$ *and the second is an isomorphism if* $\operatorname{depth} R \geq 2$ *or the defining ideal of R is generated by forms whose degrees are not multiples of the characteristic.*

In particular, the vector fields that leave X invariant, when restricted to X, correspond to the homogeneous elements in the torsionfree R-module $\operatorname{Der}_k(R)/\mathfrak{m}^{-1}$.

Proof. Consider the Euler sequence

$$\begin{array}{ccccccccc} & & & & & \nearrow & R & & \\ & & & & & & \uparrow & & \\ 0 & \longrightarrow & L & \longrightarrow & \Omega_k(R) & \longrightarrow & \mathfrak{m} & \longrightarrow & 0\,. \end{array}$$

Dualizing into R shows that the first row of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}^{-1} & \longrightarrow & \operatorname{Der}_k(R) & \longrightarrow & L^* & \longrightarrow & \operatorname{Ext}^1_R(\mathfrak{m}, R) \\ & & \uparrow & \nearrow & & & & & \\ & & R & & & & & & \end{array}$$

is exact and that $1 \in R$ maps to $\varepsilon \in \operatorname{Der}_k(R)$. Since $\operatorname{Ext}^1_R(\mathfrak{m}, R) \cong \operatorname{Ext}^2_R(k, R)$, we obtain the first asserted exact sequence.

The second exact sequence follows because $C := L/H = \operatorname{coker}\varphi$ is annihilated by $\mathfrak{m}$ and depth $R > 0$, see page 4. We also recall that $H = L$ if the defining ideal of R is generated by forms whose degrees are not multiples of the characteristic. ☐

The *singular locus* of the vector field η is the subscheme $\Sigma = V(I_2(N)) \subset \mathbb{P}_k^{n-1}$, where N is the 2 by n matrix

$$N = \begin{bmatrix} x_1 & \dots & x_n \\ a_1 & \dots & a_n \end{bmatrix} ;$$

in fact a point $P \in \mathbb{P}_k^{n-1}$ does not belong to Σ if and only if $Q := [a_1(P) : \dots : a_n(P)] \in \mathbb{P}_k^{n-1}$ and there is a unique line passing through P and Q, giving the direction defined by η at P. We observe that $I_2(N)R$, the ideal defining the subscheme $\Sigma \cap X \subset X$, is the image of the map $\mu : H \longrightarrow R$ induced by η. One usually requires that $\Sigma \cap X$ does not contain an irreducible component of X, in other words, that the ideal $\operatorname{im}\mu = I_2(N)R$ has positive height in R.

We introduce a new invariant that is going to play an important role throughout the paper.

Definition 2.5. Let R be a non-negatively graded ring and M be a finitely generated R-module. We define the *faithful initial degree* of M over R as

$$\operatorname{findeg}_R M = \inf\{\deg m \mid m \in M \text{ homogenous with } \operatorname{ann} m = 0\} .$$

Notice that $\operatorname{findeg} M \geq \operatorname{indeg} M$, and equality holds if M is torsionfree and R is a domain.

Corollary 2.6. *In addition to Setting 2.1, assume that R has no embedded associated primes. If m is the smallest degree of a vector field on $\mathbb{P}_k^{n-1}$ that leaves X invariant and whose singular locus does not contain an irreducible component of X, then*

$$m = 1 + \operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) .$$

Proof. Proposition 2.2 and Proposition 2.4 show that $m-1$ is the smallest degree of a homogenous element in $\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon$ that, when regarded as a homogenous R-linear map $H \longrightarrow R$, has the property that $\operatorname{ht}(\operatorname{im}\mu) > 0$, equivalently $\operatorname{grade} \operatorname{im}\mu > 0$, or yet equivalently $\operatorname{ann}_R \mu = 0$. ☐

In the next proposition (and the remark following it) we identify H^* with a well-known fractional ideal: the inverse of the image in R of the Jacobian ideal of a general complete intersection mapping onto R. The resulting embedding $\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon \hookrightarrow J^{-1}(2-\delta)$ will be useful to compute initial degrees.

Setting 2.7. In addition to Setting 2.1 assume that $X = \mathcal{C} \subset \mathbb{P}_k^{n-1}$ is a reduced equidimensional curve over a perfect field k. Let $f_1, \dots, f_{n-2}$ be forms in I of degrees $\delta_1, \dots, \delta_{n-2}$ that generate I generically, and let J be the ideal generated by the images in R of the maximal minors of the Jacobian matrix of $f_1, \dots, f_{n-2}$. Set $\delta = \sum_{j=1}^{n-2}(\delta_j - 1)$.

Remark 2.8. We will see, as a consequence of Theorem 3.3(a), that the forms $f_1, \dots, f_{n-2}$ in Setting 2.7 generate I generically if and only if $\operatorname{ht} J > 0$.

If I is a complete intersection, then $f_1, \dots, f_{n-2}$ can be chosen to be a minimal homogeneous generating sequence of I, in which case J is the full Jacobian ideal of R.

If k is infinite and I is generated by forms of degrees $\delta_1 \geq \ldots \geq \delta_m$, then $f_1, \dots, f_{n-2}$ can be taken to be $n-2$ general forms of degrees $\delta_1 \geq \ldots \geq \delta_{n-2}$ in I. In this case $f_1, \dots, f_{n-2}$ also form a regular sequence.

Proposition 2.9. *Adopt Setting 2.7.*

(*a*) *There exist natural homogeneous R-linear map*

$$\bigwedge^2 \Omega_k(R) \longrightarrow\!\!\!\!\rightarrow H$$

$$\bigwedge^2 \Omega_k(R) \longrightarrow\!\!\!\!\rightarrow J(\delta-2)$$

$$\bigwedge^2 \Omega_k(R) \longrightarrow \omega_R\,,$$

where the first two maps are epimorphisms and all maps are isomorphisms generically.

(*b*) *After factoring out the R-torsion of $\bigwedge^2 \Omega_k(R)$ and H, or after dualizing into R, the first two maps become isomorphisms and the last map becomes an embedding,*

$$H/\operatorname{tor}(H) \cong J(\delta-2) \cong \wedge^2\Omega_k(R)/\operatorname{tor}(\wedge^2\Omega_k(R)) \hookrightarrow \omega_R$$

and

$$H^* \cong J^{-1}(2-\delta) \cong (\wedge^2\Omega_k(R))^* \hookleftarrow \omega_R^*\,.$$

In particular,

$$\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon \hookrightarrow J^{-1}(2-\delta)\,.$$

If $\mathcal{C}$ is smooth and arithmetically Cohen-Macaulay, then the embedding $\omega_R^* \hookrightarrow (\wedge^2\Omega_k(R))^*$ is an isomorphism.

Proof. The first map is the second differential onto its image in the Koszul complex of the homomorphism $\Omega_k(R) \longrightarrow R$ corresponding to the Euler derivation. This map is homogeneous and surjective, and the module H has rank one because the Koszul complex is exact locally on the punctured spectrum of R.

The second map is a direct consequence of the fact that $\Omega_k(R)$ is a module of rank 2 generated by n elements and J is generated by the maximal minors of the matrix consisting of $n-2$ columns of a matrix presenting $\Omega_k(R)$. Indeed, let $x_1, \dots, x_n$ be the images in R of the variables of S, extend $f_1, \dots, f_{n-2}$ to a homogeneous generating sequence $f_1, \dots, f_m$ for I, let Θ be the image in R of the transpose of the Jacobian matrix of $f_1, \dots, f_m$, let Θ' be the submatrix of Θ consisting of the first $n-2$ columns of Θ, and for $1 \leq i < j \leq n$, let Δ_{ij} be the maximal minor of Θ' with rows i and j deleted. Notice that Θ is a homogeneous presentation matrix of $\Omega_k(R)$, that $I_{n-2}(\Theta') = J$, and that $\deg \Delta_{ij} = \delta$. Since $\Omega_k(R)$ is an R-module of rank 2, it follows that Θ has rank $n-2$. Now the second natural map

$$\bigwedge^2 \Omega_k(R) \longrightarrow\!\!\!\!\rightarrow J(\delta-2)$$

is the homomorphism sending $dx_i \wedge dx_j$ to $(-1)^{i+j}\Delta_{ij}$. This map is well defined because Θ is a presentation matrix of $\Omega_k(R)$ and has rank $n-2$. The map is obviously homogeneous and surjective. Also notice that J has positive grade in R because the module generated by the columns of Θ' has rank $n-2$ as it is generically equal to the syzygy module of $\Omega_k(R)$.

The third map is the canonical class of R over k (see for instance [1,14,16,31,34]). This map is homogeneous and it is an isomorphism locally at the regular prime ideals of R.

Since the first two maps are epimorphisms between modules of the same rank, namely one, we see that these maps are also isomorphisms generically. This completes the proof of part (a). Part (b) follows from (a); for the last assertion, we also use Proposition 2.4. □

As a first immediate consequence of Proposition 2.4 and Proposition 2.9 we obtain:

Corollary 2.10. *Adopt Setting 2.7 and assume that $\mathcal{C}$ is smooth and arithmetically Cohen-Macaulay. If* $\operatorname{indeg}\omega_R^* > a(R)$, *then*

$$\operatorname{Der}_k(R)/R\varepsilon \cong \omega_R^*\,.$$

Proof. From Proposition 2.4 and Proposition 2.9(b) we obtain isomorphisms $L^* \cong H^* \cong \omega_R^*$. Now again by Proposition 2.4 there is an exact sequence

$$0 \longrightarrow \operatorname{Der}_k(R)/R\varepsilon \longrightarrow \omega_R^* \longrightarrow \operatorname{Ext}_R^2(k,R)\,.$$

Thus the assertion follows once we have shown that $\operatorname{Ext}_R^2(k,R)$ is concentrated in degrees $\le a(R)$.

For this we may assume that k is infinite. Since R is Cohen-Macaulay, there exists a regular sequence x_1, x_2 consisting of linear forms in R. We have

$$\operatorname{Ext}_R^2(k,R) \cong \operatorname{Hom}_R(k, R/(x_1,x_2))(2) \cong \operatorname{socle}(R/(x_1,x_2))(2)\,,$$

and the last module is concentrated in degrees at most $a(R/(x_1,x_2)) - 2 = a(R)$. □

Corollary 2.11. *Adopt Setting 2.7. Let μ be a vector field on $\mathbb{P}_k^{n-1}$ of degree m that leaves $\mathcal{C}$ invariant and whose singular locus does not contain an irreducible component of $\mathcal{C}$, which means that* $\operatorname{ht}\operatorname{im}\mu > 0$. *Then*

$$(\operatorname{im}\mu)(m-1) \cong J(\delta-2)\,.$$

Proof. The map μ induces a homogeneous epimorphism of degree $m-1$

$$H \overset{\mu}{\longrightarrow\!\!\!\!\rightarrow} \operatorname{im}\mu\,.$$

Recall that the R-module H has rank one. Since $\operatorname{grade}\operatorname{im}\mu > 0$, the vector field μ induces a homogeneous isomorphism after factoring out the torsion of H,

$$H/\operatorname{tor}(H) \cong (\operatorname{im}\mu)(m-1)\,.$$

The assertion now follows from Proposition 2.9(b). □

3. The invariants

In this section we discuss the invariants that play a role in our estimates.

a-invariant. In many of our bounds on curves, the a-invariant replaces Castelnuovo-Mumford regularity if the curve is not arithmetically Cohen-Macaulay. The *a-invariant* of a Noetherian standard graded algebra R over a field is defined as $a(R) = -\operatorname{indeg}(\omega_R)$. Local duality implies that

$$a(R) \leq \operatorname{reg} R - \dim R \tag{3}$$

and equality holds if R is Cohen-Macaulay.

Jacobian ideals. In this paper, Jacobian ideals will play an important role. To recall the general definition, let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, $W \subset S$ a multiplicative subset, $I \subset W^{-1}S$ an ideal, and $R = (W^{-1}S)/I$. Assume that every minimal prime ideal of I has the same height g and set $D = n - g$. The *Jacobian ideal* of the k-algebra R is defined as

$$J_R = J_{R/k} = \operatorname{Fitt}_D(\Omega_k(R))\,.$$

It turns out that $D = \dim R_{\mathfrak{p}} + \operatorname{trdeg}_k \kappa(\mathfrak{p})$ for every $\mathfrak{p} \in \operatorname{Spec} R$, where $\kappa(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$, see (4). In particular, D only depends on $k \subset R$, does not change when passing to a nonzero ring of fractions, and coincides with the integers s of Theorem 3.1 and D of Theorem 3.3. Moreover, for $V \subset R$ a multiplicative closed subset, one has $J_{V^{-1}R} = V^{-1}J_R$.

We will use the following version of the Jacobi criterion.

Theorem 3.1 (Jacobi Criterion)**.** *Let $(A, \mathfrak{m}, L)$ be a local algebra essentially of finite type over a field k, with separable residue field extension $k \subset L$.*

Then A is regular if and only if $\operatorname{Fitt}_s\big(\Omega_k(A)\big) = A$ for some $s \leq \dim A + \operatorname{trdeg}_k L$. In this case, the extension $k \subset \operatorname{Quot}(A)$ is separable and $s = \dim A + \operatorname{trdeg}_k L$.

In our estimates we will also need to use partial Jacobian ideals as in Setting 2.7. The next results give Jacobi-like criteria for such ideals.

For a Noetherian ring R and $i \geq 0$ an integer, $\operatorname{Spec}(R)$ is said to be *connected in dimension i* if $i < \dim R$ and $\operatorname{Spec}(R)$ cannot be disconnected by removing a closed subset of dimension $< i$. Assume $d = \dim R > 0$, then $\operatorname{Spec}(R)$ is connected in dimension $d-1$ if R is a domain with $d < \infty$ or R is an equidimensional catenary local ring satisfying Serre's condition S_2 (for the latter case one uses Hartshorne's Connectedness Lemma [24]).

Recall that the arithmetic rank of an ideal $\mathfrak{a}$, $\operatorname{ara}(\mathfrak{a})$, is the minimal number of elements that generate $\mathfrak{a}$ up to radical.

Lemma 3.2. *Let T be a Noetherian local ring of dimension $d > 0$ and assume that T is analytically irreducible or Cohen-Macaulay or, more generally, $\operatorname{Spec}(\widehat{T})$ is connected in dimension $d-1$. Let*

$\mathfrak{a} \subset I$ be ideals and $K = \mathfrak{a} : I$. If $I_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$ for some $\mathfrak{p} \in V(I)$ and $\sqrt{I} \neq \sqrt{\mathfrak{a}}$, then

$$\operatorname{ht}(I + K) \leq \operatorname{ara}(\mathfrak{a}) + 1 \leq \mu(\mathfrak{a}) + 1.$$

Proof. We may pass to the completion of T to assume that T is a complete local ring and $\operatorname{Spec}(T)$ is connected in dimension $d-1$. Set $A = T/\mathfrak{a}$ and $s = \operatorname{ara}(\mathfrak{a})$. By Grothendieck's Connectedness Theorem [19, 3.1.7], $\operatorname{Spec}(A)$ is connected in dimension $d - 1 - \operatorname{ara}(\mathfrak{a}) = d - 1 - s$.

On the other hand, our assumptions on $\mathfrak{a}$ and I mean that $V(I) \setminus V(I+K) \neq \emptyset$ and $V(K) \setminus V(I+K) \neq \emptyset$, or equivalently, $V(IA) \setminus V(IA+KA) \neq \emptyset$ and $V(KA) \setminus V(IA+KA) \neq \emptyset$. As $\operatorname{Spec}(A) = V(IA) \cup V(KA)$, we see that $\operatorname{Spec}(A) \setminus V(IA+KA)$ is disconnected. This can only happen if $\dim T/(I+K) = \dim A/(IA+KA) \geq d-1-s$. It follows that $\operatorname{ht}(I+K) \leq s+1$. ☐

Theorem 3.3. *Let $(T, \mathfrak{n}, L)$ be a Cohen-Macaulay local ring essentially of finite type over a perfect field k. Let I be an ideal of height g and $\mathfrak{a} = (f_1, \ldots, f_g) \subset I$. Write $A = T/\mathfrak{a}$ and $R = T/I$ and assume R is equidimensional of dimension ≥ 2. Set $D = \dim R + \operatorname{trdeg}_k L$ and consider the Jacobian-like ideal $J = \operatorname{Fitt}_D(R \otimes_A \Omega_k(A)) \subset R$.*

(a) $\operatorname{ht} J \geq 1$ if and only if $I_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$ for every minimal prime $\mathfrak{p}$ of I and R satisfies Serre's condition R_0.
(b) If $\operatorname{ht} J \geq 2$ then $I = \mathfrak{a}$ is a complete intersection.
(c) $\operatorname{ht} J \geq i$ for some $i \geq 2$ if and only if $I = \mathfrak{a}$ is a complete intersection and R satisfies Serre's condition R_{i-1}.

Proof. We first prove that if $\mathfrak{p}$ is a prime ideal in $V(I)$ with residue field κ, then

$$\dim R_\mathfrak{p} + \operatorname{trdeg}_k \kappa = D\,. \tag{4}$$

Let $\mathfrak{m}$ denote the maximal ideal of R. Since R is the localization of a finitely generated k-algebra, we can write $R = R'_{\mathfrak{m}'}$, where R' is a finitely generated k-subalgebra of R and $\mathfrak{m}' = \mathfrak{m} \cap R'$. Notice that R' is equidimensional and set $\mathfrak{p}' = \mathfrak{p} \cap R'$. Now

$$\dim R_\mathfrak{p} + \operatorname{trdeg}_k \kappa = \operatorname{ht} \mathfrak{p}'R' + \dim R'/\mathfrak{p}'R' = \dim R' = \operatorname{ht} \mathfrak{m}'R' + \dim R'/\mathfrak{m}'R' = \dim R + \operatorname{trdeg}_k L\,,$$

as claimed.

It remains to prove that if $\operatorname{ht} J \geq 1$, then $I_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$ for every minimal prime $\mathfrak{p}$ of I and if $\operatorname{ht} J \geq 2$, then $I = \mathfrak{a}$. The rest follows from Theorem 3.1 and (4).

We first show that if $\mathfrak{p} \in V(I)$ and $J_\mathfrak{p} = R_\mathfrak{p}$, then $I_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$. We wish to apply Theorem 3.1 to the ring $A_\mathfrak{p}$ with $s := D$. Notice that

$$s = D = \dim R_\mathfrak{p} + \operatorname{trdeg}_k \kappa \leq \dim A_\mathfrak{p} + \operatorname{trdeg}_k \kappa$$

according to (4) and that $\operatorname{Fitt}_s(\Omega(A_\mathfrak{p})) = A_\mathfrak{p}$ because $I \subset \mathfrak{p}$. Now Theorem 3.1 implies that $A_\mathfrak{p}$ is regular and $\dim A_\mathfrak{p} = \dim R_\mathfrak{p}$. As $A_\mathfrak{p}$ is a domain mapping onto $R_\mathfrak{p}$, we conclude that $\mathfrak{a}_\mathfrak{p} = I_\mathfrak{p}$ as asserted.

Thus we have proven that if $\operatorname{ht} J \geq 1$, then $\mathfrak{a}_\mathfrak{p} = I_\mathfrak{p}$ for every minimal prime $\mathfrak{p}$ of I. On the other hand, if $\operatorname{ht} J \geq 2$ we conclude that $\mathfrak{a}_\mathfrak{p} = I_\mathfrak{p}$ for every $\mathfrak{p} \in V(I)$ with $\dim T_\mathfrak{p} \leq g+1$. So for $K := \mathfrak{a} : I$, we have $\operatorname{ht}(I+K) \geq g+2 > \mu(\mathfrak{a})+1$. Also $\mathfrak{a}_\mathfrak{p} = I_\mathfrak{p}$ for some $\mathfrak{p} \in V(I)$, in fact for every minimal prime $\mathfrak{p}$ of I. Therefore Lemma 3.2 shows that $\sqrt{\mathfrak{a}} = \sqrt{I}$. In particular, $\mathfrak{a}$ is a complete intersection. Thus every associated primes $\mathfrak{p}$ of $\mathfrak{a}$ is a minimal prime of $\mathfrak{a}$, hence a minimal prime of I because $\sqrt{\mathfrak{a}} = \sqrt{I}$. Therefore, $\mathfrak{a}_\mathfrak{p} = I_\mathfrak{p}$. Since this holds for every associated prime of $\mathfrak{a}$, we obtain $\mathfrak{a} = I$. □

Tjurina number. The estimates for plane curves in [12, 15] use the sum of the Tjurina numbers at the singular points. To allow for curves in projective spaces of arbitrary dimension, we replace the sum of the Tjurina numbers by the degree of the singular locus endowed with the scheme structure given by the Jacobian ideal, which is the multiplicity of the homogenous coordinate ring of the curve modulo its Jacobian ideal. If the curve is locally a complete intersection, as is the case for any plane curve, then the sum of the Tjurina numbers and the degree of the singular locus coincide, see Corollary 3.6.

