

RESIDUAL INTERSECTIONS OF $2 \times n$ DETERMINANTAL IDEALS

DAVID EISENBUD AND BERND ULRICH

ABSTRACT. Schemes defined by residual intersections have been extensively studied in the case when they are Cohen-Macaulay, but this is a very restrictive condition. In this paper we make the first study of a class of natural examples far from satisfying this condition, the rank 1 loci of generic $2 \times n$ matrices. Here we compute their depths and many other properties. These computations require a number of novel tools.

INTRODUCTION

If $I \subset S$ is an ideal in a Noetherian ring, and $J \subset I$ is an ideal generated by s elements, then the ideal $K := J : I$ is called an *s-residual intersection* of I if $\text{codim } K \geq s$, and a *geometric s-residual intersection* of I if, in addition, the codimension of $I + K$ is at least $s + 1$. If S is Gorenstein, I is unmixed, and $s = \text{codim } I$, and $K \neq S$ then K is said to be a *direct link* of I , and in this case the properties of K are very tightly related to those of I ; for example, if S/I is Cohen-Macaulay, then so is S/K , and if the link is geometric then the canonical module of S/K is $(I+K)/K$. However, when $s > \text{codim } I$, a case of great geometric interest, the connection is not nearly as simple. For example, much stronger conditions than S/I being Cohen-Macaulay are necessary for S/K to be Cohen-Macaulay.

Residual intersections have been studied from an enumerative point of view since the work of Chasles resolving the Steiner problem ("How many plane conics are tangent to 5 general plane conics") in 1864, and reappeared independently in the work of Brill, Noether, Macaulay and others on the Riemann-Roch and Cayley-Bacharach theorems in the 1880s and 1890s, as well as in Halphen's famous 1882 classification of projective space-curves of low degree. The enumerative theory was put into rigorous modern form by Fulton and MacPherson, and codified in Fulton's monumental 1984 work [F].

Scheme-theoretic (equivalently, ideal-theoretic) results on links have been known (in special cases) since Macaulay's work in 1913 [M], and were independently and

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more generally codified in 1974 by Peskine and Szpiro in [PS]. The modern study of the ideals defining higher residual intersections was initiated by Artin and Nagata [AN], focusing on the question of when the residual intersection is Cohen-Macaulay, and this question has been studied extensively by Huneke, Ulrich and others in [H2], [HVV], [HU1], [U], [CEU], [EU1], [H], [HN], [CNT],[EHU]. A special feature of this development is that under good circumstances not only is $R = S/K$ Cohen-Macaulay, but the canonical module is $(IR)^{s-\text{codim } I+1}$, and IR generates the class group of R .

These properties hold, for example, when I is the ideal of minors of a sufficiently general $(n-1) \times n$ matrix; but they fail already for ideals as simple as the ideal of 2×2 minors of a 2×4 matrix, and for many other simple ideals.

In this paper we initiate a detailed study of the “first” case when the strong conditions fail, namely the ideal I of 2×2 minors of a generic $2 \times n$ matrix,

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \end{pmatrix}$$

for arbitrary $n \geq 4$ (the case $n = 3$ being easy and special). We compute the depths of the sufficiently general i -residual intersections R_i for $i \leq \ell(I) - 1$ (where $\ell(I)$ is the *analytic spread* of I , the dimension of the Grassmannian $\mathbf{G}(2, n)$ plus 1), the case $i \geq \ell(I)$ being trivial, and the depths of the ideals $(IR_i)^j$ for $-1 \leq j$ (Theorems 1.1 and 1.2). A table of the depths is given in Figure 1.

The importance of the ideals $(IR_i)^j$ is suggested by Theorem 2.1 which says that under additional genericity assumptions, IR_i generates the class group of R_i . Their properties are also needed in our computation of the depths of the residual intersections themselves.

Since the proofs of our main results involve a complicated induction, it seems worth pointing out some of the main new ideas of the paper. To understand the situation, recall that two technical conditions on I have played a central role in the theory of residual intersections, culminating in the paper [U]. The first, G_s , ensures that s -residual intersection $K = (a_1, \dots, a_s) : I$ will have codimension s whenever the choice of a_1, \dots, a_s is sufficiently general. This condition is trivially satisfied for the rank 1 loci of $2 \times n$ matrices.

The second condition is much more serious, and fails dramatically for our class of ideals. It is a condition on the depths of the “thickenings” S/I^j , and may be stated as

$$\text{Ext}_S^k(S/I^j, S) = 0 \quad \text{for all } j \leq s - g + 1 \text{ and } k \geq g + j,$$

where $g = \text{codim } I$.