Let A be a local ring essentially of finite type over a field k. By $\mathrm{T}^1(A/k)$ we denote the first cotangent cohomology of the k-algebra A. If k is perfect and A is reduced, then $\mathrm{T}^1(A/k) \cong \operatorname{Ext}^1_A(\Omega_k(A), A)$. The module $\mathrm{T}^1(A/k)$ has finite length whenever k is perfect and A has an isolated singularity; this length is called the *Tjurina number* of A and denoted by $\tau(A)$. If the residue field extension is trivial, then $\tau(A)$ is the embedding dimension of the formal moduli space of A, the parameter space of the versal deformation of the k-algebra $\widehat{A}$. The *total Tjurina number* of a reduced curve $\mathcal{C} \subset \mathbb{P}^{n-1}_k$ over a perfect field is defined as $\tau(\mathcal{C}) = \sum_{p \in \operatorname{Sing}(\mathcal{C})} \tau(\mathcal{O}_{\mathcal{C},p})$.

Lemma 3.4. *Let k be a perfect field, $X \subset \mathbb{P}^{n-1}_k$ be a reduced and equidimensional subscheme, and $Y \subset X$ be a subvariety. Let R be the homogenous coordinate ring of X, let $\mathfrak{p} \subset R$ be the prime ideal defining Y, and write $T = R_\mathfrak{p}$ and $A = \mathcal{O}_{X,Y}$.*

(*a*) *$T \cong A(x)$ for every $x \in R_1 \setminus \mathfrak{p}$; any such x is transcendental over A and $A(x)$ denotes the localization of the polynomial ring $A[x]$ at the extension of the maximal ideal of A.*
(*b*) *$\Omega_k(T) \cong (\Omega_k(A) \otimes_A T) \oplus T dx$, where $T\,dx \cong T$.*
(*c*) *$J_{T/k} = J_{A/k}T$ and $\mathrm{T}^1(T/k) \cong \mathrm{T}^1(A/k) \otimes_A T$.*

Proof. Part (a) is well known, part (b) is an immediate consequence of (a), and part (c) follows from (b). □

Proposition 3.5. *Let k be a perfect field and A be a local k-algebra essentially of finite type with algebraic residue field extension. If A is a reduced complete intersection of dimension one, then*

$$\tau(A) = \lambda(A/J_A).$$

Proof. Notice that $\operatorname{projdim}_A \Omega_k(A) \leq 1$ and $\operatorname{rank}_A \Omega_k(A) = \dim A = 1$. Thus [43, Satz] implies $\lambda(A/J_A) = \lambda(\operatorname{tor}(\Omega_k(A)))$. Since A is Gorenstein, local duality gives $\lambda(\operatorname{tor}(\Omega_k(A))) = \lambda(\operatorname{Ext}^1_A(\Omega_k(A), A))$. As $\operatorname{Ext}^1_A(\Omega_k(A), A) \cong \mathrm{T}^1(A/k)$, the assertion now follows. □

The next corollary expresses the total Tjurina number of a local complete intersection curve, which is defined in terms of local invariants of the singular points, as a global invariant, the degree of the singular scheme of the curve, which can be computed without knowing the singularities. It is this global invariant that replaces the global Tjurina number in our estimates when the curves need not be a local complete intersection.

Corollary 3.6. *Let k be a perfect field and $\mathcal{C} \subset \mathbb{P}^{n-1}_k$ be a singular reduced curve that is locally a complete intersection. Write R for the homogeneous coordinate ring of $\mathcal{C}$. Then*

$$\tau(\mathcal{C}) = e(R/J_R)\,.$$

Proof. Write $\mathfrak{m}$ for the homogenous maximal ideal of R. In this case we have

$$\begin{aligned}
\tau(\mathcal{C}) &= \sum_{p\in \operatorname{Sing}(\mathcal{C})} \lambda_{\mathcal{O}_{\mathcal{C},p}}(\mathcal{O}_{\mathcal{C},p}/J_{\mathcal{O}_{\mathcal{C},p}}) && \text{by Proposition 3.5}\\
&= \sum_{\mathfrak{p}\in V(J_R)\setminus\{\mathfrak{m}\}} \lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/J_{R_\mathfrak{p}}) && \text{by Lemma 3.4(c)}\\
&= e(R/J_R) && \text{by the associativity formula for multiplicity.}
\end{aligned}$$

□

Loewy multiplicity. Besides the multiplicity of the homogeneous coordinate ring of a curve modulo its Jacobian ideal, we will also consider what we call the Loewy multiplicity, which is defined by replacing length by Loewy length in the associativity formula for multiplicity.

The *Loewy length* of a module M of finite length over a local ring $(A, \mathfrak{m})$ is the smallest integer $s \geq 0$ so that $\mathfrak{m}^s M = 0$. The Loewy length satisfies the inequality $\ell\ell(M) \leq \lambda(M)$, which is an equality if and only if M and $\mathfrak{m}M$ are cyclic if and only if every submodule of M is cyclic. We will use a strengthening of this inequality:

Proposition 3.7. *Let A be the local ring of a point on a reduced plane curve over a perfect field and write $e = e(A)$. Then*

$$\ell\ell(A/J_A) \leq \lambda(A/J_A) - \binom{e-1}{2}.$$

Proof. We may assume that $A = S/(f)$, where $S = k[x,y]_{(x,y)}$. Let $\mathfrak{n}$ be the maximal ideal of S and $\mathfrak{m}$ be the maximal ideal of $B := A/J_A$. We have $f \in \mathfrak{n}^e$ and so $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \subset \mathfrak{n}^{e-1}$. It follows that $\mathfrak{m}^i/\mathfrak{m}^{i+1} \cong \mathfrak{n}^i/\mathfrak{n}^{i+1} \cong k^{i+1}$ for $i \leq e-2$. Therefore

$$\lambda(A/J_A) - \ell\ell(A/J_A) \geq \sum_{i=0}^{e-2}(\lambda(\mathfrak{m}^i/\mathfrak{m}^{i+1}) - 1) \geq \binom{e-1}{2}.$$

□

Let M be a finitely generated module over a Noetherian ring R, where either R is local or else M is graded and R is positively graded over an Artinian local ring. Recall that

$$e(M) = \sum \lambda(M_{\mathfrak{p}}) \cdot e(R/\mathfrak{p})\,,$$

where $\mathfrak{p}$ ranges over all prime ideals of maximal dimension in $\mathrm{Supp}(M)$. Analogously, we define the *Loewy multiplicity* of M as

$$\mathrm{Lmult}(M) := \sum \ell\ell(M_{\mathfrak{p}}) \cdot e(R/\mathfrak{p})\,.$$

Clearly, $\mathrm{Lmult}(M) \leq e(M)$ and equality holds if and only if for every $\mathfrak{p}$ as above, every R_p-submodule of $M_{\mathfrak{p}}$ is cyclic.

Singularity degree and genus. If A is an analytically unramified Noetherian local ring, with integral closure $\overline{A}$, then the A-module $\overline{A}/A$ has finite length if and only if A is normal locally on the punctured spectrum. In this case, $\sigma(A) := \lambda(\overline{A}/A)$ is called the *singularity degree* of A. The singularity degree of a reduced curve $\mathcal{C} \subset \mathbb{P}^{n-1}_k$ is defined as

$$\sigma(\mathcal{C}) = \sum_{p \in \mathrm{Sing}(\mathcal{C})} \sigma(\mathcal{O}_{\mathcal{C},p})\,.$$

An argument as in the proof of Corollary 3.6 shows that $\sigma(\mathcal{C}) = e(\overline{R}/R)$ if $\mathcal{C}$ is singular, where R is the homogeneous coordinate ring of $\mathcal{C}$.

The singularity degree of a curve is closely related to its arithmetic and geometric genus. Let $X \subset \mathbb{P}^{n-1}_k$ be a reduced subscheme over a field k, with homogeneous coordinate ring R. Let $\mathfrak{m}$ be the homogeneous maximal ideal and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the minimal primes of R, and write $R_i = R/\mathfrak{p}_i$. Each $\overline{R_i}$ are positively graded algebra over a finite field extension k_i of k, and $\overline{R}$ is a positively graded algebra over the Artinian ring $K := \times k_i$. A suitable Veronese subring of $\overline{R}$ is a standard graded algebra over K and is the homogeneous coordinate ring of the normalization of X embedded into a projective space over K.

We describe, in passing, an embedding of the normalization into a projective space over the field k, when k is algebraically closed. In this case, $k_i = k$ and we may define the natural projections $\pi_i : \overline{R_i} \twoheadrightarrow k$. We consider the fiber product $\widetilde{R} := \{(a_i) \in \times \overline{R_i} \mid \pi_i(a_i) = \pi_j(a_j)\ \forall\ i \neq j\}$. One has $R \subset \widetilde{R} \subset \overline{R}$ and $\mathfrak{m}\overline{R} \subset \widetilde{R}$. The ring $\widetilde{R}$ is a positively graded algebra over the field k, and one sees that a Veronese subring of $\widetilde{R}$ is the homogeneous coordinate ring of the normalization of X embedded into a projective space over k. If X is equidimensional, then $\omega_{\widetilde{R}} \cong \omega_{\overline{R}} := \times \omega_{\overline{R_i}}$. Also notice that the degree zero component of the canonical module is unchanged by passing to a Veronese subring.

Now let $\mathcal{C} \subset \mathbb{P}^{n-1}_k$ be a reduced curve over a field, with homogeneous coordinate ring R. The *arithmetic genus* p_a of $\mathcal{C}$ is 1 minus the constant term of the Hilbert polynomial of R. If k is algebraically closed, the *geometric genus* p_g of $\mathcal{C}$ can be defined as $\dim_k[\omega_{\overline{R}}]_0 = \dim_k[\omega_{\widetilde{R}}]_0$.

The following lemma and proposition are well known, we give a proof for the convenience of the reader.

Lemma 3.8. *Let $\mathcal{C} \subset \mathbb{P}^n_k$ be a reduced arithmetically Cohen-Macaulay curve over a field k, with arithmetic genus p_a and homogeneous coordinate ring R. Then*

$$p_a = \dim_k [\omega_R]_0 \, .$$

Proof. Let $\mathfrak{m}$ be the homogeneous maximal ideal of R, and let h and p denote the Hilbert function and the Hilbert polynomial of R, respectively. One has

$$p_a = h(0) - p(0) = \dim_k [H^2_{\mathfrak{m}}(R)]_0 = \dim_k [\omega_R]_0 \, ,$$

where the second equality follows from the Grothendieck-Serre formula and the third equality is a consequence of local duality. □

Proposition 3.9. *Let k be an algebraically closed field and $\mathcal{C} \subset \mathbb{P}^{n-1}_k$ be a reduced curve with s irreducible components, arithmetic genus p_a, geometric genus p_g, and singularity degree σ. One has*

$$p_a - p_g = \sigma - s + 1 \, .$$

Proof. Let R be the homogeneous coordinate ring of $\mathcal{C}$ and $\overline{R}$ its integral closure. We may assume that R and $\overline{R}$ are standard graded after passing to Veronese subrings; this does not change the local rings of $\mathcal{C}$, the constant term of the Hilbert polynomial of R, which is $1 - p_a$, and the degree zero component of $\omega_{\overline{R}}$. So $p_g = \dim_k [\omega_{\overline{R}}]_0$.

We compare the constant terms of the Hilbert polynomials of the graded R-modules in the exact sequence

$$0 \longrightarrow R \longrightarrow \overline{R} \longrightarrow \overline{R}/R \longrightarrow 0 \, .$$

The constant term of the Hilbert polynomial of R is $1 - p_a$. One has $\overline{R} = \times_{i=1}^{s} \overline{R_i}$, where $\overline{R_i}$ are standard graded Cohen-Macaulay algebras over k. Applying Lemma 3.8 we see that the constant term of the Hilbert polynomial of $\overline{R}$ is

$$\sum_{i=1}^{s} (1 - \dim_k [\omega_{\overline{R_i}}]_0) = s - \dim_k [\omega_{\overline{R}}]_0 = s - p_g \, .$$

Finally, the constant term of the Hilbert polynomial of $\overline{R}/R$ is σ. Now the additivity of the Hilbert polynomial in short exact sequences implies that

$$s - p_g = 1 - p_a + \sigma,$$

as claimed. □

4. Lower bounds for hypersurfaces and curves with at most planar singularities

One of the main results of this section is an estimate for the degree of a vector field in terms of the a-invariant of R, the number of singular points, and the Loewy multiplicity modulo the Jacobian ideal, see Theorem 4.10. It says that if $\mathcal{C}$ has only plane singularities, then

$$\operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \geq a(R) + 1 + |\operatorname{Sing}(\mathcal{C})| - \operatorname{Lmult}(R/\operatorname{Jac}(R))\,.$$

If $\mathcal{C}$ has only ordinary nodes as singularities, then $|\operatorname{Sing}(\mathcal{C})| - \operatorname{Lmult}(R/\operatorname{Jac}(R)) = 0$ and we obtain the inequality $\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \geq a(R) + 1$, which we prove to be an equality when $\mathcal{C}$ is arithmetically Gorenstein, see Corollary 6.2. The case of ordinary nodes had been treated before with the additional assumption that, first, $\mathcal{C}$ is a plane curve [7], then, $\mathcal{C}$ is a complete intersection [4], and, finally, $\mathcal{C}$ is arithmetically Cohen-Macaulay [16, 17].

Proposition 4.1. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field and let $\mathfrak{m}_S$ denote its maximal homogeneous ideal. Let f be a homogeneous polynomial of degree d and assume that d is not a multiple of the characteristic. Denote the partial derivatives of f by $f_1, \ldots, f_n$ and let $\mathcal{B}$, $\mathcal{Z}$, and $\mathcal{H}$ be the modules of first boundaries, cycles, and homology of the Koszul complex of $f_1, \ldots, f_n$. Write $R = S/(f)$.*

(*a*) *There are natural epimorphisms of homogeneous S-modules*

$$\mathcal{Z}(d) \twoheadrightarrow \operatorname{Der}_k(R)/R\varepsilon \twoheadrightarrow (\mathcal{Z}/\mathfrak{m}_S\mathcal{B})(d)\ ;$$

in particular,

$$k \otimes_S \mathcal{Z}(d) \cong k \otimes_R \operatorname{Der}_k(R)/R\varepsilon$$

(*b*)
$$\mu(\operatorname{Der}_k(R)/R\varepsilon) = \beta_S^2\left(\frac{S}{(f_1, \ldots, f_n)}\right) \leq (d-1)^n$$

(*c*)
$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \operatorname{indeg}(\mathcal{Z}) - d = \min\{d-2, \operatorname{indeg}(\mathcal{H}) - d\}.$$

Proof. Dualizing the exact sequence

$$R(-d) \xrightarrow{[f_1,\ldots,f_n]^t} R^n(-1) \longrightarrow \Omega_k(R) \longrightarrow 0$$

into R, we obtain

$$0 \longrightarrow \operatorname{Der}_k(R) \longrightarrow \oplus_{i=1}^n R\,\partial/\partial x_i \xrightarrow{[f_1,\ldots,f_n]} R(d)\ . \tag{5}$$

There is a homogeneous map

$$\varphi : \mathcal{Z}(d) \longrightarrow \operatorname{Der}_k(R)$$

induced by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Der}_k(R) & \longrightarrow & \oplus_{i=1}^n R\,\partial/\partial x_i & \xrightarrow{[f_1,\ldots,f_n]} & R(d) \\ & & \varphi\big\uparrow & & \big\uparrow\!\!\uparrow & & \big\uparrow\!\!\uparrow \\ 0 & \longrightarrow & \mathcal{Z}(d) & \longrightarrow & \oplus_{i=1}^n S e_i & \xrightarrow{[f_1,\ldots,f_n]} & S(d) \end{array} \tag{6}$$

where the homogeneous basis elements e_i have degree -1.

We claim that the composition ψ

$$\mathcal{Z}(d) \longrightarrow \mathrm{Der}_k(R) \longrightarrow\!\!\!\!\rightarrow \mathrm{Der}_k(R)/R\varepsilon$$

is surjective. Let $g = \sum b_i\,\partial/\partial x_i \in \mathrm{Der}_k(R)$. According to (5), $\sum b_i f_i = cf$ for some $c \in S$. Using the Euler relation $f = \frac{1}{d}(x_1 f_1 + \ldots + x_n f_n)$, we obtain $\sum (b_i - \frac{c}{d}x_i) f_i = 0$. Therefore $\sum (b_i - \frac{c}{d}x_i)e_i \in \mathcal{Z}(d)$ and $\varphi(\sum (b_i - \frac{c}{d}x_i)e_i) = g - \frac{c}{d}\varepsilon$.

To show part (a) it suffices to show that $\ker\psi \subset \mathfrak{m}_S\mathcal{B}$. The diagram (6) shows that

$$\ker\psi = (fS^n + S\sum_{i=1}^n x_i e_i) \cap \mathcal{Z}(d)\,. \tag{7}$$

Let $z = \sum_{i=1}^n \alpha_i e_i \in \ker\psi$. According to (7) there exists $a_1, \ldots, a_n, b$ in S so that

$$z = \sum_{i=1}^n a_i f e_i + b\sum_{i=1}^n x_i e_i\,. \tag{8}$$

Since z is a syzygy of $f_1, \ldots, f_n$ we obtain

$$0 = \sum_{i=1}^n \alpha_i f_i = \sum_{i=1}^n a_i f f_i + b\sum_{i=1}^n x_i f_i = \sum_{i=1}^n a_i f f_i + dbf\,.$$

Dividing by f it follows that

$$b = -\frac{1}{d}\sum_{i=1}^n a_i f_i\,.$$

Substituting the above expression of b into (8) and using the Euler relation we see that

$$\alpha_i = \sum_{j=1}^n \frac{1}{d}(a_i x_j - a_j x_i) f_j\,.$$

Since the $n \times n$ matrix $(a_i x_j - a_j x_i)$ is alternating and has entries in $\mathfrak{m}_S$ it follows that $z \in \mathfrak{m}_S\mathcal{B}(d)$.

The equality in (b) and the first equality in (c) are direct consequences of the isomorphism in (a). The inequality in (b) follows from [11, 3.10]. For the second equality in (c), notice that $\mathrm{indeg}(\mathcal{Z}) - d \le \mathrm{indeg}(\mathcal{B}) - d = d - 2$ and $\mathrm{indeg}(\mathcal{Z}) \le \mathrm{indeg}(\mathcal{H})$ and that either $\mathrm{indeg}(\mathcal{Z}) = \mathrm{indeg}(\mathcal{B})$ or else $\mathrm{indeg}(\mathcal{Z}) = \mathrm{indeg}(\mathcal{H})$. □

The next theorem is a generalization of [18, Theorem 2.5] from plane curves to hypersurfaces.

Theorem 4.2 (The hypersurface case)**.** *Let k be a perfect field and $X \subset \mathbb{P}_k^{n-1}$ be a reduced hypersurface of degree d. Let R be the homogenous coordinate ring of X and J_R the Jacobian ideal. Assume $n \geq 3$ and that d is not a multiple of the characteristic. If X has at most isolated singularities, then*

$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \begin{cases} \min\{d-2, (n-1)(d-2) - a(R/J_R) - 2\} & \\ d-2 & \textit{if } X \textit{ is smooth} \end{cases}.$$

Proof. We may assume that k is infinite. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables, and write $R = S/(f)$ with f a form of degree d. Denote the partial derivatives of f by $f_1, \ldots, f_n$ and set $\mathcal{J} = (f_1, \ldots, f_n)$. Notice that J_R is the image of $\mathcal{J}$ in R. Since d is a unit in k, the Euler relation shows that $f \in \mathcal{J}$. Therefore $\operatorname{ht} \mathcal{J} \geq n-1$, and after a linear change of variables we can assume that $f_1, \ldots, f_{n-1}$ form a regular sequence.

Let $\mathcal{H}$ be the first homology module of the Koszul complex of $f_1, \ldots, f_n$. If X is smooth, then $\mathcal{H} = 0$ and the claim follows from Proposition 4.1(c). Otherwise $\operatorname{ht} \mathcal{J} = n-1$ and

$$\mathcal{H}(d-1) = \frac{(f_1, \ldots, f_{n-1}) :_S \mathcal{J}}{(f_1, \ldots, f_{n-1})},$$

as $f_1, \ldots, f_{n-1}$ form a regular sequence. Now

$$\begin{aligned} \frac{(f_1, \ldots, f_{n-1}) :_S \mathcal{J}}{(f_1, \ldots, f_{n-1})} &\cong \omega_{S/\mathcal{J}}(n - (n-1)(d-1)) \\ &= \omega_{R/J_R}(1 - (n-1)(d-2)) \end{aligned}$$

and the theorem follows again by Proposition 4.1(c) because $\operatorname{indeg} \omega_{R/J_R} = -a(R/J_R)$. □

The connection between the degree of vector fields and the degree of syzygies of Jacobian ideals, which is used in the previous proof, was already observed in [16, Remark 9]. The use of the conductor and the integral closure in the proof of Proposition 4.4 below was inspired by [13, proof of Corollary 5.1].

Lemma 4.3. *Let R be a standard graded Noetherian algebra over a field and $\mathfrak{b}_1, \ldots, \mathfrak{b}_t$ homogeneous ideals of R so that the rings $R/\mathfrak{b}_i$ have dimension 1. Then*

$$a(R/\mathfrak{b}_1 \cdot \ldots \cdot \mathfrak{b}_t) \leq \sum_{i=1}^{t} a(R/\mathfrak{b}_i) + 2t - 2.$$

Proof. Let S be a standard graded polynomial ring over the ground field of R, mapping homogeneously onto R, and let $\tilde{\mathfrak{b}}_i$ denote the preimage of $\mathfrak{b}_i$ in S. Since $S/\tilde{\mathfrak{b}}_1 \cdot \ldots \cdot \tilde{\mathfrak{b}}_t \longrightarrow R/\mathfrak{b}_1 \cdot \ldots \cdot \mathfrak{b}_t$ is a surjection of rings having the same dimension, $a(R/\mathfrak{b}_1 \cdot \ldots \cdot \mathfrak{b}_t) \leq a(S/\tilde{\mathfrak{b}}_1 \cdot \ldots \cdot \tilde{\mathfrak{b}}_t)$. Hence it suffices to prove our assertion for the case that R is a polynomial ring, which we now assume.

If $\mathfrak{b}^{\mathrm{unm}}$ denotes the unmixed part of a homogeneous ideal $\mathfrak{b}$ of the polynomial ring R such that $R/\mathfrak{b}$ has dimension 1, then $R/\mathfrak{b}^{\mathrm{unm}}$ is Cohen-Macaulay and

$$a(R/\mathfrak{b}) = a(R/\mathfrak{b}^{\mathrm{unm}}) = \operatorname{reg}(R/\mathfrak{b}^{\mathrm{unm}}) - 1 = \operatorname{reg}(\mathfrak{b}^{\mathrm{unm}}) - 2 \leq \operatorname{reg}(\mathfrak{b}) - 2.$$

Thus

$$a(R/\mathfrak{b}_1 \cdot \ldots \cdot \mathfrak{b}_t) \le \operatorname{reg}(\mathfrak{b}_1 \cdot \ldots \cdot \mathfrak{b}_t) - 2.$$

On the other hand according to [40, Theorem 1.8],

$$\operatorname{reg}(\mathfrak{b}_1 \cdot \ldots \cdot \mathfrak{b}_t) \le \sum_{i=1}^{t} \operatorname{reg}(\mathfrak{b}_i) = 2t + \sum_{i=1}^{t} a(R/\mathfrak{b}_i),$$

which completes the proof. □

Proposition 4.4 (The plane curve case)**.** *Let k be an algebraically closed field and $\mathcal{C} \subset \mathbb{P}^2_k$ be a reduced curve of degree d. Let R be the homogenous coordinate ring of $\mathcal{C}$ and J_R be the Jacobian ideal. Assume that d is not a multiple of the characteristic. One has*

$$\begin{aligned}
\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) &= \begin{cases} \min\{d-2, 2d-6-a(R/J_R)\} & \\ d-2 & \text{if } \mathcal{C} \text{ is smooth} \end{cases} \\
&\ge \begin{cases} d-3+|\operatorname{Sing}(\mathcal{C})| - \operatorname{Lmult}(R/J_R) & \\ d-2+|\operatorname{Sing}(\mathcal{C})| - \operatorname{Lmult}(R/J_R) & \text{if } \mathcal{C} \text{ is irreducible} \\ d-2 & \text{if } \mathcal{C} \text{ is smooth} \end{cases} \\
&\ge \begin{cases} d-3+|\operatorname{Sing}(\mathcal{C})| + \sum_{p\in \operatorname{Sing}(\mathcal{C})} \binom{e(\mathcal{O}_{\mathcal{C},p})-1}{2} - \tau(\mathcal{C}) & \\ d-2+|\operatorname{Sing}(\mathcal{C})| + \sum_{p\in \operatorname{Sing}(\mathcal{C})} \binom{e(\mathcal{O}_{\mathcal{C},p})-1}{2} - \tau(\mathcal{C}) & \text{if } \mathcal{C} \text{ is irreducible} \\ d-2 & \text{if } \mathcal{C} \text{ is smooth} \end{cases}.
\end{aligned}$$

Proof. The equality is a special case of Theorem 4.2, and the second inequality is a consequence of the bound $\operatorname{Lmult}(R/J_R) \le \tau(\mathcal{C}) - \sum_{p\in \operatorname{Sing}(\mathcal{C})} \binom{e(\mathcal{O}_{\mathcal{C},p})-1}{2}$, which follows from Corollary 3.6 and Proposition 3.7.

To prove the first inequality, we estimate $a(R/J_R)$ when $\mathcal{C}$ is singular. Set $J = J_R$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes of J, which are necessarily of height one; they correspond to the singular points of $\mathcal{C}$ and therefore $t = |\operatorname{Sing}(\mathcal{C})|$. Write $s_i = \ell\ell(R_{\mathfrak{p}_i}/J_{\mathfrak{p}_i})$ for the Loewy length of $R_{\mathfrak{p}_i}/J_{\mathfrak{p}_i}$. Let $\mathfrak{f} = R :_R \overline{R}$ denote the conductor of R and notice that $\mathfrak{f} \subset \sqrt{J}$. One has

$$J^{\text{unm}} \supset \cap\, \mathfrak{p}_i^{s_i} \supset \sqrt{J} \cdot \mathfrak{p}_1^{s_1-1} \cdots\ \mathfrak{p}_t^{s_t-1} \supset \mathfrak{f} \cdot \mathfrak{p}_1^{s_1-1} \cdots\ \mathfrak{p}_t^{s_t-1}.$$

It follows that

$$\begin{aligned}
a(R/J) = a(R/J^{\text{unm}}) &\le a(R/\mathfrak{f} \cdot \mathfrak{p}_1^{s_1-1} \cdots\ \mathfrak{p}_t^{s_t-1}) \\
&\le a(R/\mathfrak{f}) + \sum_{i=1}^{t} (s_i - 1) \quad \text{by Lemma 4.3} \\
&= a(R/\mathfrak{f}) + \operatorname{Lmult}(R/J) - |\operatorname{Sing}(\mathcal{C})|\,.
\end{aligned}$$

To estimate $a(R/\mathfrak{f})$ we dualize the short exact sequence

$$0 \to \mathfrak{f} \to R \to R/\mathfrak{f} \to 0$$

into $\omega_R \cong R(d-3)$. As $\operatorname{Hom}_R(\mathfrak{f}, R) = \overline{R}$, we obtain $\omega_{R/\mathfrak{f}} = (\overline{R}/R)(d-3)$. It follows that $a(R/\mathfrak{f}) = -\operatorname{indeg}(\overline{R}/R) + d - 3$. Therefore

$$a(R/J) \leq -\operatorname{indeg}(\overline{R}/R) + d - 3 + \operatorname{Lmult}(R/J) - |\operatorname{Sing}(\mathcal{C})|.$$

Finally, we have $\operatorname{indeg}(\overline{R}/R) \geq 0$ because R is reduced, and $\operatorname{indeg}(\overline{R}/R) \geq 1$ if R is a domain since k is algebraically closed. □

The bounds of Proposition 4.4 can be generalized to curves which are not necessarily planar, see Theorem 4.10. For plane curves on the other hand, they can be improved as follows.

Remark 4.5. We use the assumptions and the notation of Proposition 4.4 and its proof. We write $s = \max\{s_i\}$ and consider the sets of reduced points $U_j = \bigcup_{s_i \geq j} V(\mathfrak{p}_i) \subset \mathbb{P}^2_k$. One has

$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \quad \geq \quad \begin{cases} \min\{d-2, 2d-4-\sum_{j=1}^{s} \operatorname{reg}(U_j)\} & \\ d-2 & \text{if } \mathcal{C} \text{ is smooth.} \end{cases}$$

Proof. Write $I_j = I(U_j)$ for the reduced defining ideal of U_j and notice that $J^{unm} \supset \prod_{j=1}^{s} I_j$. Now apply Lemma 4.3 as in the previous proof to see that

$$a(R/J) \leq \sum_{j=1}^{s} a(R/I_j) + 2s - 2 = \sum_{j=1}^{s} \operatorname{reg}(U_j) - 2.$$

□

In order to deal with subschemes that are not necessarily arithmetically Gorenstein, we introduce the following notion:

Definition 4.6. Let R be a standard graded algebra over a field. We say that R has the *generalized Cayley-Bacharach property* if $\operatorname{findeg} \omega_R = \operatorname{indeg} \omega_R$.

Examples of rings with the generalized Cayley-Bacharach property are domains, Gorenstein rings, and more generally level rings. If the ground field is algebraically closed and R is reduced and one-dimensional, then R has the generalized Cayley-Bacharach property if and only if the corresponding set of points in projective space has the Cayley-Bacharach property in the usual sense (see [23]).

The next theorem is inspired by work of Esteves [16, Theorem 17]. We do not require arithmetic Cohen-Macaulayness as in [16] and our proof is short and elementary. We will say that a ring extension $A \subset R$ is *birational* if R is a torsionfree A-module and the induced map $\operatorname{Quot}(A) \longrightarrow \operatorname{Quot}(R)$ is an isomorphism. The following fact, which is a special case of [44, Proposition 5.2], will be used frequently.

Lemma 4.7. *Let k be an infinite perfect field and let R be a reduced and equidimensional k-algebra of dimension D generated by $y_1, \dots, y_n$. If A is the k-subalgebra generated by $D+1$ general k-linear combinations of $y_1, \dots, y_n$, then $A \subset R$ is a finite and birational extension and the induced map $\mathrm{Quot}(A) \longrightarrow \mathrm{Quot}(A) \otimes_A R$ is an isomorphism.*

Theorem 4.8. *Let k be perfect field and let $A \subset R$ be a finite and birational homogeneous extension of standard graded k-algebras. Assume that R is reduced and equidimensional of dimension at least two, with maximal homogenous ideal $\mathfrak{m}$, and that A is Gorenstein.*

One has

$$\mathrm{findeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) \geq \mathrm{indeg}(\mathrm{Der}_k(A)/A\varepsilon_A) - a(A) + a(R)$$

and

$$\mathrm{findeg}(\mathrm{Der}_k(A)/A\varepsilon_A) \geq \mathrm{indeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) - a(A) + a(R)\,.$$

If in addition R has the generalized Cayley-Bacharach property, then

$$\mathrm{findeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) \geq \mathrm{findeg}(\mathrm{Der}_k(A)/A\varepsilon_A) - a(A) + a(R)$$

$$\mathrm{indeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) \geq \mathrm{indeg}(\mathrm{Der}_k(A)/A\varepsilon_A) - a(A) + a(R)$$

and

$$\mathrm{findeg}(\mathrm{Der}_k(A)/A\varepsilon_A) \geq \mathrm{findeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) - a(A) + a(R)$$
$$\mathrm{indeg}(\mathrm{Der}_k(A)/A\varepsilon_A) \geq \mathrm{indeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) - a(A) + a(R)\,.$$

Notice that the inverse of the maximal homogeneous ideal of A is equal to A since $\mathrm{depth}\, A \geq 2$.

Proof. We consider the conductor

$$\mathfrak{f} = A :_A R \cong \mathrm{Hom}_A(R, A) \cong \mathrm{Hom}_A(R, \omega_A)(-a(A)) \cong \omega_R(-a(A))\,.$$

Notice that $\mathrm{indeg}\, \mathfrak{f} = a(A) - a(R)$. In addition, $\mathfrak{f}$ annihilates the R-module $\Omega_A(R)$; indeed, if $d : R \longrightarrow \Omega_A(R)$ denotes the universal A-derivation of R, then for any $a \in \mathfrak{f}$ and $r \in R$,

$$a\, d(r) = d(ar) - r\, d(a) = 0$$

since both ar and a are in A.

In the exact sequence

$$R \otimes_A \Omega_k(A) \xrightarrow{\alpha} \Omega_k(R) \longrightarrow \Omega_A(R) \longrightarrow 0\,,$$

$\ker \alpha$ and $\Omega_A(R)$ are R-torsion modules because the extension $A \subset R$ is birational. Thus this sequence induces an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{Hom}_R(\Omega_A(R), R) & \longrightarrow & \mathrm{Hom}_R(\Omega_k(R), R) & \longrightarrow & \mathrm{Hom}_A(\Omega_k(A), R) & \longrightarrow & \mathrm{Ext}^1_R(\Omega_A(R), R) \\ \| & & \wr\| & & \wr\| & & \\ 0 & & \mathrm{Der}_k(R) & \xrightarrow{\quad\beta\quad} & \mathrm{Der}_k(A, R) & & \end{array}$$

Observe that $\beta(\varepsilon_R) = \varepsilon_A$ and that $\operatorname{Ext}^1_R(\Omega_A(R), R)$ is annihilated by $\mathfrak{f}$. Thus we obtain an embedding

$$\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R \overset{\varphi}{\hookrightarrow} \operatorname{Der}_k(A, R)/\mathfrak{m}^{-1}\varepsilon_A$$

whose cokernel is annihilated by $\mathfrak{f}$.

On the other hand, the obvious inclusion of $\operatorname{Der}_k(A) \subset \operatorname{Der}_k(A, R)$ induces an A-linear map

$$\psi : \operatorname{Der}_k(A)/A\varepsilon_A \longrightarrow \operatorname{Der}_k(A, R)/\mathfrak{m}^{-1}\varepsilon_A\,.$$

Since the extension $A \subset R$ is birational, this map is generically injective and hence it is injective because $\operatorname{Der}_k(A)/A\varepsilon_A$ is torsionfree as an A-module according to Proposition 2.4.

Now we have embeddings

$$\begin{array}{ccc} \operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R & \overset{\varphi}{\hookrightarrow} & \operatorname{Der}_k(A, R)/\mathfrak{m}^{-1}\varepsilon_A \\ & & \psi \big\uparrow \\ & & \operatorname{Der}_k(A)/A\varepsilon_A\,, \end{array} \tag{9}$$

where the cokernels of both φ and ψ are annihilated by $\mathfrak{f}$. Thus we obtain containments

$$\mathfrak{f} \cdot \operatorname{im}\varphi \subset \operatorname{im}\psi \tag{10}$$

$$\mathfrak{f} \cdot \operatorname{im}\psi \subset \operatorname{im}\varphi\,. \tag{11}$$

The inclusion (10) shows that

$$\begin{aligned} \operatorname{indeg}\mathfrak{f} + \operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) &\geq \operatorname{indeg}(\operatorname{Der}_k(A)/A\varepsilon_A) \\ \operatorname{findeg}\mathfrak{f} + \operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) &\geq \operatorname{indeg}(\operatorname{Der}_k(A)/A\varepsilon_A) \\ \operatorname{findeg}\mathfrak{f} + \operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) &\geq \operatorname{findeg}(\operatorname{Der}_k(A)/A\varepsilon_A)\ ; \end{aligned}$$

in the second inequality we use the fact that $\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R$ is torsionfree as an R-module (see Proposition 2.4). The inclusion (11) implies the same inequalities with the roles of $\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R$ and $\operatorname{Der}_k(A)/A\varepsilon_A$ reversed.

Finally recall that $\operatorname{indeg}\mathfrak{f} = a(A) - a(R)$ and that $\operatorname{findeg}\mathfrak{f} = a(A) - a(R)$ if R has the generalized Cayley-Bacharach property. □

Proposition 4.9. *In addition to Setting 2.1 assume that R is reduced. Let $\mathfrak{p}$ be a minimal prime ideal of R so that $R_\mathfrak{p}$ is a field and write $R' = R/\mathfrak{p}$ and $\mathfrak{m}' = \mathfrak{m}/\mathfrak{p}$. Then* $\operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) \geq \operatorname{findeg}(\operatorname{Der}_k(R')/\mathfrak{m}'^{-1}\varepsilon_{R'})$.

Proof. Since R is reduced, every derivation in $\operatorname{Der}_k(R)$ induces a derivation in $\operatorname{Der}_k(R')$, see for instance [9, page 614]. This gives a natural map $\operatorname{Der}_k(R) \longrightarrow \operatorname{Der}_k(R')$. The projection $R \twoheadrightarrow R'$ induces a map $\varphi : \operatorname{Quot}(R) \longrightarrow \operatorname{Quot}(R')$, which is surjective since R is reduced. It follows that $\varphi(\mathfrak{m}^{-1}) \subset \mathfrak{m}'^{-1}$. Combining these facts we obtain a natural map

$$\psi : \operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R \longrightarrow \operatorname{Der}_k(R')/\mathfrak{m}'^{-1}\varepsilon_{R'}\,.$$

Notice that $\psi_{\mathfrak{p}}$ is an isomorphism since R is reduced. It follows that if $\nu \in \mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R$ with $\mathrm{ann}_R\,\nu = 0$, then $\mathrm{ann}'_R\,\psi(\nu) = 0$, which proves the lemma. □

In some items of the next theorem we will assume that the curve $\mathcal{C}$ is locally irreducible. By this we mean that the local ring at every point of $\mathcal{C}$ is a domain, or equivalently, that $\mathcal{C}$ is the disjoint union of its irreducible components.