The first advance that enables our computations is [CEU, Corollary 4.2], where we showed that the s -residual intersection of the ideal I satisfies Serre’s condition S_2 if the much weaker vanishing condition

$$\text{Ext}_S^{g+j}(S/I^j, S) = 0 \quad \text{for } 1 \leq j \leq s - g + 1$$

holds. By [RWW, Theorem 4.3] this condition is satisfied in our case.

We use this to compute the depths of the ideals $(IR_i)^j$ for large j , but need a form of duality to compute even the depth of R_i itself. For this we need to prove that the canonical module, even in this non-Cohen-Macaulay case, is given by the formula $\omega_R = (IR)^{s-g+1}$. The usual proofs do not work in our case, so instead we use our [EU1, Theorem 4.1], which gives a map $(IR)^{s-g+1} \rightarrow \omega_R$ defined under very weak hypotheses. In our case these are satisfied, and we are able to show that the map is an isomorphism under much weaker conditions than were known before.

Other important inputs to our proof include the representation theoretic analysis of free resolutions of powers of I in [RWW] and a surprising equality between certain betti numbers of S/I^j and S/I^{j+2} whose proof was shown to us by Raicu after we had discovered the phenomenon experimentally using [M2].

It may be asked why we treat the case of a $2 \times n$ matrix but not $3 \times n$ or larger matrices. The fact is that residual intersections in most of these cases simply do not exist: the Artin-Nagata condition G_s , which is an assumption on the local number of generators at various primes, is rarely satisfied in determinantal cases beyond $2 \times n$.

In [EHU] we made an analysis of the $(\ell(I) - 1)$ -residual intersections of $I_2(X)$, and various generalizations, showing in particular that for large j the module $(IR_{2n-4})^j$ is Cohen-Macaulay (a special case of Theorem 1.2(e) below). In Section 3 we give a general bound on the asymptotic depth of the powers of an arbitrary ideal I modulo its s -residual intersection, showing that except for $s = \ell(I) - 1$, the high powers are never Cohen-Macaulay modules.

In [EU2], in progress, we study the Rees algebras of the ideals $IR_i \subset R_i$.

Note on computation. The research that went into this paper was heavily dependent on computations using Macaulay2, without which we would not have guessed the rather unintuitive form of the result of the main Theorem. In making these computations, we found it useful to use a rather sparsely generated sequence of ideals J —this allowed us to make the difficult residual ideal computations in many more cases. The idea is that there is a minimal reduction of the ideal of $m \times m$ minors of a generic $m \times n$ matrix (or its initial ideal) consisting of the $\ell = m(n - m) + 1$ sums of these minors whose column numbers add to a given number (see [DEP, Theorem 6.3]). For example, for a 2×5 matrix with columns $1, \dots, 5$ the sums of minors may be represented by:

$$(1, 2), (1, 3), (1, 4) + (2, 3), (1, 5) + (2, 4), (2, 5) + (3, 4), (3, 5), (4, 5).$$

The initial subsequence of i elements of this sequence, in the case $m = 2$, generates an ideal J_i that makes the computation of the residual intersection $J_i : I$ relatively quick.

1. DEPTHS OF RESIDUAL INTERSECTIONS

We use the following notation throughout this paper: Let

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \end{pmatrix}$$

be a generic $2 \times n$ matrix over the ring $S = k[x_{1,1}, \dots, x_{2,n}]$, where k is a field of characteristic 0 and $n \geq 4$ (the case $n = 3$ being easy and special). Write \mathfrak{m} for the maximal homogeneous ideal of S , and let $I = I_2(X)$ be the ideal of 2×2 minors of X . The codimension and analytic spread of I are $g := n - 1$, $\ell := 2n - 3$ respectively. Let a_1, \dots, a_ℓ be quadrics in I that generate a minimal reduction. Set $J_i = (a_1, \dots, a_i) \subset I$ and $R_i = S/(J_i : I)$ and assume that $K_i := J_i : I$ is a geometric i -residual intersection for $i \leq \ell - 1$, which will be the case for a general choice of the a_i . We henceforth assume that $0 \leq i \leq \ell - 1$ because the ℓ -residual intersection would be primary to the homogeneous maximal ideal. We set $(IR_i)^0 := R_i$, and if $j < 0$ we write $(IR_i)^j$ for the inverse ideal $((IR_i)^{-j})^{-1} = R_i :_{\text{Quot}(R_i)} (IR_i)^{-j}$.

Theorem 1.1. *With hypotheses as above:*

- (a) *The rings R_i are Cohen-Macaulay on the punctured spectrum. They are Cohen-Macaulay if $i \leq g$ and, at least for $n \geq 5$, Buchsbaum if $i = g + 1$. If $i \geq g + 1$ then*

$$\text{depth } R_i = 2n - 3 - i = \dim R_i - 3,$$

and the local cohomology of R_i is nonzero only in cohomological degrees $\text{depth } R_i$ and $\dim R_i$.