Theorem 4.10 (The case of curves with plane singularities)**.** *Let k be an algebraically closed field and $\mathcal{C} \subset \mathbb{P}^{n-1}_k$ be a reduced curve of degree d. Let R be the homogeneous coordinate ring of $\mathcal{C}$, with maximal homogeneous ideal $\mathfrak{m}$, and let J_R be the Jacobian ideal. Assume d is not a multiple of the characteristic. If $\mathcal{C}$ has at most plane singularities, then*

$$\begin{aligned}\mathrm{findeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \quad &\geq \begin{cases} a(R) + |\,\mathrm{Sing}(\mathcal{C})| - \mathrm{Lmult}(R/J_R) \\ a(R) + 1 + |\,\mathrm{Sing}(\mathcal{C})| - \mathrm{Lmult}(R/J_R) & \textit{if } \mathcal{C} \textit{ is locally irreducible}\end{cases} \\ &\geq \begin{cases} a(R) + |\,\mathrm{Sing}(\mathcal{C})| + \sum\limits_{p\in \mathrm{Sing}(\mathcal{C})} \binom{e(\mathcal{O}_{\mathcal{C},p})-1}{2} - \tau(\mathcal{C}) \\ a(R) + 1 + |\,\mathrm{Sing}(\mathcal{C})| + \sum\limits_{p\in \mathrm{Sing}(\mathcal{C})} \binom{e(\mathcal{O}_{\mathcal{C},p})-1}{2} - \tau(\mathcal{C}) & \textit{if } \mathcal{C} \textit{ is locally irreducible.}\end{cases}\end{aligned}$$

Proof. It suffices to prove the first inequality. We may assume that $n \geq 3$. Let x, y, z be general linear forms in R and write $A = k[x, y, z] \subset R$. Notice $A \subset R$ is a finite and birational homogeneous extension of standard graded k-algebras by Lemma 4.7, and A is the homogeneous coordinate ring of a plane curve $\mathcal{D}$. Write J_A for the Jacobian ideal of A. By Theorem 4.8 and Proposition 4.4 one has

$$\mathrm{findeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) \geq \mathrm{indeg}(\mathrm{Der}_k(A)/A\varepsilon_A) - a(A) + a(R)$$

and

$$\mathrm{indeg}(\mathrm{Der}_k(A)/A\varepsilon_A) \geq \begin{cases} e(A) - 3 + |\,\mathrm{Sing}(\mathcal{D})| - \mathrm{Lmult}(A/J_A) \\ e(A) - 2 + |\,\mathrm{Sing}(\mathcal{D})| - \mathrm{Lmult}(A/J_A) & \text{if } \mathcal{D} \text{ is irreducible}.\end{cases}$$

Since $\mathcal{D}$ is a plane curve we have $a(A) = e(A) - 3$. Thus we obtain

$$\mathrm{findeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) \geq \begin{cases} a(R) + |\,\mathrm{Sing}(\mathcal{D})| - \mathrm{Lmult}(A/J_A) \\ a(R) + 1 + |\,\mathrm{Sing}(\mathcal{D})| - \mathrm{Lmult}(A/J_A) & \text{if } \mathcal{D} \text{ is irreducible}.\end{cases}$$

Next we show that $|\,\mathrm{Sing}(\mathcal{D})| - \mathrm{Lmult}(A/J_A) = |\,\mathrm{Sing}(\mathcal{C})| - \mathrm{Lmult}(R/J_R)$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the distinct minimal prime ideals of J_R having height one. As $\mathrm{edim}\, R_{\mathfrak{p}_i} = 2$, it follows that $R_{\mathfrak{p}_i} = A_{\mathfrak{p}_i \cap A}$, see [44, Proposition 5.2] for instance. In particular, $\ell\ell((R/J_R)_{\mathfrak{p}_i}) = \ell\ell((A/J_A)_{\mathfrak{p}_i\cap A})$. The ring A may acquire additional prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ of height one where it is not regular, but they all correspond to ordinary nodes of $\mathcal{D}$, see [25, Chapter IV, Proposition 3.5 and Theorem 3.10], in other words $\ell\ell((A/J_A)_{\mathfrak{q}_i}) = 1$. It follows that

$$\mathrm{Lmult}(A/J_A) - \mathrm{Lmult}(R/J_R) = s = |\,\mathrm{Sing}(\mathcal{D})| - |\,\mathrm{Sing}(\mathcal{C})|\,,$$

as required. This completes the proof of the first inequality if the assumption of $\mathcal{C}$ being local irreducible is replaced by $\mathcal{C}$ being irreducible.

It remains to reduce the locally irreducible case to the irreducible case. Thus assume that $\mathcal{C}$ is locally irreducible and let $\wp_1, \ldots, \wp_r$ be the minimal prime ideals of R. Consider the exact sequence of R-modules

$$0 \longrightarrow R \stackrel{\iota}{\longrightarrow} \times_{i=1}^{r}(R/\wp_i) \longrightarrow N \longrightarrow 0\,.$$

Since $\mathcal{C}$ is locally irreducible, the map ι is an isomorphism locally on the punctured spectrum of R, so N is a module of finite length. It follows that $\omega_R \cong \times_{i=1}^{r}\omega_{R/\wp_i}$ and therefore

$$a(R) = \max\{a(R/\wp_i)\,|\,1 \le i \le r\}\,.$$

Now let $\wp$ be a minimal prime of R such that $a(R) = a(R/\wp)$, write $R' = R/\wp$, and let $\mathcal{C}'$ be the corresponding irreducible curve. We obtain

$$\begin{aligned}
\mathrm{findeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) &\ge \mathrm{findeg}(\mathrm{Der}_k(R')/\mathfrak{m}'^{-1}\varepsilon_{R'}) && \text{by Proposition 4.9}\\
&\ge a(R') + 1 + |\,\mathrm{Sing}(\mathcal{C}')| - \mathrm{Lmult}(R'/J_{R'}) && \text{since } \mathcal{C}' \text{ is irreducible}\\
&= a(R) + 1 + |\,\mathrm{Sing}(\mathcal{C}')| - \mathrm{Lmult}(R'/J_{R'})\\
&\ge a(R) + 1 + |\,\mathrm{Sing}(\mathcal{C})| - \mathrm{Lmult}(R/J_R)\,,
\end{aligned}$$

where the last inequality holds because $\mathcal{C}$ is the disjoint union of its irreducible components. □

Remark 4.11. If in addition to the assumption of Theorem 4.10, the ring R satisfies the generalized Cayley-Bacharach property, then according to Theorem 4.8

$$\begin{aligned}
\mathrm{indeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) &\ge a(R) + |\,\mathrm{Sing}(\mathcal{C})| - \mathrm{Lmult}(R/J_R)\\
&\ge a(R) + |\,\mathrm{Sing}(\mathcal{C})| + \sum_{p \in \mathrm{Sing}(\mathcal{C})} \binom{e(\mathcal{O}_{\mathcal{C},p}) - 1}{2} - \tau(\mathcal{C})\,.
\end{aligned}$$

The next result was first proved for plane curves in [7], then for complete intersection curves in [4], and finally for arithmetically Cohen-Macaulay curves in [16, Theorem 1].

Corollary 4.12 (The case of curves with ordinary nodes)**.** *Let k be an algebraically closed field and $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ be a reduced curve of degree d. Let R be the homogenous coordinate ring of $\mathcal{C}$ with maximal homogeneous ideal $\mathfrak{m}$. Assume d is not a multiple of the characteristic. If $\mathcal{C}$ has at most ordinary nodes as singularities, then*

$$\mathrm{findeg}(\mathrm{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \ge \begin{cases} a(R) \\ a(R)+1 & \textit{if } \mathcal{C} \textit{ is locally irreducible}\,. \end{cases}$$

Proof. The assertion follows from Theorem 4.10, because $\operatorname{Lmult}(R/J_R) = |\operatorname{Sing}(\mathcal{C})|$ if (and only if) $\mathcal{C}$ has only ordinary nodes as singularities. □

Corollary 4.13. *Let k be a perfect field and $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ be a curve of degree d. Let R be the homogenous coordinate ring of $\mathcal{C}$ with maximal homogeneous ideal $\mathfrak{m}$. Assume d is not a multiple of the characteristic. If $\mathcal{C}$ is smooth, then*

$$\operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \geq a(R) + 1\,.$$

Corollary 4.14. *Let k be a perfect field and $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ be a curve of degree d. Let R be the homogenous coordinate ring of $\mathcal{C}$ with maximal homogeneous ideal $\mathfrak{m}$. Assume d is not a multiple of the characteristic. If $\mathcal{C}$ is smooth and arithemetically Gorenstein, then*

$$\operatorname{Der}_k(R)/R\varepsilon \cong \mathfrak{m}(-a(R))\,.$$

In particular, $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) = a(R) + 1\,.$

Proof. First notice that R is a domain, hence $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon)$. As in the proof of Corollary 2.10 we have an exact sequence

$$0 \longrightarrow \operatorname{Der}_k(R)/R\varepsilon \longrightarrow \omega_R^* \longrightarrow \operatorname{Ext}_R^2(k, R)\,,$$

where $\operatorname{Ext}_R^2(k, R)$ is concentrated in degrees $\leq a(R)$. Since $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \geq a(R) + 1$ by Corollary 4.13, we conclude that

$$\operatorname{Der}_k(R)/R\varepsilon \cong (\omega_R^*)_{\geq a(R)+1}\,.$$

Now the assertion follows because, $\omega_R^* \cong R(-a(R))$. □

5. Lower bounds in terms of algebraic and geometric genus

The main results of this section are the estimates on the degrees of vector fields of Theorem 5.6 and Theorem 5.10. Our estimates will require Corollary 5.3 below, a remarkable lower bound for the Castelnuovo-Mumford regularity of $R/(\operatorname{im}\mu)^{\mathrm{sat}}$ that was proved in [17, 4.5]. As in [17], we deduce this bound from the nonvanishing of a map between cohomology modules. Our proof of the nonvanishing, Theorem 5.2, uses general properties of the Koszul complex and of regular differential forms, and is different from the proofs of the corresponding results [17, 2.1 and 2.2]. In Section 7 we will apply Theorem 5.2 to obtain structural information about the module $\operatorname{Der}_k(R)$ and the natural map $\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon \longrightarrow L^*$, see Proposition 7.1 and Proposition 7.4.

Lemma 5.1. *Let k be a field, let d and n be integers with $1 < d < n$, let $x_1, \ldots, x_n$ be variables over k, and consider the standard graded polynomial rings $A = k[x_1, \ldots, x_d] \subset D = k[x_1, \ldots, x_n]$ with homogeneous maximal ideals $\mathfrak{n}$ and $\mathfrak{N}$, respectively. By $B_\bullet(A)$ and $B_\bullet(D)$ we denote the boundaries in the Koszul complexes $K_\bullet(A) = K_\bullet(x_1, \ldots, x_d; A)$ and $K_\bullet(D) = K_\bullet(x_1, \ldots, x_n; D)$.*

(*a*) *There exists a homogeneous A-linear map* δ *fitting in the commutative diagram*

$$\begin{array}{ccc} H^d_{\mathfrak{N}}(B_{d-1}(D)) & \xrightarrow{\alpha} & H^d_{\mathfrak{n}}(B_{d-1}(D)) \\ \delta \uparrow & & \uparrow \beta \\ H^{d-1}_{\mathfrak{n}}(B_{d-2}(A)) & \xrightarrow{\gamma} & H^d_{\mathfrak{n}}(B_{d-1}(A)) \,. \end{array}$$

Here α *is the natural map arising from the fact that* $\mathfrak{n} \subset \mathfrak{N}$, β *is induced by the morphism of complexes* $K_\bullet(A) \longrightarrow K_\bullet(D)$, *and* γ *is the connecting homomorphism in the long exact sequence associated to the exact sequence* $0 \to B_{d-1}(A) \to K_{d-1}(A) \to B_{d-2}(A) \to 0$.

(*b*) *The map* γ *as in item (a) is an isomorphism in degree zero.*

Proof. To prove part (a) we first show that the map α is injective. Write $B = B_{d-1}(D)$ and $\mathfrak{N}_i = (x_1, \ldots, x_i)D$, and notice that $H^d_{\mathfrak{n}}(B) = H^d_{\mathfrak{N}_d}(B)$. By [2, 8.1.2], for $n \ge i > d$ there are natural exact sequences

$$H^{d-1}_{\mathfrak{N}_{i-1}}(B_{x_i}) \longrightarrow H^d_{\mathfrak{N}_i}(B) \longrightarrow H^d_{\mathfrak{N}_{i-1}}(B)\,.$$

As $B_{x_i} \cong \oplus D_{x_i}$ and $\operatorname{grade}(\mathfrak{N}_{i-1} D_{x_i}) \ge i-1 \ge d$ it follows that $H^{d-1}_{\mathfrak{N}_{i-1}}(B_{x_i}) = 0$, hence $H^d_{\mathfrak{N}_i}(B) \hookrightarrow H^d_{\mathfrak{N}_{i-1}}(B)$ for $n \ge i > d$. This shows that α is injective.

The morphism of complexes $K_\bullet(A) \longrightarrow K_\bullet(D)$ and the naturality of the long exact sequence of local cohomology gives a commutative diagram

$$\begin{array}{ccc} H^{d-1}_{\mathfrak{n}}(B_{d-2}(D)) & \xrightarrow{\varepsilon} & H^d_{\mathfrak{n}}(B_{d-1}(D)) \\ \uparrow & & \uparrow \\ H^{d-1}_{\mathfrak{n}}(B_{d-2}(A)) & \xrightarrow{\gamma} & H^d_{\mathfrak{n}}(B_{d-1}(A)) \,. \end{array}$$

Since α is an isomorphism onto its image, the existence of δ will follow once we have shown that $\operatorname{im} \varepsilon \subset \operatorname{im} \alpha$. In fact, we are now going to prove that

$$\operatorname{im} \varepsilon = \operatorname{soc}_D(H^d_{\mathfrak{n}}(B_{d-1}(D))) = \operatorname{im} \alpha\,.$$

The acyclicity of $K_\bullet(A)$ and $K_\bullet(D)$ imply that for $0 \le i \le d-2$,

$$H^{i+1}_{\mathfrak{n}}(B_i(A)) \cong H^0(k) = k \qquad \text{as graded } A\text{-modules, and}$$
$$H^{i+1}_{\mathfrak{n}}(B_i(D)) \cong H^0(k) = k \qquad \text{as graded } D\text{-modules}\,.$$

In particular, we have homogeneous D-isomorphisms

$$H^{i+1}_{\mathfrak{N}}(B_i(D)) \cong k \quad \text{for} \quad 0 \le i \le n-2\,.$$

The long exact sequence of local cohomology gives an exact sequence of graded D-modules

$$\begin{array}{ccccccc} H^{d-1}_{\mathfrak{n}}(K_{d-1}(D)) & \longrightarrow & H^{d-1}_{\mathfrak{n}}(B_{d-2}(D)) & \xrightarrow{\varepsilon} & H^d_{\mathfrak{n}}(B_{d-1}(D)) & \longrightarrow & H^d_{\mathfrak{n}}(K_{d-1}(D)) \\ \| & & \wr\| & & & & \wr\| \\ 0 & & k & & & & H^d_{\mathfrak{n}}(K_{d-1}(A)) \otimes_A D\,. \end{array}$$

As x_{d+1} is a non zerodivisor on the D-module $H^d_{\mathfrak{n}}(K_{d-1}(A)) \otimes_A D$, this module has trivial socle. It follows that

$$\operatorname{soc}_D(H^d_{\mathfrak{n}}(B_{d-1}(D))) = \operatorname{im} \varepsilon \cong k\,.$$

On the other hand,

$$k \cong H^d_{\mathfrak{N}}(B_{d-1}(D)) \overset{\alpha}{\hookrightarrow} H^d_{\mathfrak{n}}(B_{d-1}(D))\,.$$

This shows that

$$0 \neq \operatorname{im} \alpha \subset \operatorname{soc}_D(H^d_{\mathfrak{n}}(B_{d-1}(D)))\,.$$

This inclusion is an equality since the socle is one-dimensional. It follows that $\operatorname{im} \varepsilon = \operatorname{im} \alpha$.

We prove part (b). As before, the long exact sequence of local cohomology gives an exact sequence of graded A-modules

$$\begin{array}{ccccc} H^{d-1}_{\mathfrak{n}}(K_{d-1}(A)) & \longrightarrow & H^{d-1}_{\mathfrak{n}}(B_{d-2}(A)) & \xrightarrow{\ \gamma\ } & H^d_{\mathfrak{n}}(B_{d-1}(A)) \\ \| & & \wr\| & & \wr\| \\ 0 & & k & & H^d_{\mathfrak{n}}(K_d(A))\ . \end{array}$$

Since $H^d_{\mathfrak{n}}(K_d(A)) \cong H^d_{\mathfrak{n}}(A(-d)) \cong k[x_1^{-1}, \ldots, x_d^{-1}]$, we see that γ is an isomorphism in degree zero. □

Let k be a field and R be a standard graded Noetherian k-algebra with homogeneous maximal ideal $\mathfrak{m}$. Set $\Omega := \Omega_k(R)$. Let $K_\bullet = K_\bullet(R)$ be the Koszul complex of the functional $\Omega \longrightarrow R$ corresponding to the Euler derivation of R over k. Write $Z_\bullet = Z_\bullet(R)$ for the cycles of $K_\bullet$. If R is regular, then this notation is consistent with the one introduced in Lemma 5.1. As $Z_\bullet$ is a graded commutative R-algebra, there is a natural homomorphism

$$\bigwedge\nolimits^\bullet Z_1 \longrightarrow Z_\bullet\,.$$

Moreover, the complex $K_\bullet$ is acyclic if R is regular. It is also acyclic if k has characteristic zero, since the differential of the de Rham complex produces a k-linear contracting homotopy in positive internal degree. If T is a flat R-algebra with $\mathfrak{m}T = T$, then $K_\bullet \otimes_R T$ is split-exact and the map $(\wedge^\bullet Z_1) \otimes_R T \longrightarrow Z_\bullet \otimes_R T$ is an isomorphism. In particular, the kernel and cokernel of $\wedge^\bullet Z_1 \longrightarrow Z_\bullet$ have dimension zero.

Any morphism of positively graded Noetherian k-algebras $S \longrightarrow R$ induces homomorphisms of differential graded algebras $K_\bullet(S) \longrightarrow K_\bullet(R)$ and then $Z_\bullet(S) \longrightarrow Z_\bullet(R)$.

Theorem 5.2. *Let k be a field, let R and S be standard graded k-algebras with homogeneous maximal ideals $\mathfrak{m}$ and $\mathfrak{m}_S$, respectively, and assume that the multiplicity of R is not a multiple of the characteristic of k. If $S \to R$ is a homogeneous homomorphism that is module finite and $d := \dim R \geq 2$, then the induced maps*

$$H^d_{\mathfrak{m}_S}(Z_{d-1}(S))_0 \longrightarrow H^d_{\mathfrak{m}}(Z_{d-1}(R))_0 \qquad \text{and} \qquad H^d_{\mathfrak{m}_S}(\wedge^{d-1} Z_1(S))_0 \longrightarrow H^d_{\mathfrak{m}}(\wedge^{d-1} Z_1(R))_0$$

are nonzero.

Proof. Since $d \geq 2$ and the natural maps $\wedge^{d-1} Z_1(S) \longrightarrow Z_{d-1}(S)$ and $\wedge^{d-1} Z_1(R) \longrightarrow Z_{d-1}(R)$ have zero-dimensional kernels and cokernels, it follows that these maps become isomorphisms after applying $H^d_{\mathfrak{m}_S}$ and $H^d_{\mathfrak{m}}$, respectively. Thus it suffices to prove that the first map in the statement of the theorem is not zero.

To show that this map is not zero, we may pass to the algebraic closure of k to assume that k is infinite and perfect. Our assumption on the multiplicity of R and the associativity formula for multiplicities imply that, for some prime ideal $\mathfrak{p}$ of R with $\dim R/\mathfrak{p} = d$, the multiplicity e of $R/\mathfrak{p}$ is not a multiple of the characteristic of k. Write $n = \dim S$ and let $x_1, \ldots, x_n$ be general linear forms in S. We consider the polynomial subrings $A = k[x_1, \ldots, x_d] \subset D = k[x_1, \ldots, x_n]$ of S, and we denote their maximal ideals by $\mathfrak{n}$ and $\mathfrak{N}$, respectively. Notice that D is a Noether normalization of S and A is a Noether normalization of R and of $R/\mathfrak{p}$. Moreover, $\operatorname{rank}_A R/\mathfrak{p} = e$ is a unit in k.

Since $D \subset S$ and $A \subset R$ are integral extensions, it follows that $H^i_{\mathfrak{m}_S} \simeq H^i_{\mathfrak{N}S} \simeq H^i_{\mathfrak{N}}$ and $H^i_{\mathfrak{m}} \simeq H^i_{\mathfrak{n}R} \simeq H^i_{\mathfrak{n}}$. Thus it remains to show that the map $H^d_{\mathfrak{N}}(Z_{d-1}(S)) \longrightarrow H^d_{\mathfrak{n}}(Z_{d-1}(R))$ is nonzero in degree zero. Composing this map with the natural homomorphisms $H^d_{\mathfrak{N}}(Z_{d-1}(D)) \longrightarrow H^d_{\mathfrak{N}}(Z_{d-1}(S))$ from the right and $H^d_{\mathfrak{n}}(Z_{d-1}(R)) \longrightarrow H^d_{\mathfrak{n}}(Z_{d-1}(R/\mathfrak{p}))$ from the left yields a homomorphism $H^d_{\mathfrak{N}}(Z_{d-1}(D)) \longrightarrow H^d_{\mathfrak{n}}(Z_{d-1}(R/\mathfrak{p}))$, and it suffices to prove that this last map is nonzero in degree zero. Replacing R by $R/\mathfrak{p}$ we may now assume that R is a domain, with Noether normalization A. We need to prove that

$$H^d_{\mathfrak{N}}(Z_{d-1}(D)) \longrightarrow H^d_{\mathfrak{n}}(Z_{d-1}(R))$$

is nonzero in degree zero.

Recall that the complexes $K_\bullet(D)$ and $K_\bullet(A)$ are acyclic and that $d \geq 2$. We use Lemma 5.1 and the natural maps $D \to R$ and $A \to R$ to obtain a commutative diagram

$$\begin{array}{ccccc} H^d_{\mathfrak{N}}(Z_{d-1}(D)) & \xrightarrow{\alpha} & H^d_{\mathfrak{n}}(Z_{d-1}(D)) & \xrightarrow{g} & H^d_{\mathfrak{n}}(Z_{d-1}(R)) \\ \uparrow & & \uparrow & \nearrow_{f} & \\ H^{d-1}_{\mathfrak{n}}(B_{d-2}(A)) & \xrightarrow{\gamma} & H^d_{\mathfrak{n}}(Z_{d-1}(A)) & & \end{array},$$

where γ is an isomorphism in degree zero. We need to prove that the composition $g \circ \alpha$ is not zero in degree zero. As γ is an isomorphism in degree zero, this will follow once we have shown that f is nonzero in degree zero.

The morphism of complexes $K_\bullet(A) \longrightarrow K_\bullet(R)$ induces a commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Z_{d+1}(R) & \longrightarrow & K_d(R) & \longrightarrow & Z_{d-1}(R) & \longrightarrow & H_{d-1}(K_\bullet(R)) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & & & \\ & & K_d(A) & \xrightarrow{\cong} & Z_{d-1}(A) & & & & & & \end{array}.$$

The bottom map is an isomorphism because $K_\bullet(A)$ is acyclic and has length d. Recall that the complex $K_\bullet(R)$ is exact locally on the punctured spectrum. The module $\Omega_k(R)$ has rank d because

$K \subset L$ is a separable algebraic field extension. Thus $Z_d(R)$ has rank zero and therefore its dimension is $< d$. In addition, the module $H_{d-1}(K_\bullet(R))$ has finite length, hence dimension $< d-1$. Thus we obtain an induced commutative diagram

$$\begin{array}{ccc} H^d_{\mathfrak{n}}(K_d(R)) & \xrightarrow{\cong} & H^d_{\mathfrak{n}}(Z_{d-1}(R)) \\ {\scriptstyle h}\uparrow & & {\scriptstyle f}\uparrow \\ H^d_{\mathfrak{n}}(K_d(A)) & \xrightarrow{\cong} & H^d_{\mathfrak{n}}(Z_{d-1}(A)) \,. \end{array}$$

It remains to show that h is nonzero in degree zero.

Let $K \subset L$ be the extension of quotient fields of A and R. This field extension has degree e and is separable since e is a unit in k. Since $A \subset R$ is a separable Noether normalization, we can consider the complementary module

$$\mathfrak{C}_A(R) = \{z \in L \mid \operatorname{Tr}_{L/K}(zR) \subset A\} \,,$$

which is a finitely generated graded R-module. The image of the natural map

$$\wedge^d \Omega_k(R) \longrightarrow \wedge^d \Omega_k(L) = L\ dx_1 \wedge \ldots \wedge dx_d \cong L$$

is contained in $\mathfrak{C}_A(R)$, see for instance [30, Theorem 9.7]. Hence we obtain a homogeneous R-linear map

$$\mathfrak{c}_R : \wedge^d \Omega_k(R) \longrightarrow \mathfrak{C}_A(R)\ dx_1 \wedge \ldots \wedge dx_d \,.$$

Likewise we have

$$\mathfrak{c}_A : \wedge^d \Omega_k(A) = A\ dx_1 \wedge \ldots \wedge dx_d \longrightarrow \mathfrak{C}_A(A)\ dx_1 \wedge \ldots \wedge dx_d = A\ dx_1 \wedge \ldots \wedge dx_d \,,$$

which is the identity map. Notice that $\operatorname{Tr}_{L/K}(\mathfrak{C}_A(R)) \subset A$ by definition of the complementary module.

Now we have a diagram of homogenous A-linear maps

$$\begin{array}{ccc} K_d(R) = \wedge^d \Omega_k(R) & \xrightarrow{\mathfrak{c}_R} & \mathfrak{C}_A(R)\ dx_1 \wedge \ldots \wedge dx_d \\ \uparrow & & \downarrow {\scriptstyle \frac{1}{e}\cdot \operatorname{Tr}_{L/K}\ dx_1\wedge\ldots\wedge dx_d} \\ K_d(A) = \wedge^d \Omega_k(A) & \xrightarrow{\mathfrak{c}_A} & A\ dx_1 \wedge \ldots \wedge dx_d \,. \end{array}$$

This diagram commutes, as can be seen by following the element $dx_1 \wedge \ldots \wedge dx_d \in \wedge^d \Omega_k(A)$ and using the fact that $\frac{1}{e} \cdot \operatorname{Tr}_{L/K}(1) = 1$. Applying the functor $H^d_{\mathfrak{n}}$ to this diagram, the left vertical map becomes h, and it suffices to prove that $H^d_{\mathfrak{n}}(\mathfrak{c}_A)$ is nonzero in degree zero. However, this map is the identity map and $H^d_{\mathfrak{n}}(A(-d)) = k[x_1^{-1}, \ldots, x_d^{-1}]$, which is the field k in degree zero. □

In the remainder of this section we use Theorem 5.2 to estimate the degree of the singular locus of vector fields. These estimates in turn will lead to bounds on the degree of the vector fields themselves.

Corollary 5.3. *Adopt Setting 2.7 and assume that the degree of $\mathcal{C}$ is not a multiple of the characteristic of k. Let η be a vector field on $\mathbb{P}_k^{n-1}$ of degree m leaving $\mathcal{C}$ invariant whose singular locus does not contain an irreducible component of $\mathcal{C}$. This vector field induces a homogeneous R-linear map $\mu : H \to R$ of degree $m-1$ such that $\operatorname{ht} \operatorname{im} \mu > 0$. Let $L = Z_1(R)$ be as in (2).*

(a) *The natural maps*

$$H^1_{\mathfrak{m}_S}(S/\operatorname{im}\eta) \longrightarrow H^1_{\mathfrak{m}}(R/\operatorname{im}\mu) \quad \text{and} \quad H^1_{\mathfrak{m}}(R/\operatorname{im}\mu) \longrightarrow H^2_{\mathfrak{m}}(\operatorname{im}\mu)$$

are both nonzero in degree $m-1$.

(b) $\dim(R/\operatorname{im}\mu) = 1$, $\operatorname{reg} R/(\operatorname{im}\mu)^{\mathrm{sat}} \geq m$, *and* $e(R/\operatorname{im}\mu) \geq m+1$.

(c) *If* $[H^2_{\mathfrak{m}}(L)]_0 \cong k$, *then* $m \geq a(R) + 2$.

Proof. Let $Z = Z_1(S)$ be as in (2). According to Theorem 5.2 the natural map $H^2_{\mathfrak{m}_S}(Z) \to H^2_{\mathfrak{m}}(L)$ is not zero in degree zero.

There is a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\ \eta\ } & \operatorname{im}\eta \\ \downarrow & & \downarrow \\ H & \xrightarrow{\ \mu\ } & \operatorname{im}\mu \\ \downarrow & & \\ L & & \end{array}$$

where the horizontal maps are homogeneous of degree $m-1$. We also have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im}\eta & \longrightarrow & S & \longrightarrow & S/\operatorname{im}\eta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{im}\mu & \longrightarrow & R & \longrightarrow & R/\operatorname{im}\mu & \longrightarrow & 0 \end{array}$$

Together, these diagrams induce a commutative diagram

$$\begin{array}{ccccc} & & & & H^2_{\mathfrak{m}_S}(Z) \\ & & & \swarrow & \downarrow g \\ H^1_{\mathfrak{m}_S}(S/\operatorname{im}\eta) & \xrightarrow{\ \alpha\ } & H^2_{\mathfrak{m}_S}(\operatorname{im}\eta) & & H^2_{\mathfrak{m}}(H) \\ \downarrow & & \downarrow & \swarrow_{\beta} & \downarrow h \\ H^1_{\mathfrak{m}}(R/\operatorname{im}\mu) & \xrightarrow{\ \gamma\ } & H^2_{\mathfrak{m}}(\operatorname{im}\mu) & & H^2_{\mathfrak{m}}(L) \end{array}$$

In this diagram, the two diagonal maps are homogenous of degree $m-1$, the map α is bijective because $\operatorname{depth} S \geq 3$, the map β is bijective since $\dim(\ker\mu) \leq 1$ due to the assumption that $\operatorname{grade} \operatorname{im}\mu > 0$, and h is bijective as L/H is a module of finite length (see page 4). As $h \circ g$ is not zero in degree zero, the same holds for g. Now the diagram readily implies part (a).

From (a) we obtain, in particular, that $[H^1_{\mathfrak{m}}(R/\operatorname{im}\mu)]_{m-1} \neq 0$. Now the assertions about dimension and regularity in part (b) follow immediately. As to the claim about the multiplicity, $e(R/\operatorname{im}\mu) = e(R/(\operatorname{im}\mu)^{\mathrm{sat}})$ and the ring $R/(\operatorname{im}\mu)^{\mathrm{sat}}$ is Cohen-Macaulay, hence its multiplicity is bounded below by its regularity plus 1.

In the setting of (c), the diagram shows that $[H^2_{\mathfrak{m}}(\operatorname{im}\mu)]_{m-1} \cong k$. Thus, since $[\gamma]_{m-1} \neq 0$, this map is surjective, and then the long exact sequence of local cohomology implies that $[H^2_{\mathfrak{m}}(R)]_{m-1} = 0$. Therefore $[H^2_{\mathfrak{m}}(R)]_j = 0$ for all $j \geq m-1$, showing that $m-1 > a(R)$ as asserted. ☐

The multiplicity estimate in Corollary 5.3(b) can be improved substantially if the curve $\mathcal{C}$ is arithmetically Gorenstein:

Proposition 5.4. *We use the hypotheses and notation of Corollary 5.3, and write a for the a-invariant of R and p_g for the geometric genus of $\mathcal{C}$. If R is Gorenstein, then*

$$e(R/\operatorname{im}\mu) \geq \dim_k(R_{m-\delta+a+1}) + \delta - a - 1 - p_g.$$

Proof. We first observe that the natural map $S/(f_1,\ldots,f_{n-2}) \longrightarrow R$ is a surjection of rings having the same dimension. Thus

$$a = a(R) \leq a(S/(f_1,\ldots,f_{n-2})) = -n + \sum_{j=1}^{n-2} \delta_j = \delta - 2.$$

This inequality and the regularity estimate in Corollary 5.3(b) give

$$m - \delta + a + 1 \leq m \leq \operatorname{reg} R/(\operatorname{im}\mu)^{\mathrm{sat}}.$$

Again by Corollary 5.3(b), the standard graded algebra $R/(\operatorname{im}\mu)^{\mathrm{sat}}$ is one-dimensional and therefore Cohen-Macaulay. Thus its Hilbert function increases strictly up to degree $\operatorname{reg} R/(\operatorname{im}\mu)^{\mathrm{sat}}$ and is equal to the multiplicity afterward. As $e((R/(\operatorname{im}\mu)^{\mathrm{sat}}) = e(R/\operatorname{im}\mu)$, it follows that

$$e(R/\operatorname{im}\mu) \geq \dim_k((R/(\operatorname{im}\mu)^{\mathrm{sat}})_{m-\delta+a+1}) + \delta - a - 1.$$

Write $t = m - \delta + 1$. It remains to prove that

$$\dim_k((\operatorname{im}\mu)^{\mathrm{sat}})_{t+a} \leq p_g.$$

Indeed, Corollary 2.11 shows that $\operatorname{im}\mu \cong J(-t)$, and since R is Cohen-Macaulay of dimension ≥ 2, this isomorphism induces an isomorphism

$$(\operatorname{im}\mu)^{\mathrm{sat}} \cong J^{\mathrm{sat}}(-t).$$

On the other hand, J is contained in J_R, the Jacobian ideal of R. Let $\overline{R}$ denote the integral closure of R, and $\mathfrak{f} := R :_R \overline{R}$ the conductor. As is classically known, see e.g. [35], one has $J_R \subset \mathfrak{f}$. Thus, as $\mathfrak{f}$ is unmixed,

$$J^{\mathrm{sat}} \subset \mathfrak{f}.$$

In turn, since R is Gorenstein,

$$\mathfrak{f} \cong \operatorname{Hom}_R(\overline{R}, R) \cong \operatorname{Hom}_R(\overline{R}, \omega_R(-a)) \cong \omega_{\overline{R}}(-a).$$

Combining these facts we conclude that

$$\dim_k((\operatorname{im}\mu)^{\mathrm{sat}})_{t+a} = \dim_k(J^{\mathrm{sat}})_a \leq \dim_k \mathfrak{f}_a = \dim_k(\omega_{\overline{R}})_0 = p_g,$$

as required □

The main application in this section, Theorem 5.6, generalizes results of du Plessis and Wall and of Esteves and Kleiman [12,15] for the case of plane curves. In this paper, it is an easy consequence of Corollary 2.11 and Corollary 5.3. Our proof was inspired by an argument in [15, proof of Proposition 5.2].

Lemma 5.5. *Let R be an equidimensional Noetherian standard graded algebra over a field, with* $\operatorname{depth} R > 0$*, let $\mathfrak{a}$ and $\mathfrak{b}$ be homogeneous ideals of height one, and assume that $\mathfrak{a} \cong \mathfrak{b}(-n)$ for some $n \in \mathbb{Z}$. Then*

$$e(R/\mathfrak{a}) = e(R/\mathfrak{b}) + n \cdot e(R)\,.$$

Proof. By symmetry we may assume that $n \geq 0$, and by induction one reduces to the case $n = 1$. We may further suppose that the ground field is infinite, and hence there exists a linear form $x \in R$ that is non zerodivisor. Thus $\mathfrak{a} \cong \mathfrak{b}(-1)$ and $x\mathfrak{b}$ have the same Hilbert function, and so do $R/\mathfrak{a}$ and $R/x\mathfrak{b}$, which gives $e(R/\mathfrak{a}) = e(R/x\mathfrak{b})$. So it suffices to show that $e(R/x\mathfrak{b}) = e(R/\mathfrak{b}) + e(R)$.

Consider the exact sequence

$$0 \longrightarrow b/x\mathfrak{b} \longrightarrow R/x\mathfrak{b} \longrightarrow R/\mathfrak{b} \longrightarrow 0\,.$$

The three R-modules in this sequence have the same dimension, because $\mathfrak{b}$ is in no minimal prime ideal of R and therefore $\operatorname{ann}_R \mathfrak{b} \subset \sqrt{0}$. Thus,

$$e(R/x\mathfrak{b}) = e(R/\mathfrak{b}) + e(\mathfrak{b}/x\mathfrak{b})\,.$$

On the other hand, $e(\mathfrak{b}/x\mathfrak{b}) = e(\mathfrak{b})$ since the linear form x is a non zerodivisor on $\mathfrak{b}$, and $e(\mathfrak{b}) = e(R)$ by the associativity formula because $\mathfrak{b}$ is in no minimal prime ideal of R. □

The estimates in the next theorem use the multiplicity of R/J, where J is a partial Jacobian ideal as defined in Setting 2.7.

Theorem 5.6. *In addition to Setting 2.7, assume that the degree d of $\mathcal{C}$ is not a multiple of the characteristic. One has*

$$\operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \geq \begin{cases} a(R)+1 & \text{if } \mathcal{C} \text{ is a smooth complete intersection} \\ \delta - 2 - \frac{e(R/J)-\delta}{d-1} & \text{otherwise}\,. \end{cases}$$

Proof. If $\mathcal{C}$ is a smooth complete intersection the assertion follows from Corollary 4.13. Otherwise $\operatorname{ht} J = 1$ by Theorem 3.3 (c).

Let μ and m be as in Corollary 2.11 and assume that m is minimal. Recall that

$$\operatorname{im}\mu \cong J(\delta - m - 1)$$

by that corollary. We use Corollary 5.3(b), which says that $\operatorname{ht}\operatorname{im}\mu = 1$ and

$$e(R/\operatorname{im}\mu) \geq m + 1\,.$$

Now combining Lemma 5.5 with the two displayed formulas, we obtain

$$m + 1 \leq e(R/\operatorname{im}\mu) = e(R/J) + (m + 1 - \delta)e(R)\,, \tag{12}$$

as required. □

Part (a) of the next corollary is essentially [12, Theorem 3.2] and [15, Corollary 6.4]. The estimate of part (b) is often sharper for plane curves of small genus.

Corollary 5.7. *Let k be a perfect field and $\mathcal{C} \subset \mathbb{P}^2_k$ be a reduced curve of degree d that is not smooth. Assume that d is not a multiple of the characteristic. Let τ and p_g denote the total Tjurina number and the geometric genus of $\mathcal{C}$ and let R be the homogeneous coordinate ring of $\mathcal{C}$. One has*

(*a*) $\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) \geq d - 2 - \frac{\tau}{d-1}$;

(*b*) $\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) \geq d - \frac{3}{2} - \sqrt{2\tau(\mathcal{C}) + 2p_g - d^2 + 3d - \frac{7}{4}}$.

Proof. Part (a) follows from Theorem 5.6. We prove part (b). Let m be the minimal degree of a vector field leaving $\mathcal{C}$ invariant and recall that $\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) = m - 1$. We start from the inequalities (12), but replace the multiplicity estimate of Corollary 5.3(b) by the one of Proposition 5.4 to obtain

$$\dim_k(R_{m-\delta+a+1}) + \delta - a - 1 - p_g \leq e(R/\operatorname{im}\mu) = e(R/J_R) + (m + 1 - \delta) \cdot e(R)\,.$$

Notice that $\delta = d - 1$, $a = d - 3$, and $m - \delta + a + 1 = m - 1 \leq d - 2$, where the last inequality will be proved in Theorem 6.1(d). Thus $\dim_k(R_{m-\delta+a+1}) = \dim_k(S_{m-1}) = \binom{m+1}{2}$, and we obtain

$$m^2 - (2d - 1)m - 2e(R/J_R) - 2p_g + 2d^2 - 4d + 2 \leq 0\,.$$

Since there exists a vector field of degree m whose singular locus does not contain an irreducible component of $\mathcal{C}$, the polynomial in m on the left-hand side has a real root and the smallest real root is

$$d - \frac{1}{2} - \sqrt{2e(R/J_R) + 2p_g - d^2 + 3d - \frac{7}{4}}\,.$$

□

In addition to the assumptions of Corollary 5.7 suppose that $\mathcal{C}$ is a rational curve, that is $p_g = 0$. If moreover $\mathcal{C}$ has only ordinary nodes as singularities, then the lower bound in part (a) of the corollary gives $\frac{a(R)+1}{2}$, whereas the bound in (b) gives $a(R) + 1$, which is the exact value for $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon)$ proved in Corollary 4.12. For rational curves in general, the bound in (b) is better than the bound in (a) if and only if $\tau < \binom{d-1}{2} + \alpha - \frac{1}{2}\sqrt{3\alpha^2+\alpha}$, where $\alpha = (d-1)^2$. For a rational curve with only ordinary nodes and ordinary cusps as singularities, this inequality holds if and only if the number of cusps is less than $\alpha - \frac{1}{2}\sqrt{3\alpha^2+\alpha}$.

It will follow from Theorem 6.1(e) below that if $\mathcal{C}$ is a smooth complete intersection, then the equality $\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) = a(R)+1$ holds in Theorem 5.6. Therefore we are not going to consider this case in the remainder of this section.

Theorem 5.8. *Let k be an algebraically closed field of characteristic zero and $X \subset \mathbb{P}_k^{n-1}$ be an equidimensional subscheme of dimension s, where $1 \le s \le 3$. Assume that X is locally a complete intersection and has only isolated singularities. If X is defined scheme theoretically by an ideal I generated by forms of degrees $\le t$, let Z be a complete intersection of dimension s defined by general forms of degree t in I, and let Y be the link of X with respect to Z.*

(a) *Y and $X \cap Y$ are nonsingular;*
(b) *$\operatorname{Sing}(Z)$ is the disjoint union of $\operatorname{Sing}(X)$ and $X \cap Y$; if $p \in \operatorname{Sing}(X)$, then $\mathcal{O}_{Z,p} \cong \mathcal{O}_{X,p}$, and if $p \in X \cap Y$, then $\widehat{\mathcal{O}_{Z,p}} \cong k[\![x_1,\ldots,x_{s+1}]\!]/(x_1x_2)$.*

Proof. Let $S = k[x_1,\ldots,x_n]$ be the homogenous coordinate ring of $\mathbb{P}_k^{n-1}$, let $\mathfrak{a}$ be the saturated ideal of Z, and $K = \mathfrak{a} : I$ be the saturated ideal of Y. Replacing I by $I_{\ge t}$, we may assume that I is generated by forms $f_1,\ldots,f_m$ of degree t. We write $g = \operatorname{ht} I = n-s-1$. We may assume that the ideals I and $\mathfrak{a}$ are equal locally at every minimal prime of I. Therefore $\mathfrak{a} = I \cap K$, the ideal K is unmixed of height g, and all associated primes $\neq \mathfrak{m}_s$ of $I+K$ have height $g+1$, as can be seen from the exact sequence

$$0 \longrightarrow S/\mathfrak{a} \longrightarrow S/I \oplus S/K \longrightarrow S/(I+K) \longrightarrow 0\,.$$

We now prove (a). The ideal $\mathfrak{a}$ is generated by g k-linear combinations $\sum \lambda_{ij} f_j$ with $\underline{\lambda} = (\lambda_{ij})$ a general point in $\mathbb{A}_k^{gm}$.

We consider the polynomial rings $U = k[\{u_{ij} \mid 1 \le i \le g, 1 \le j \le m\}]$ and $\widetilde{S} = S \otimes_k U$, and the $\widetilde{S}$-ideals $\widetilde{\mathfrak{a}} = (\sum u_{ij} f_j \mid 1 \le i \le g)$ and $\widetilde{K} = \widetilde{\mathfrak{a}} :_{\widetilde{S}} I$. There are natural maps

$$\psi_1 : U \longrightarrow \widetilde{T} := \widetilde{S}/\widetilde{K} \quad \text{and} \quad \psi_2 : U \longrightarrow \widetilde{P} := \widetilde{S}/(I\widetilde{S} + \widetilde{K})\,.$$

According to [29, 2.4(b)], the generic fiber of ψ_1 satisfies Serre's condition R_s and the generic fiber of ψ_2 satisfies R_{s-1}.

Let Q be the quotient field of U, and write $T_Q = \widetilde{T} \otimes_U Q$ and $P_L = \widetilde{P} \otimes_U Q$. The rings T_Q and P_Q are standard graded Q-algebras of dimension at most $s+1$ and s, respectively. Since these rings satisfy R_s and R_{s-1}, respectively, they are regular locally on the punctured spectrum. Thus by Theorem 3.1 there exits an integer ℓ such that

$$(x_1,\ldots,x_n)^\ell \, T_Q \subset \operatorname{Fitt}_{s+1}(\Omega_Q(T_Q)) \quad \text{and} \quad (x_1,\ldots,x_n)^\ell \, P_Q \subset \operatorname{Fitt}_s(\Omega_Q(P_Q))\,.$$

Hence for some nonzero polynomial $h \in U$,

$$h\,(x_1,\dots,x_n)^\ell\,\widetilde{T} \subset \mathrm{Fitt}_{s+1}(\Omega_U(\widetilde{T})) \quad \text{and} \quad h\,(x_1,\dots,x_n)^\ell\,\widetilde{P} \subset \mathrm{Fitt}_{s}(\Omega_U(\widetilde{P}))\,. \tag{13}$$

For a point $\underline{\lambda} = (\lambda_{ij}) \in \mathbb{A}_k^{gm}$, we write $k(\underline{\lambda}) = U/(u_{ij} - \lambda_{ij})$, $S_\lambda = \widetilde{S} \otimes_U k_\lambda$, $T_{\underline{\lambda}} = \widetilde{T} \otimes_U k(\underline{\lambda})$, and $P_{\underline{\lambda}} = \widetilde{P} \otimes_U k(\underline{\lambda})$. It follows from (13) that whenever $h(\underline{\lambda}) \neq 0$, then

$$(x_1,\dots,x_n)^\ell\,T_{\underline{\lambda}} \subset \mathrm{Fitt}_{s+1}(\Omega_U(\widetilde{T})) \otimes_U k(\underline{\lambda}) = \mathrm{Fitt}_{s+1}(\Omega_k(T_{\underline{\lambda}}))$$
$$(x_1,\dots,x_n)^\ell\,P_{\underline{\lambda}} \subset \mathrm{Fitt}_{s}(\Omega_U(\widetilde{P})) \otimes_U k(\underline{\lambda}) = \mathrm{Fitt}_{s+1}(\Omega_k(P_{\underline{\lambda}}))\,.$$

We conclude that locally on the punctured spectrum, $\Omega_k(T_{\underline{\lambda}})$ is generated by $s+1$ elements and $\Omega_k(P_{\underline{\lambda}})$ is generated by s elements.

On the other hand, we may assume that $\mathfrak{a} = \widetilde{\mathfrak{a}} S_\lambda$. Hence there is a natural epimorphism of S-algebras

$$T_\lambda = S_\lambda/\widetilde{K} S_\lambda \longrightarrow\!\!\!\!\!\rightarrow S_\lambda/(\widetilde{\mathfrak{a}} S_\lambda : I) = S/K\,,$$

and likewise $P_\lambda \twoheadrightarrow S/(I+K)$. Thus locally on the punctured spectrum, the modules $\Omega_k(S/K)$ and $\Omega_k(S/(I+K))$ too are generated by $s+1$ and s elements, respectively. As S/K and $S/(I+K)$ are equidimensional k-algebra of dimension $s+1$ and s, respectively, it follows that both rings are regular on the punctured spectrum, see Theorem 3.1.

For the proof of part (b), recall that $Z = X \cup Y$. Let $p \in Z$. If $p \notin Y$, then $\mathcal{O}_{Z,p} \cong \mathcal{O}_{X,p}$. If $p \notin X$, then $\mathcal{O}_{Z,p} \cong \mathcal{O}_{Y,p}$, which is regular by part (a). If $p \in X \cap Y$, then $\mathcal{O}_{Z,p}$ has at least two distinct minimal primes, hence cannot be regular. This shows that $\mathrm{Sing}(Z) = \mathrm{Sing}(X) \cup (X \cap Y)$.

By the general choice of $\mathfrak{a}$, we have $I_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$ for the finitely many prime ideals $\mathfrak{p}$ corresponding to the singular points of X (here we also use the fact that $\mathfrak{a}$ is also general in $I_\mathfrak{p}$, see [38, 2.5(a)]). So $\mathfrak{p} \not\supset \mathfrak{a} : I = K$. Thus for every $p \in \mathrm{Sing}(X)$, $p \notin Y$ and so $\mathcal{O}_{Z,p} \cong \mathcal{O}_{X,p}$. Moreover, $\mathrm{Sing}(X)$ and $X \cap Y$ are disjoint.

It remains to prove the claim about $\mathcal{O}_{Z,p}$ for $p \in X \cap Y$. By part (a) and since $p \notin \mathrm{Sing}(X)$, the rings $\mathcal{O}_{X,p}, \mathcal{O}_{Y,p}, \mathcal{O}_{X\cap Y,p}$ are regular. Write $S' = \mathcal{O}_{\mathbb{P}^{n-1},p}$, and let I', K', $\mathfrak{a}'$ be the defining ideals in S' of $\mathcal{O}_{X,p}, \mathcal{O}_{Y,p}, \mathcal{O}_{Z,p}$. It suffices to find a regular system of parameters $x_1,\dots,x_{n-1}$ of S' such that $\mathfrak{a}' = (x_1,\dots,x_{g-1}, x_g x_{g+1})$.

Recall that the ideals I' and K' have height g and are geometrically linked by $\mathfrak{a}'$. Since S'/K' is Gorenstein, we have $I'/\mathfrak{a}' \cong \omega_{S'/K'} \cong S'/K'$ is cyclic, so $g-1$ generators of $\mathfrak{a}'$ are part of a minimal generating set of I'. Call these elements $x_1,\dots,x_{g-1}$. Since S'/I' is regular, these elements are part of a regular system of parameters of S' and $I' = (x_1,\dots,x_g)$ with $x_1,\dots,x_g$ part of a regular system of parameters. Notice $\mathfrak{a}' = (x_1,\dots,x_{g-1}, y x_g)$ for some $y \in S'$. Now $K' = \mathfrak{a}' : I' = (x_1,\dots,x_{g-1}, y)$, so $I' + K' = (x_1,\dots,x_g, y)$. Since this ideal has height $g+1$ and $S'/K' + I'$ is regular, $x_1,\dots,x_g, y$ form a part of a regular system of parameters of S', as claimed. ☐

Remark 5.9. Following the approach of [8, 4.4] one sees that Theorem 5.8 still holds when Z is not necessarily defined by general forms of the same degree.

In the next theorem we assume that the curve $\mathcal{C}$ is not a smooth complete intersection because otherwise we know from Corollary 4.14 that $\operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) = a(R) + 1$.

Theorem 5.10. *Let k be an algebraically closed field of characteristic zero and $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ be a reduced curve of degree d that is locally a complete intersection. Assume that $\mathcal{C}$ is not a smooth complete intersection. Let τ and p_a denote the total Tjurina number and the arithmetic genus of $\mathcal{C}$ and let R be the homogeneous coordinate ring of $\mathcal{C}$ with maximal homogeneous ideal $\mathfrak{m}$.*

(*a*) *If R is a domain or, more generally, R has the generalized Cayley-Bacharach property, then*
$$\operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \geq \frac{d}{d-1}\, a(R) - \frac{\tau - 2}{d-1}\,.$$

(*b*) *If R is Cohen-Macaulay, then*
$$\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) \geq \frac{2p_a - \tau}{d-1}\,.$$

Proof. We deduce the theorem from Theorem 5.6. In Setting 2.7 we choose elements $f_1, \ldots, f_{n-2}$ that satisfy the conclusion of Theorem 5.8 with $X = \mathcal{C}$ and $Z = V(f_1, \ldots, f_{n-2})$. For the proof of parts (a) and (b) we are going to estimate and compute, respectively, $e(R/J)$. We write $\mathfrak{a} = (f_1, \ldots, f_{n-2})$ and $A = S/\mathfrak{a}$. By Theorem 5.8, $\operatorname{Sing}(Z) = \operatorname{Sing}(\mathcal{C}) \cup (\mathcal{C} \cap Y)$ and for every $p \in \operatorname{Sing}(Z)$ either $\mathcal{O}_{Z,p} \cong \mathcal{O}_{\mathcal{C},p}$ or else $p \in \mathcal{C} \cap Y$ and $\widehat{\mathcal{O}_{Z,p}} \cong k[\![x_1, x_2]\!]/(x_1x_2)$. In the latter case the Jacobian ideal $J_{\mathcal{O}_{Z,p}}$ of $\mathcal{O}_{Z,p}$ is the maximal ideal and therefore $\mathcal{O}_{\mathcal{C},p}/J_{\mathcal{O}_{Z,p}}\mathcal{O}_{\mathcal{C},p} \cong k$. It follows that

$$e(R/J) = \tau + \deg(\mathcal{C} \cap Y). \tag{14}$$

We are now going to estimate and compute, respectively, the degree of $\mathcal{C} \cap Y$. We write $K = \mathfrak{a} : I$ for the saturated ideal defining the link Y. The subscheme $\mathcal{C} \cap Y$ is defined by the ideal $I + K$, and $S/(I + K) \cong R/KR$, so $\deg(\mathcal{C} \cap Y) = e(R/KR)$. On the other hand, $\omega_R \cong (KR)(\delta - 2)$.

We now prove part (a). Since R has the generalized Cayley-Bacharach property, we have $\operatorname{findeg} \omega_R = \operatorname{indeg} \omega_R = -a(R)$. Therefore KR contains a homogeneous R-regular element of degree $\delta - 2 - a(R)$. Since moreover $\operatorname{ht} KR = 1$, it follows that

$$\deg(\mathcal{C} \cap Y) = e(R/KR) \leq d(\delta - 2 - a(R))\,. \tag{15}$$

Now the assertion follows by combining (15), (14), and Theorem 5.6.

To prove part (b) we write the Hilbert series of R as $H_R(t) = \frac{q(t)}{(1-t)^2}$. Since R is Cohen-Macaulay, the Hilbert series of ω_R is $H_{\omega_R}(t) = \frac{t^2 q(t^{-1})}{(1-t)^2}$. Therefore

$$H_{R/KR} = H_R - t^{\delta-2} H_{\omega_R} = \frac{Q(t)}{(1-t)^2}\,,$$

where $Q(t) = q(t) - t^{\delta} q(t^{-1})$. Since $\dim R/KR = 1$, we have

$$e(R/KR) = -Q'(1) = \delta q(1) - 2q'(1) = \delta e_0 - 2e_1\,, \tag{16}$$

where $e_0 = d$ and e_1 is the first Hilbert coefficient of R. On the other hand $p_a = e_1 - e_0 + 1$. Now the conclusion follows from (16), (14), the equality $\deg(\mathcal{C} \cap Y) = e(R/KR)$, and Theorem 5.6. ☐

To illustrate the above bounds we are going to present a family of curves $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ for which the inequality in Theorem 5.8(b) is an equality, see Proposition 5.12 and in particular part (c). We will use the following lemma:

Lemma 5.11. *Let k be a perfect field and A be a Noetherian positively graded k-algebra generated by n homogeneous elements of degrees $\delta_1, \ldots, \delta_n$ none of which is a multiple of the characteristic. Assume A is a reduced complete intersection of dimension 1. Write $\mathfrak{m}$ for the maximal homogeneous ideal, a for the a-invariant, J_A for the Jacobian ideal, and $\mathfrak{f}$ for the conductor of A.*

One has $J_A \cong \mathfrak{m}(-a)$. If in addition k is algebraically closed and A is a domain, then

$$\tau(A) = \frac{a+1}{\gcd(\delta_1, \ldots, \delta_n)} = 2\,\lambda(A/\mathfrak{f}) = 2\sigma(A)\,.$$

Proof. Write $A \cong S/(f_1, \ldots, f_{n-1})$, where $S = k[x_1, \ldots, x_n]$ is a positively graded polynomial ring with $\deg x_i = \delta_i$ and f_i are homogeneous polynomials of degree d_i. Write y_i for the image of x_i in A, Θ for the Jacobian matrix of $f_1, \ldots, f_{n-1}$ with entries in A, and $\Delta_1, \ldots, \Delta_n$ for the signed maximal minors of Θ. Since the image of the matrix Θ has rank $n-1$ over A and since both vectors

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_n \end{bmatrix}$$

are in the kernel of Θ and their images have rank 1, it follows that these vectors are proportional, by multiplication with a quotient of two homogeneous non-zerodivisors in A. On the other hand, $\mathfrak{m}$ is generated in degrees δ_i and J_A is generated in degrees $(\sum d_j - \sum \delta_j) + \delta_i = a + \delta_i$. It follows that $J_A \cong \mathfrak{m}(-a)$.

If k is algebraically closed and A is a domain, then the integral closure $\overline{A}$ is a graded polynomial ring $k[t]$, where t has degree $\gcd(\delta_1, \ldots, \delta_n)$. After regrading we may assume that this degree is 1. We may also assume that $A \neq \overline{A}$. The ring A is a monomial subalgebra of $\overline{A} = k[t]$, and computing local cohomology with support in $\mathfrak{m}$ one sees that t^a is the highest degree monomial in $\overline{A} \setminus A$. Thus $\overline{A}\, t^{a+1} = \mathfrak{f}$, and it follows that

$$a + 1 = \lambda(\overline{A}/\mathfrak{f}) = 2\,\lambda(A/\mathfrak{f}) = 2\sigma(A),$$

where the last two equalities hold because A is Gorenstein.

On the other hand, the isomorphism $J_A \cong \mathfrak{m}(-a)$ reads as $J_A = \mathfrak{m}\, t^a$. We also recall Proposition 3.5, which applies since A is a complete intersection. Thus we obtain

$$\tau(A) = \lambda(A/J_A) = \lambda(A/\mathfrak{m}\, t^a) = \lambda(A/\mathfrak{m}) + \lambda(\mathfrak{m}/\mathfrak{m}\, t^a) = a + 1,$$

as required. ☐

Proposition 5.12. *Let k be an algebraically closed field of characteristic zero and let r be a positive integer. Let $\mathcal{C} \subset \mathbb{P}^{n-1}_k$ be the curve defined by the ideal I of $S = k[x_1, \ldots, x_n]$ generated by the maximal minors of the matrix*

$$\begin{bmatrix} x_1 & x_2 & \ldots & \ldots & x_{n-2} & x_{n-1} \\ x_2^r & \ldots & \ldots & x_{n-2}^r & x_{n-1}^r & x_1^r + x_n^r \end{bmatrix}. \tag{17}$$

(*a*) *The curve $\mathcal{C}$ has degree*

$$d = r^{n-2} + r^{n-3} + \ldots + 1\,,$$

arithmetic genus

$$p_a = \frac{r}{2}((n-2)r^{n-2} - r^{n-3} - \ldots - 1) = \frac{1}{2}((n-2)\,r^{n-1} - d + 1)\,,$$

geometric genus

$$p_g = \frac{r}{2}(r^{n-2} - 1)\,,$$

total Turina number

$$\tau = r^2((n-3)r^{n-3} - r^{n-4} - \ldots - 1) = (n-3)\,r^{n-1} - d + r + 1\,,$$

and singularity degree

$$\sigma = \frac{\tau}{2} = \frac{1}{2}((n-3)\,r^{n-1} - d + r + 1)\,;$$

(*b*) *If $r > 1$, the set $\{(1 : 0 : \ldots : 0 : \rho_i) \mid \rho_i^r = -1\}$ is the singular locus of $\mathcal{C}$ and for every singular point p of $\mathcal{C}$,*

$$\widehat{\mathcal{O}_{\mathcal{C},p}} \cong k[\![\, t^{r^{n-3}}, t^{r^{n-3}+r^{n-4}}, \ldots, t^{r^{n-3}+\ldots+1} \,]\!]\,;$$

(*c*) *Let $R = S/I$ be the homogeneous coordinate ring of $\mathcal{C}$ and let y_i denote the images of x_i in R. The element*

$$\sum_{i=2}^{n-1}(r^{n-2} + \ldots + r^{n-i})\, y_i y_n^{r-1} \frac{\partial}{\partial x_i} + d\,(y_1^r + y_n^r)\frac{\partial}{\partial x_n} \in \bigoplus_{i=1}^{n} R\frac{\partial}{\partial x_i}$$

gives a minimal generator of $\operatorname{Der}_k(R)/R\varepsilon$. In particular

$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) = r - 1 = \frac{2p_a - \tau}{d-1}\,.$$

Proof. The formula for the degree of $\mathcal{C}$ follows by applying [26] to a regular sequence of n forms of degree r.

We begin by proving part (b). Observe that y_1, y_n is a regular sequence on R. To show the claim about the singular locus, let $\mathfrak{p} \in V(J_R)$ with $\dim R/\mathfrak{p} = 1$. We claim that

$$(y_2, \ldots, y_{n-1}, y_1^r + y_n^r) \subset \mathfrak{p}. \tag{18}$$

To this end we first prove that $y_1 \notin \mathfrak{p}$. Suppose $y_1 \in \mathfrak{p}$. Modulo $y_1 R$, the $(n-2)\times(n-2)$ subblock of the Jacobian matrix over R that uses the partial derivatives with respect to $x_1, \ldots, x_{n-2}$ and the minors of (17) involving the last column is an upper triangular matrix with y_n^r along the diagonal.

Hence $y_n \in \mathfrak{p}$. Now we see from (17) that $(y_2, \ldots, y_{n-1})$ is also in $\mathfrak{p}$. Thus $(y_1, \ldots, y_n) \in \mathfrak{p}$, which is impossible because $\dim R/\mathfrak{p} = 1$.

Next we show that $y_n \notin \mathfrak{p}$. One easily sees that

$$R_{y_1}/(y_n) \cong k[x_1, x_1^{-1}, x_{n-1}]/(x_{n-1}^d - x_1^d),$$

which is reduced because the characteristic is zero. Now suppose that $y_n \in \mathfrak{p}$. Since both ideals $(y_n) \subset \mathfrak{p}$ have height one and $y_n R_{\mathfrak{p}}$ is radical, it follows that $\mathfrak{p}R_{\mathfrak{p}} = (y_n)R_{\mathfrak{p}}$. So $R_{\mathfrak{p}}$ is a DVR, which is impossible since $\mathfrak{p} \in V(J_R)$.

Now, the $(n-2)\times(n-2)$ subblock of the Jacobian matrix over R that uses the partial derivatives with respect to $x_3, \ldots, x_n$ and the minors of (17) involving the first column turns out to be a lower triangular matrix with diagonal entries $ry_1y_i^{r-1}$, where $3 \leq i \leq n$. It follows that $y_1y_3\cdots y_n \in \mathfrak{p}$. Hence, as both y_1 and y_n are not in $\mathfrak{p}$, we obtain that $y_i \in \mathfrak{p}$ for some i with $3 \leq i \leq n-1$. Reducing modulo the ideal (y_i), one sees that the maximal minors of the matrix

$$\left(\begin{array}{ccccc|ccccc} y_1 & y_2 & \cdots & \cdots & y_i & 0 & y_{i+1} & \cdots & \cdots & y_{n-1} \\ y_2^r & \cdots & \cdots & y_i^r & 0 & y_{i+1}^r & \cdots & \cdots & y_{n-1}^r & y_1^r + y_n^r \end{array}\right)$$

are in $\mathfrak{p}$. Therefore $(y_2, \ldots, y_{n-1}) \in \mathfrak{p}$ and $y_1(y_1^r + y_n^r) \in \mathfrak{p}$. As $y_1 \notin \mathfrak{p}$, it follows that

$$(y_2, \ldots, y_{n-1}, y_1^r + y_n^r) \subset \mathfrak{p}\,,$$

as asserted.

Recall that $y_1 \notin \mathfrak{p}$. Claim (18) gives the containment $\operatorname{Sing}(\mathcal{C}) \subset \{(1:0:\ldots:0:\rho_i) \mid \rho_i^r = -1\}$. To prove equality and the remaining assertion of part (b) it suffices to show that for every point $p = (1:0:\ldots:0:\rho)$ with $\rho^r = -1$ one has $\widehat{\mathcal{O}_{\mathcal{C},p}} \cong k[\![\, t^{r^{n-3}}, t^{r^{n-3}+r^{n-4}}, \ldots, t^{r^{n-3}+\,\cdots\,+1}\,]\!]$. Writing $z_i = \frac{x_i}{x_1}$ for $2 \leq i \leq n$ we obtain $\mathcal{O}_{\mathcal{C},p} = k[z_2, \ldots, z_n]_{(z_2,\ldots,z_{n-1},z_n-\rho)}/H$, where H is generated by the maximal minors of the matrix

$$\begin{pmatrix} 1 & z_2 & \cdots & \cdots & z_{n-1} \\ z_2^r & \cdots & \cdots & z_{n-1}^r & z_n^r + 1 \end{pmatrix}.$$

The ideal H contains the element $z_n^r + 1 - z_2^r z_{n-1} = (z_n - \rho)v - z_2^r z_{n-1}$, where v is a unit, which shows that the maximal ideal of $\mathcal{O}_{\mathcal{C},p}$ is generated by the images of $z_2, \ldots, z_{n-1}$. Now the Cohen structure theorem gives the natural surjection

$$\varphi : B := k[\![z_2, \ldots, z_{n-1}\,]\!]/K \longrightarrow\!\!\!\!\!\rightarrow \widehat{\mathcal{O}_{\mathcal{C},p}}\,,$$

where K is the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} 1 & z_2 & \cdots & \cdots & z_{n-2} \\ z_2^r & \cdots & \cdots & z_{n-3}^r & z_{n-1}^r \end{pmatrix}.$$

On the other hand, there is a natural surjection

$$\psi : B \twoheadrightarrow C := k[\![t^{r^{n-3}}, t^{r^{n-3}+r^{n-4}}, \ldots, t^{r^{n-3}+\ldots+1}]\!] .$$

The ideal K is generated by the $n-3$ elements $z_i^r - z_2^r z_{i-1}$ for $3 \leq i \leq n-1$. It follows that z_2 is a non zerodivisor on B and that B/z_2B has multiplicity r^{n-3}. Therefore $e(B) \leq r^{n-3} = e(C)$. As B and C are Cohen-Macaulay rings of the same dimension, ψ is an isomorphism. In particular, B is a domain, which then shows that φ is an isomorphism. This completes the proof of part (b).

We now prove part (a). According to part (b) the total Tjurina number and the singularity degree of $\mathcal{C}$ are

$$\tau = r \cdot \tau(B) \quad \text{and} \quad \sigma(\mathcal{C}) = r \cdot \sigma(B) ,$$

where B is the ring defined in the proof of part (b). Write $A = k[z_2, \ldots, z_{n-1}]/K$ where K is the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} 1 & z_2 & \ldots & \ldots & z_{n-2} \\ z_2^r & \ldots & \ldots & z_{n-3}^r & z_{n-1}^r \end{pmatrix} .$$

Giving the variables z_i degree $\deg z_i := r^{n-3} + \ldots + r^{n-i-1}$, the ideal K is generated by the homogenous regular sequence $z_i^r - z_2^r z_{i-1}$, where $3 \leq i \leq n-1$. In particular

$$a(A) = \sum_{i=3}^{n-1} r \cdot \deg z_i - \sum_{i=2}^{n-2} \deg z_i = (n-3)r^{n-2} - r^{n-3} - \ldots - 1 .$$

To compute $\tau(A)$ and $\sigma(A)$ we apply Lemma 5.11. Since $\gcd(\deg z_2, \ldots, \deg z_{n-1}) = 1$, it follows that

$$\tau(A) = 2 \cdot \sigma(A) = a(A) + 1 = r((n-3)r^{n-3} - r^{n-4} - \ldots - 1) .$$

Since $B = \widehat{A}$, the asserted equality for the total Tjurina number and the singularity degree of $\mathcal{C}$ now follow.

To compute the arithmetic genus of $\mathcal{C}$, we pass to a rational curve $\mathcal{C}' \subset \mathbb{P}_k^{n-1}$ with homogenous coordinate ring R', so that R and R' have the same Hilbert function. We take $\mathcal{C}'$ to be the curve defined by the maximal minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & \ldots & \ldots & x_{n-1} \\ x_2^r & \ldots & \ldots & x_{n-1}^r & x_n^r \end{pmatrix} . \tag{19}$$

Clearly R and R' have the same Hilbert function.

We claim that $\mathcal{C}'$ is parametrized by the map $F : \mathbb{P}^1 \longrightarrow \mathbb{P}_k^{n-1}$, where

$$F = (s^{r^{n-2}+\ldots+r+1} : t^{r^{n-2}} s^{r^{n-3}+\ldots+r+1} : \ldots\ldots : t^{r^{n-2}+\ldots+r} s : t^{r^{n-2}+\ldots+r+1}) .$$

Let $C := k[s^{r^{n-2}+\ldots+r+1}, \ldots, t^{r^{n-2}+\ldots+r} s, t^{r^{n-2}+\ldots+r+1}]$ be the homogenous coordinate ring of the image of F. Since $\operatorname{im} F$ is a monomial curve, it is covered by two affine charts obtained by setting $t = 1$ or $s = 1$, respectively. If we set $t = 1$, then the affine coordinate ring is the polynomial ring

$k[s]$, which shows that this chart is smooth and F is birational onto its image (for the latter see also [32, 4.6(3)]). The other affine chart has at most one singular point, namely $(1,0,\ldots,0)$. Since the map F is birational onto its image, it follows that $\deg \operatorname{im} F = r^{n-2}+\ldots+r+1 = d$. As the multiplicity of R' is also d, the natural surjection

$$\phi : R' \longrightarrow\!\!\!\!\!\rightarrow C$$

shows that ϕ is an isomorphism. Thus $\mathcal{C}'$ is a rational curve with at most one singular point, namely, $p=(1,0,\ldots,0)$.

Now

$$p_a(\mathcal{C}) = p_a(\mathcal{C}') = p_a(\mathcal{C}') - p_g(\mathcal{C}') = \sigma(\mathcal{C}') = \sigma(\mathcal{O}_{\mathcal{C}',p})\,, \tag{20}$$

where the first equality obtains because R and R' have the same Hilbert function, the second equality holds because $\mathcal{C}'$ is rational, and the third equality follows from Proposition 3.9 since $\mathcal{C}'$ is irreducible.

To compute $\sigma(\mathcal{O}_{\mathcal{C}',p})$, we let A be the coordinate ring of the affine chart obtained by setting $s=1$ and we write $z_i = \frac{x_i}{x_1}$ for $2 \le i \le n$. Notice that $A := k[z_2,\ldots,z_n]/H$, where H is generated by the maximal minors of the matrix

$$\begin{pmatrix} 1 & z_2 & \ldots & \ldots & z_{n-1} \\ z_2^r & \ldots & \ldots & z_{n-1}^r & z_n^r \end{pmatrix}.$$

Giving the variables z_i degree $\deg z_i = t^{r^{n-2}+\ldots+r^{n-i}}$, the ideal H is generated by the homogenous regular sequence $z_i^r - z_2^r z_{i-1}$, where $3 \le i \le n$. In particular

$$a(A) = \sum_{i=3}^{n} r\cdot \deg z_i - \sum_{i=2}^{n} \deg z_i = (n-2)r^{n-1} - d\,.$$

Thus by Lemma 5.11, $\sigma(A) = \frac{a(A)+1}{2} = \frac{1}{2}((n-2)\,r^{n-1} - d + 1)$. On the other hand $\mathcal{O}_{\mathcal{C}',p} = A_{(z_2,\ldots,z_n)A}$. Now (20) gives the asserted equality for $p_a(\mathcal{C})$.

The assertion about p_g follows from Proposition 3.9 and the formulas for p_a and σ.

We now prove part (c). To see that the vector

$$\zeta = \sum_{i=2}^{n-1} (r^{n-2}+\ldots+r^{n-i})\, y_i y_n^{r-1} \frac{\partial}{\partial x_i} + d\,(y_1^r + y_n^r) \frac{\partial}{\partial x_n} \in \bigoplus_{i=1}^{n} R \frac{\partial}{\partial x_i}$$

belongs to $\operatorname{Der}_k(R)$ one has to check that ζ is in the null space of the Jacobian matrix over the ring R. To show that ζ annihilates the row corresponding to the ij minor of (17) one uses that the same minor is zero in R. Also notice that ζ is not a multiple of the Euler derivation, hence its image in $\operatorname{Der}_k(R)/R\varepsilon$ is non zero. This element is homogenous of degree $r-1$, and therefore $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \le r-1$.

On the other hand, R is a domain because R is Cohen-Macaulay and locally a domain by part (b). Therefore $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon)$. According to Theorem 5.10(b)

$$\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) \geq \frac{2p_a - \tau}{d-1} = r - 1\,,$$

where the last equality holds by part (a). Thus $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) = r-1$ and the image of ζ is a minimal generator of $\operatorname{Der}_k(R)/R\varepsilon$. □

6. Upper bounds

Recall that a Cohen-Macaulay positively graded algebra R over a field is called *nearly Gorenstein* if the homogenous maximal ideal $\mathfrak{m}$ of R is contained in the trace of ω_R, the image of the natural map $\omega_R^* \otimes \omega_R \longrightarrow R$. Clearly every Gorenstein ring is nearly Gorenstein.

Theorem 6.1. *Let k be a perfect field and $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ be a reduced curve of degree d that is arithmetically Cohen-Macaulay. Let R be the homogenous coordinate ring of $\mathcal{C}$ with maximal homogeneous ideal $\mathfrak{m}$. One has*

(*a*) $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \leq \max\{\operatorname{indeg} \omega_R^*,\, a(R)+1\}$;
(*b*) $\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) \leq \max\{\operatorname{findeg} \omega_R^*,\, a(R)+1\}$;
(*c*) *if $\mathcal{C}$ is smooth and d is not a multiple of the characteristic, then*

$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) = \max\{\operatorname{indeg} \omega_R^*,\, a(R)+1\}\,;$$

(*d*) *if R is nearly Gorenstein or, more generally, the trace of ω_R is not contained in $\mathfrak{m}^2$, then*

$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \leq a(R)+1\,;$$

(*e*) *if R is Gorenstein, then*

$$\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) \leq a(R)+1\,.$$

Proof. We prove parts (a), (b), and (c). Since $\operatorname{depth} R \geq 2$, Proposition 2.4 gives an exact sequence

$$0 \longrightarrow \operatorname{Der}_k(R)/R\varepsilon \longrightarrow H^* \longrightarrow \operatorname{Ext}^2(k,R)\,.$$

This sequence shows, in particular, that $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \geq \operatorname{indeg} H^*$. Recall from the proof of Corollary 2.10 that $\operatorname{Ext}^2_R(k,R)$ is concentrated in degrees at most $a(R)$. Since $\operatorname{depth} H^* > 0$, we conclude that

$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \leq \max\{\operatorname{indeg} H^*,\, a(R)+1\}\,,$$

and likewise for the faithful initial degree.

Now parts (a) and (b) follow because

$$H^* \hookleftarrow \omega_R^*$$

by Proposition 2.9(b).

For part (c) we first notice that $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon)$ because R is a domain and $\operatorname{Der}_k R/R\varepsilon$ is torsionfree by Proposition 2.4. Furthermore $\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) \geq a(R)+1$ by Corollary 4.13. Thus we are done if $\max\{\operatorname{findeg} \omega_R^*,\ a(R)+1\} = a(R)+1$. Otherwise, $\operatorname{Der}_k(R)/R\varepsilon \cong \omega_R^*$ by Corollary 2.10, and the assertion follows again.

(d) Assume that the trace of ω_R is not contained in $\mathfrak{m}^2$, and let $x \neq 0$ be a linear form in the trace of ω_R. There exist homogenous non-zero elements $\varphi_i \in \omega_R^*$ and $w_i \in \omega_R$ such that $x = \sum \varphi_i(w_i)$ and $\varphi_i(w_i)$ are linear forms. As $\deg w_i \geq \operatorname{indeg} \omega_R = -a(R)$, it follows that $\deg \varphi_i \leq a(R)+1$. Thus $\operatorname{indeg} \omega_R^* \leq a(R)+1$. Now the assertion follows from (a).

(e) Since R is Gorenstein, we have $\omega_R^* \cong R(-a(R))$. Thus $\operatorname{findeg} \omega_R^* = a(R)$ and the assertion follows part (b). ☐

Corollary 6.2. *Let k be an algebraically closed field and $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ be an irreducible curve of degree d. Let R be the homogenous coordinate ring of $\mathcal{C}$. Assume d is not a multiple of the characteristic. If $\mathcal{C}$ is arithmetically nearly Gorenstein and has at most ordinary nodes as singularities, then*

$$\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) = a(R)+1\,.$$

Proof. This follows from Corollary 4.12 and Theorem 6.1. ☐

Theorem 6.3. *Let k be a perfect field and $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ be a reduced curve of degree d. Let R be the homogenous coordinate ring of $\mathcal{C}$ with maximal homogeneous ideal $\mathfrak{m}$. Then*

(a) $\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon) \leq 2d-5-a(R)$;

(b) $\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) \leq d-2$ *if* $\operatorname{char} k = 0$ *and $\mathcal{C}$ is smooth and arithmetically Cohen-Macaulay.*

Proof. We may assume that k is infinite and $n \geq 3$. Let x_1, x_2, x_3 be general linear forms in R and consider the subalgebra $A = k[x_1, x_2, x_3]$ of R. Notice that $A \subset R$ is a finite and birational extension by Lemma 4.7 and that A is a hypersurface ring. Thus Theorem 6.1(d) gives

$$\operatorname{findeg}(\operatorname{Der}_k(A)/A\varepsilon_A) \leq a(A)+1\,.$$

Also observe that $e(A) = e(R)$ by Lemma 4.7, hence $a(A) = e(A)-3 = e(R)-3 = d-3$.

Now part (a) follows because

$$\operatorname{indeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) \leq \operatorname{findeg}(\operatorname{Der}_k(A)/A\varepsilon_A) + a(A) - a(R)$$

by Theorem 4.8. If the assumptions of part (b) are satisfied, then R is the integral closure of A and R is a domain. Hence every derivation of A can be extended to a derivation of R, according to [39, Theorem, page 168]. From (9) in the proof of Theorem 4.8 we see that there are embeddings

$$\operatorname{Der}_k(A)/A\varepsilon_A \longrightarrow \operatorname{Der}_k(A,R)/\mathfrak{m}^{-1}\varepsilon_A \xleftarrow{\ \varphi\ } \operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R\,.$$

Since every derivation of A can be extended to a derivation of R, the map φ is an isomorphism. Thus we obtain an embedding $\operatorname{Der}_k(A)/A\varepsilon_A \hookrightarrow \operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R$. As $\operatorname{depth} R > 0$, this embedding shows that

$$\operatorname{findeg}(\operatorname{Der}_k(R)/\mathfrak{m}^{-1}\varepsilon_R) \leq \operatorname{findeg}(\operatorname{Der}_k(A)/A\varepsilon_A)\,,$$

which proves part (b). □

Proposition 6.4. *Let k be a perfect field and $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ be a reduced curve that is arithmetically Cohen-Macaulay. Let R be the homogenous coordinate ring of $\mathcal{C}$. One has*

$$\operatorname{findeg}(\operatorname{Der}_k(R)/R\varepsilon) \leq \max\{r(R)\cdot(a(R)+2)-2\,,\,a(R)+1\}\,.$$

Proof. In view of Theorem 6.1(b) it suffices to prove that $\operatorname{findeg}\omega_R^* \leq r(R)\cdot(a(R)+2)-2$. We may assume that $n \geq 3$ and that $\mathcal{C} \subset \mathbb{P}_k^{n-1}$ is non-degenerate. We write S for the coordinate ring of $\mathbb{P}_k^{n-1}$ and consider a minimal homogeneous free S-resolution $\mathbb{F}_\bullet$ of R. Since R is Cohen-Macaulay, the resolution $\mathbb{F}_\bullet$ has length $n-2$ and F_{n-2} is generated in degrees at most $a(R)+n$. Moreover, $\operatorname{indeg} F_{n-3} \geq n-2$ because the curve is non-degenerate. It follows that the entries of φ, the last matrix in the resolution $\mathbb{F}_\bullet$, have degrees at most $a(R)+2$.

Let α be a general homogeneous minimal generator of ω_R of maximal degree. Observe that $\operatorname{ann}_R \alpha = 0$. Moreover, the graded module $\omega_R/R\alpha$ is minimally generated by $r(R)-1$ homogeneous elements and is presented by the transpose of φ, with one row removed. This is a matrix with $r(R)-1$ rows and homogeneous entries of degrees at most $a(R)+2$. The ideal $\mathfrak{a}$ of $(r(R)-1)\times(r(R)-1)$ minors of this matrix satisfies $\mathfrak{a} \subset \operatorname{ann}_R(\omega_R/R\alpha) \subset \sqrt{\mathfrak{a}}$, has positive grade, and is generated by forms of degrees at most $(r(R)-1)(a(R)+2)$. Thus, there exists a homogeneous non-zerodivisor $b \in \mathfrak{a} \subset \operatorname{ann}_R(\omega_R/R\alpha)$ with $\deg b \leq (r(R)-1)(a(R)+2)$.

Now the exact sequence

$$0 \longrightarrow \omega_R^* \longrightarrow (R\alpha)^* \longrightarrow \operatorname{Ext}^1_R(\omega_R/R\alpha, R)$$

shows that

$$b(R\alpha)^* \subset \omega_R^*\,.$$

Since $(R\alpha)^* \cong R(\deg\alpha)$ and $\deg\alpha \geq \operatorname{indeg}\omega_R = -a(R)$, it follows that $\operatorname{findeg}(R\alpha)^* \leq a(R)$. As moreover b is a non-zerodivisor, we conclude that

$$\operatorname{findeg}\omega_R^* \leq \operatorname{findeg} b(R\alpha)^* = \deg b + \operatorname{findeg}(R\alpha)^* \leq (r(R)-1)(a(R)+2)+a(R)\,,$$

as required. □

We finish this section by providing the minimal graded free resolution of the module $\operatorname{Der}_k(R)/R\varepsilon$ for the case of a smooth arithmetically Cohen-Macaulay curve in $\mathbb{P}_k^3$. From this we obtain, for instance, the initial degree, the minimal number of generators, and the entire Hilbert series of $\operatorname{Der}_k(R)/R\varepsilon$. In particular, we see that the upper bound of Theorem 6.1(d) fails dramatically without the nearly Gorenstein assumption.

Theorem 6.5. *Let k be a perfect field and $\mathcal{C} \subset \mathbb{P}_k^3$ be a curve of degree d that is smooth and arithmetically Cohen-Macaulay. Let R be the homogenous coordinate ring of $\mathcal{C}$ and $S = k[x_1,\ldots,x_4]$*

be the homogenous coordinate ring of $\mathbb{P}^3_k$. *Let*

$$F_\bullet : 0 \longrightarrow F_2 = \oplus_{j=1}^{n-1} S(-b_j) \xrightarrow{\varphi} F_1 = \oplus_{i=1}^{n} S(-a_i) \longrightarrow S$$

be the minimal homogenous S*-free resolution of* R. *We may assume that* $a_1 \leq \ldots \leq a_n$.

(*a*)

$$\operatorname{Der}_k(R)/R\varepsilon \cong \begin{cases} \mathfrak{m}(-a(R)) & \text{if } n=2 \text{ and } d \text{ is not a multiple of the characteristic} \\ \omega_R^* & n \geq 3\,. \end{cases}$$

(*b*)

$$\operatorname{indeg}(\operatorname{Der}_k(R)/R\varepsilon) = \begin{cases} a_1 + a_2 - 3 & \text{if } n=2 \text{ and } d \text{ is not a multiple of the characteristic} \\ a_1 + a_2 - 4 & n \geq 3\,. \end{cases}$$

(*c*) *If* $n \geq 3$, *then the minimal homogenous* S*-free resolution of* $\operatorname{Der}_k(R)/R\varepsilon$ *is of the form*

$$0 \longrightarrow \begin{matrix} \bigoplus\limits_{2\leq j_1 \leq j_2 \leq n-1} S(-b_{j_1} - b_{j_2} + 4) \\ \bigoplus \\ \bigoplus\limits_{1 \leq j \leq n-1} S(-b_1 - b_j + 4) \end{matrix} \longrightarrow \begin{matrix} \bigoplus\limits_{\substack{2\leq j \leq n-1 \\ 1 \leq i \leq n}} S(-b_j - a_i + 4) \\ \bigoplus \\ \bigoplus\limits_{1 \leq i \leq n} S(-b_1 - a_i + 4) \end{matrix} \longrightarrow \bigoplus_{1 \leq i_1 < i_2 \leq n} S(-a_{i_1} - a_{i_2} + 4)$$

(*d*) *Assume* $n \geq 3$ *and let* ψ *be the* $n-2$ *by* n *matrix obtained by deleting the first column of* φ. *One has* $\operatorname{ht} I_{n-2}(\psi) = 3$ *and*

$$\operatorname{Der}_k(R)/R\varepsilon \cong \omega_R^* \cong \frac{I_{n-2}(\psi)}{I_{n-1}(\varphi)}(4 - b_1)\,.$$

(*e*) *Assume* $n \geq 3$. *Write* $F_2 = F_{21} \oplus F_{22}$ *where* F_{21} *is generated by the first basis element of* F_2 *and* F_{22} *is generated by the remaining basis elements, so that* $\psi : F_{22} \longrightarrow F_1$, *and let* $\pi : F_2 \twoheadrightarrow F_{22}$ *be the natural projection. Write* $-^\vee = \operatorname{Hom}_S(-, S)$. *Set* $b = \sum\limits_{j=1}^{n-1} b_j$ *and* $b' = \sum\limits_{j=2}^{n-1} b_j$. *Consider the diagram*

$$\begin{array}{ccccccc} 0 \longrightarrow D_2(F_{22}) \otimes \bigwedge^{n} F_1^\vee(-b') & \longrightarrow & F_{22} \otimes \bigwedge^{n-1} F_1^\vee(-b') & \longrightarrow & \bigwedge^{n-2} F_1^\vee(-b') \\ & & \uparrow{\scriptstyle -\pi\otimes\delta} & & \uparrow{\scriptstyle \delta} \\ & 0 \longrightarrow & F_2 \otimes \bigwedge^{n} F_1^\vee(-b) & \longrightarrow & \bigwedge^{n-1} F_1^\vee(-b)\,, \end{array}$$

where the first and the second row are the truncated Eagon-Northcott complexes associated to the matrices ψ *and* φ, *respectively, and* δ *is the differential in the Koszul complex of the sequence consisting of the entries of the first column of* φ. *These vertical maps give a morphism of* *complexes $u_\bullet$, and the mapping cone $C(u_\bullet)$ is a minimal homogeneous S-free resolution of* $\mathrm{Der}_k(R)/R\varepsilon$.

Proof. We first prove the claim about the height and the second isomorphism in part (d). Notice that

$$F_\bullet^\vee : 0 \longrightarrow S(-4) \longrightarrow F_1^\vee(-4) \xrightarrow{\varphi^\vee} F_2^\vee(-4) = \oplus_{j=1}^{n-1} S(b_j-4)$$

is a minimal homogeneous free S-resolution of ω_R. Since ω_R is a torsionfree R-module and R is a domain, the image $\alpha \in \omega_R$ of the first basis element of $F_2^\vee(-4)$ generates a submodule $R\alpha \cong R(b_1 - 4)$. Notice that $\dim(\omega_R/R\alpha) \le 1$. As an S-module, $\omega_R/R\alpha$ is presented by the n by $n-2$ matrix $\psi^\vee$. It follows that $\operatorname{ht} I_{n-2}(\psi) = \operatorname{ht} I_{n-2}(\psi^\vee) \ge 3$. Therefore $\operatorname{ann}_S(\omega_R/R\alpha) = I_{n-2}(\psi^\vee) = I_{n-2}(\psi)$ according to [3, Theorem page 232].

On the other hand, a shift of ω_R is isomorphic to a homogenous ideal K of R. Let $\beta \in K$ be the element corresponding to α. One has

$$\begin{aligned}\omega_R^*(b_1-4) &\cong \mathrm{Hom}(\omega R, R\alpha) \cong \mathrm{Hom}(K, R\beta) \cong R\beta :_R K\\ &= \operatorname{ann}_R(K/R\beta) = \operatorname{ann}_R(\omega_R/R\alpha) = \frac{\operatorname{ann}_S(\omega_R/R\alpha)}{I}\\ &= \frac{I_{n-2}(\psi)}{I_{n-1}(\varphi)}.\end{aligned}$$

Next we prove parts (a) and (b), which will also completes the proof of (d). If $n = 2$, the assertions follow from Corollary 4.14. Hence we may assume that $n \ge 3$. The second isomorphism in (d) shows that $\operatorname{indeg} \omega_R^* = \operatorname{indeg} I_{n-2}(\psi) + b_1 - 4$. On the other hand, $\operatorname{indeg} I_{n-2}(\psi) = \sum_{j=2}^{n-1} b_j - \sum_{i=2}^{n} a_i = a_1 + a_2 - b_1$. The last equality holds because $\sum_{j=1}^{n-1} b_j = \sum_{i=1}^{n} a_i$, by the Hilbert-Burch theorem. We conclude that $\operatorname{indeg} \omega_R^* = a_1 + a_2 - 4$. Now parts (a) and (b) follow from Corollary 2.10 once we have shown that $a_1 + a_2 - 4 > a(R)$.

To this end we may assume that $b_1 \le \ldots \le b_{n-1}$. Hence $a(R) = b_{n-1} - 4$ and we need to prove that $a_1 + a_2 > b_{n-1}$. We consider the degree matrix associated to φ, which is the $n-1$ by n matrix with entries $u_{ij} = b_j - a_i$. Notice that $\varphi_{ij} = 0$ if $u_{ij} \le 0$. It easily follows that $u_{j+2,j} > 0$ for all j since I is a prime ideal (see also [22, page 3142]). As $a_2 = u_{1,n-1} + \sum_{j=1}^{n-2} u_{j+2,j}$ and $n \ge 3$, we see that $a_2 > b_{n-1} - a_1$.

We now prove (e). Since $\operatorname{ht} I_{n-2}(\psi) \ge 3$ by part (d), the two truncated Eagon-Northcott complexes are minimal homogeneous S-free resolutions of $I_{n-2}(\psi)$ and $I_{n-1}(\varphi)$, respectively. One easily checks that $u_\bullet$ is a morphism of complexes. Thus $C(u_\bullet)$ is a homogeneous S-free resolution of $\frac{I_{n-2}(\psi)}{I_{n-1}(\varphi)}$. It is minimal because the matrices of the vertical maps have entries in $\mathfrak{m}_S$. We deduce part (c) from (e), repeatedly using the equality $\sum_{j=1}^{n-1} b_j = \sum_{i=1}^{n} a_i$. □

7. The Euler derivation in the module of derivation

In proving the graded case of the Zariski-Lipman conjecture, Hochster showed that, for any Noetherian positively graded algebra R over a field of characteristic zero, the Euler derivation is a minimal generator of $\operatorname{Der}_k(R)$, unless R is a polynomial ring over a subalgebra [28, pg 412]. One may wonder whether the Euler derivation can generate a free direct summand. In this section we use the results from Section 5 to address this issue and the related question of whether the natural map $\operatorname{Der}_k(R) \to L^*$ of Proposition 2.4 can be surjective.

Proposition 7.1. *Let R be a two-dimensional Noetherian standard graded algebra over a field k, with homogeneous maximal ideal $\mathfrak{m}$, and assume that the multiplicity of R is not a multiple of the characteristic of k. If R is Gorenstein, then the natural map $\operatorname{Der}_k(R) \to L^*$ is not surjective and the Euler derivation does not generate a free direct summand of $\operatorname{Der}_k(R)$.*

Proof. Let S be a polynomial ring over k of dimension ≥ 3 with homogeneous maximal ideal $\mathfrak{m}_S$ that maps homogeneously onto R. The commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & \Omega_k(S) & \longrightarrow & \mathfrak{m}_S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & \Omega_k(R) & \longrightarrow & \mathfrak{m} & \longrightarrow & 0 \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} H^1_{\mathfrak{m}_S}(\mathfrak{m}_S) & \xrightarrow{\alpha} & H^2_{\mathfrak{m}_S}(Z) \\ \downarrow & & \downarrow \beta \\ H^1_{\mathfrak{m}}(\mathfrak{m}) & \xrightarrow{\gamma} & H^2_{\mathfrak{m}}(L)\ . \end{array}$$

Here α is an isomorphism since $\operatorname{depth}_{\mathfrak{m}_S} \Omega_k(S) \geq 3$, and β is nonzero by Theorem 5.2. We conclude that γ is nonzero.

We use the exact sequence of Proposition 2.4

$$0 \longrightarrow \operatorname{Der}_k(R)/R\,\varepsilon \xrightarrow{\delta} L^* \xrightarrow{\nu} \operatorname{Ext}^2_R(k, R) \cong \operatorname{Ext}^1_R(\mathfrak{m}, R)$$

that was also induced by the exact sequence $0 \to L \to \Omega_k(R) \to \mathfrak{m} \to 0$. Since R is Gorenstein and $\gamma \neq 0$, local duality shows that $\nu \neq 0$. Thus δ is not surjective. Moreover, $\operatorname{depth} \operatorname{Der}_k(R)/R\,\varepsilon = 1$ by the depth lemma because $\operatorname{im} \nu$ has depth zero. Thus $R\,\varepsilon$ cannot be a direct summand of $\operatorname{Der}_k(R)$, a module of depth two. □

Surprisingly, if R is Cohen–Macaulay, but not Gorenstein then the natural map $\operatorname{Der}_k(R) \to L^*$ can be surjective in dimension two. This is always the case for the coordinate rings of rational normal curves of degree $n \geq 3$ in $\mathbb{P}^n_k$, as we will see in Proposition 7.4 below.

Lemma 7.2. *Let T be a standard graded Noetherian domain with* $\operatorname{grade} T_+ \geq 2$*. Let $s \in \mathbb{N}$ be invertible in T and denote the sth Veronese functor by $-^{(s)}$. Then*

$$\operatorname{Der}_{T_0}(T^{(s)}) = (\operatorname{Der}_{T_0}(T))^{(s)}\,.$$

Proof. We may assume that $T_+ \neq 0$ and $s \geq 2$. We write $R = T^{(s)}$ and consider the exact sequence

$$T \otimes_R \Omega_{T_0}(R) \xrightarrow{\varphi} \Omega_{T_0}(T) \longrightarrow \Omega_R(T) \longrightarrow 0 \tag{21}$$

We first prove that $\operatorname{Supp}(\Omega_R(T)) \subset V(T_+)$. Let $d : T \to \Omega_{T_0}(T)$ denote the universal derivation. Let $\mathfrak{p} \in \operatorname{Spec}(T) \setminus V(T_+)$ and ℓ an arbitrary linear form in T. We need to show that $d(\ell) \in (\operatorname{im}\varphi)_{\mathfrak{p}}$. We choose $x \in T_1 \setminus \mathfrak{p}$. Since $s\,x^{s-1}d(x) = d(x^s) \in \operatorname{im}\varphi$ it follows that $d(x) \in (\operatorname{im}\varphi)_{\mathfrak{p}}$. Now the containment $(s-1)\,x^{s-2}\ell\, d(x) + x^{s-1}d(\ell) = d(\ell x^{s-1}) \in \operatorname{im}\varphi$ implies that $d(\ell) \in (\operatorname{im}\varphi)_{\mathfrak{p}}$.

Let K and L be the quotient fields of R and T, respectively. Since $\Omega_K(L) = 0$ by the above, the field extension $K \subset L$ is separable algebraic. Therefore $\varphi \otimes_T L$ is an isomorphism, which shows that $\ker\varphi$ is a torsion module. Again, since $\operatorname{Supp}(\Omega_R(T)) \subset V(T_+)$, it follows that $\operatorname{grade}\Omega_R(T) \geq 2$, hence $\operatorname{Ext}^1_T(\Omega_R(T), T) = 0$. Now, dualizing the sequence (21) into T gives the identification

$$\operatorname{Der}_{T_0}(T) = \operatorname{Der}_{T_0}(R, T)\,.$$

Thus $\operatorname{Der}_{T_0}(R) \subset \operatorname{Der}_{T_0}(T)$, and a degree argument immediately yields the desired equality. ☐

Corollary 7.3. *Let $T = T_0[z_1, \ldots, z_t]$ be a standard graded polynomial ring with $t \geq 2$, and let $s \geq 2$ be invertible in T. Then $\operatorname{Der}_{T_0}(T^{(s)})$ is the $T^{(s)}$-submodule of $\operatorname{Der}_{T_0}(T) = \oplus_{i=1}^{t}\, T\,\frac{\partial}{\partial z_i}$ minimally generated by the homogeneous elements $z_i\frac{\partial}{\partial z_j}$ for $1 \leq i, j \leq t$.*

Proof. Applying Lemma 7.2 we obtain

$$\operatorname{Der}_{T_0}(T^{(s)}) = (\operatorname{Der}_{T_0}(T))^{(s)} = T^{(s)}[\operatorname{Der}_{T_0}(T)]_0\,,$$

where the last equality holds because $\operatorname{Der}_{T_0}(T)$ is generated in degree -1 and $-s < -1 \leq 0$. ☐

The coordinate ring of the rational normal curve of degree n in $\mathbb{P}^n_k$ is of the form $R = S/I$, where $S = k[x_0, \ldots, x_n]$ and I is the ideal generated by the maximal minors of the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 & \ldots & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & \ldots & x_{n-1} & x_n \end{bmatrix}. \tag{22}$$

We write y_i for the images of x_i in R.

Proposition 7.4. *Let R be the coordinate ring of the rational normal curve of degree $n \geq 3$ in $\mathbb{P}^n_k$, and assume that the characteristic of the field k is not a divisor of n.*

(*a*) *The module* $\operatorname{Der}_k(R)$ *is the* R*-submodule of* $\operatorname{Der}_k(S,R) = \oplus_{i=0}^n R\frac{\partial}{\partial x_i}$ *minimally generated by the following* 4 *homogeneous elements of degree* 0*,*

$$\sum_{i=0}^{n-1}(n-i)\,y_i\,\frac{\partial}{\partial x_i}\;, \qquad \sum_{i=0}^{n-1}(n-i)\,y_{i+1}\,\frac{\partial}{\partial x_i}\;, \qquad \sum_{i=1}^{n} i\,y_{i-1}\,\frac{\partial}{\partial x_i}\;, \qquad \sum_{i=1}^{n} i\,y_i\,\frac{\partial}{\partial x_i}\;;$$

in particular, $\operatorname{Der}_k(R)/R\varepsilon$ *is minimally generated by three homogeneous element of degree zero.*

(*b*) $L^* \cong (y_0, y_1, y_2)(1)$.

(*c*) *The natural map* $\operatorname{Der}_k(R) \to L^*$ *is surjective.*

Proof. We consider the polynomial ring $T = k[u,v]$, where the variables u and v are given degree $\frac{1}{n}$. By mapping y_i to $u^{n-i}v^i$, one identifies R with the Veronese subring $k[\{u^{n-i}v^i\}] = \oplus_{j\in\mathbb{Z}_{\geq 0}}T_j$ of T. By Corollary 7.3 the R-submodule $\operatorname{Der}_k(R)$ of $\operatorname{Der}_k(T)$ is minimally generated by the elements of degree zero $u\frac{\partial}{\partial u}$, $v\frac{\partial}{\partial u}$, $u\frac{\partial}{\partial v}$, $v\frac{\partial}{\partial v}$.

Consider the natural map $S \to T$ of k-algebras with $x_i \mapsto u^{n-i}v^i$. It induces a T-linear map $\Omega_k(S)\otimes T \to \Omega_k(T)$ with $dx_i \mapsto (n-i)u^{n-i-1}v^i du + iu^{n-i}v^{i-1}dv$. Dualizing into T, we obtain a map $\operatorname{Der}_k(T) \to \operatorname{Der}_k(S,T)$ with $\frac{\partial}{\partial u} \mapsto \sum_{i=0}^{n-1}(n-i)\,u^{n-i-1}v^i\,\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial v} \mapsto \sum_{i=1}^{n} i\,u^{n-i}v^{i-1}\,\frac{\partial}{\partial x_i}$.

Using the identification of $\operatorname{Der}_k(R)$ as an R-submodule of $\operatorname{Der}_k(T)$ and $\operatorname{Der}_k(S,T)$,

$$\begin{array}{ccc} \operatorname{Der}_k(T) & \xrightarrow{\qquad\qquad} & \operatorname{Der}_k(S,T) \\ & \supset \qquad \subset & \\ & \operatorname{Der}_k(R) & \end{array}$$

the generators

$$u\frac{\partial}{\partial u}\;, \qquad v\frac{\partial}{\partial u}\;, \qquad u\frac{\partial}{\partial v}\;, \qquad v\frac{\partial}{\partial v}$$

of $\operatorname{Der}_k(R)$ become

$$\sum_{i=0}^{n-1}(n-i)\,y_i\,\frac{\partial}{\partial x_i}\;, \qquad \sum_{i=0}^{n-1}(n-i)\,y_{i+1}\,\frac{\partial}{\partial x_i}\;, \qquad \sum_{i=1}^{n} i\,y_{i-1}\,\frac{\partial}{\partial x_i}\;, \qquad \sum_{i=1}^{n} i\,y_i\,\frac{\partial}{\partial x_i}\;.$$

We now prove (b). We may assume that k is perfect. As the rational normal curve is smooth, Proposition 2.4 and Proposition 2.9(b) imply that $L^* \cong \omega_R^*$.

Since R is a determinantal ring, one has $\omega_R \cong (y_0,y_1)^{n-2}(n-3)$. It follows that

$$\begin{aligned} \omega_R^*(-1) &\cong y_0^{n-2}R :_R (y_0,y_1)^{n-2}R \\ &= (u^{n(n-2)}T\cap R) :_R (u^n, u^{n-1}v)^{n-2}R \\ &= (u^{n(n-2)}T :_T (u^n, u^{n-1}v)^{n-2}R)\cap R \\ &= (u^{n(n-2)}T :_T (u^n, u^{n-1}v)^{n-2}T)\cap R \\ &= (u^{n-2}T)\cap R \\ &= (y_0,y_1,y_2). \end{aligned}$$

To prove part (c) recall that according to Proposition 2.4, the natural map $\mathrm{Der}_k(R) \longrightarrow L^*$ induces an embedding $\mathrm{Der}_k(R)/R\varepsilon \hookrightarrow L^*$. Now use that $\mathrm{Der}_k(R)/R\varepsilon$ is minimally generated by 3 homogenous elements of degree zero according to (a) and L^* is generated by 3 homogeneous elements of degree zero by (b). □

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