- (b) *If $g+1 \leq i \leq \ell-2$ the ideal IR_i is unmixed of codimension 1, $\text{depth } R_i/(IR_i) \geq \text{depth } R_i$, and $\text{depth}(IR_i)^{-1}/R_i \geq \text{depth } R_i$.*
(c) *For $i \geq 0$ the canonical module of R_i is*

$$\omega_{R_i} = (IR_i)^{i-g+1}(2(i-g-1)),$$

noting that $(IR_i)^{i-g+1} = R_i$ if $i \leq g - 1$.

- (d) *For $i \leq g - 1$ the ring R_i has regularity i , for $i = g$ it has regularity $i - 2$, and for $i = g + 1 \geq 5$ it has regularity $i - 3$.*

Conjecture 1.1. *For $i \geq g + 1 \geq 5$ the regularity of R_i is $i - 3$.*

Theorem 1.2. *With hypotheses as above and $j \geq -1$:*

- (a) *$(IR_i)^j \cong I^j/J_i I^{j-1}$ for $j > 0$.*
(b) *The modules $(IR_i)^j$ are locally Cohen-Macaulay on the punctured spectrum for $j \leq i - g + 2$.*
(c) *The local cohomology of the modules IR_i , R_i and $(IR_i)^{-1}$ are 0 except at the depth, given below, and dimension, $2n - i$.*

(d) If $j \leq i - g + 2$ and one of the conditions

(i) $i \leq \ell - 2$; or

(ii) $i = \ell - 1$ and $j \geq i - g$,

is satisfied, then the multiplication maps $(IR_i)^j \otimes (IR_i)^{(i-g+1)-j} \rightarrow (IR_i)^{i-g+1}$ induce duality isomorphisms

$$\mathrm{Hom}((IR_i)^{(i-g+1)-j}, \omega_{R_i}) \cong (IR_i)^j(2(i-g-1)), \quad (1)$$

and thus $(IR_i)^j$ satisfies Serre's condition S_2 . Furthermore, if $2 \leq p \leq \dim R_i - 1$, then

$$\mathrm{H}_m^{\dim R_i + 1 - p}((IR_i)^{(i-g+1)-j})^\vee \cong \mathrm{H}_m^p((IR_i)^j)(2(i-g-1)), \quad (2)$$

where $-\vee$ denotes k -dual.

(e)

$$\mathrm{depth}(IR_i)^j = \begin{cases} \dim R_i & \text{if } j \leq 0 \text{ and } i \leq g & (3) \\ \dim R_i - 3 & \text{if } -1 \leq j \leq \min\{1, i - g - 1\} & (4) \\ n + 2 & \text{if } j = 1 \text{ and } i \leq g - 1 & (5) \\ \dim R_i & \text{if } j = 1 \text{ and } g \leq i \leq g + 1 & (6) \\ \min\{\dim R_i - 3, 4\} & \text{if } 2 \leq j \leq i - g - 1 & (7) \\ 4 & \text{otherwise.} & (8) \end{cases}$$

Remark: In the cases $g = 4, i = 4$ (linkage) and the case $g = 3, i = 2$ the ideal $(IR_i)^3$ can have an associated prime of codimension 2 in R_i (by computation in an example.) Thus the S_2 conclusion of Theorem 1.2(d) can fail outside the given range.

As a consequence, we see that $M_{i,j}$ is a maximal Cohen-Macaulay R_i -module if and only if:

- $j = 0$ and $i \leq g$; or
- $j = 1$ and $g - 1 \leq i \leq g + 1$; or
- $j \geq n - 3$ and $i = \ell - 1$.

In the proofs we will use the following statements, which are essentially in the literature:

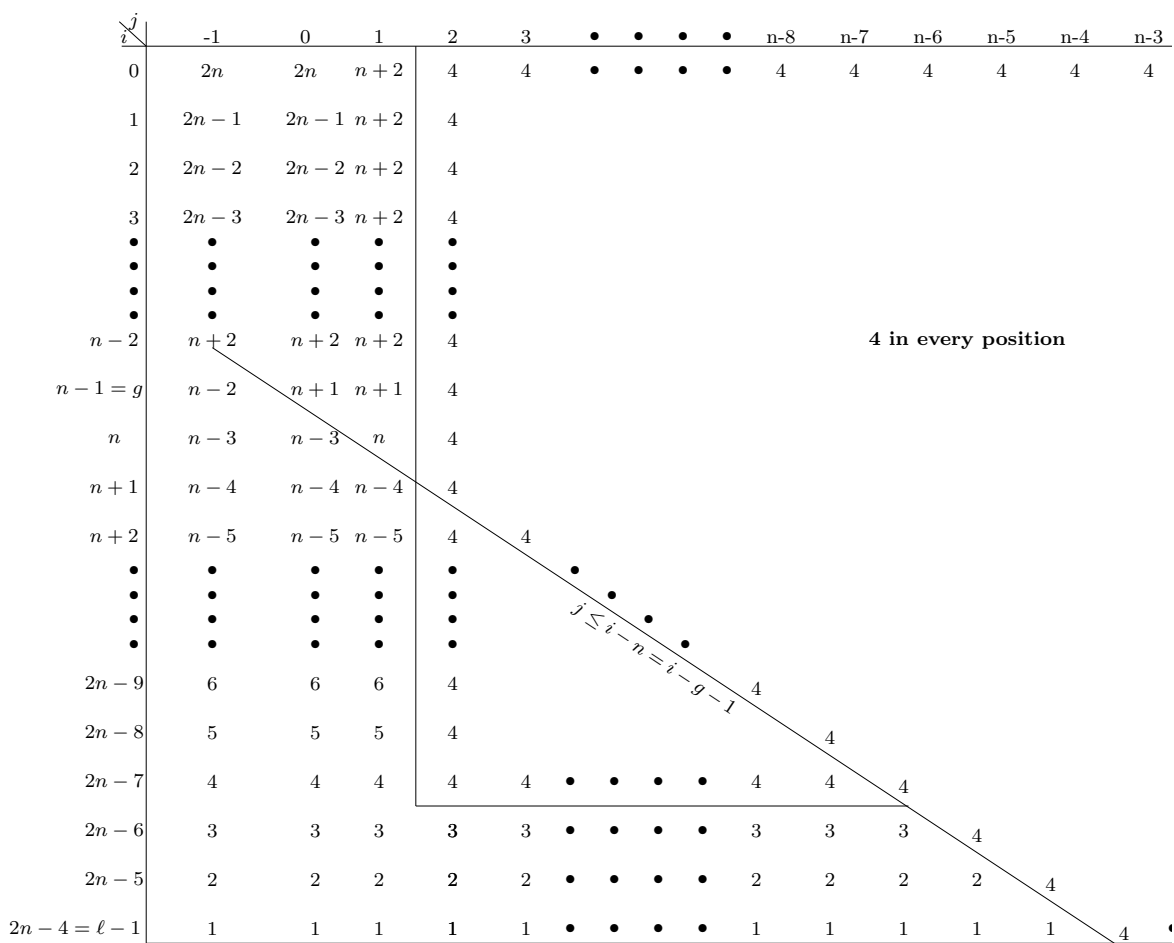
Recall that an ideal I is called weakly s -residually S_2 if, for any geometric i -residual intersection K of I with $i \leq s$, the ring S/K satisfies Serre's condition S_2 .

Also, recall that an ideal J is said to be of linear type if the natural surjections

$$\mathrm{Sym}_j(J) \twoheadrightarrow J^j$$

are isomorphisms.

FIGURE 1. Depth of $(IR_i)^j$



Proposition 1.3. *Let S be any Cohen-Macaulay ring, and $J := (a_1, \dots, a_s) \subsetneq I \subset S$ any ideals. Set $J_i := (a_1, \dots, a_i)$. Suppose that I is weakly $(s-2)$ -residually S_2 and that for $i < s$ the ideal $K_i = I : J_i$ is a geometric i -residual intersection of I . For $i < s$ we have:*

- (a) $J_i : a_{i+1} = J_i : I$.
- (b) $(J_i : I) \cap I = J_i$.
- (c) K_i is unmixed of codimension i .
- (d) The element a_{i+1} is a nonzerodivisor on S/K_i .
- (e) $J_i \cap J^j = J_i J^{j-1}$ if $j > 0$.
- (f) $J/J_i \subset S/J_i$ is an ideal of linear type.

Proof. From the assumptions we see that the ideal I satisfies the condition G_s . Because I is weakly $(s-2)$ -residually S_2 , it is $(s-1)$ -parsimonious and $(s-1)$ thrifty in the sense of [CEU, Proposition 3.1]. This means that for every $0 \leq k \leq s-1$ we have

$$\begin{aligned} J_k : a_{k+1} &= J_k : I && \text{(parsimonious)} \\ (J_k : I) \cap I &= J_k && \text{(thrifty)}. \end{aligned}$$

In particular, this proves parts (a) and (b).

Note that K_i is a proper ideal because $J \neq I$. Thus we may apply [CEU, Proposition 3.3(a)] and conclude that K_i is unmixed of codimension exactly i . Also, [CEU, Proposition 3.3(b)] is the statement (d).

We postpone the proof of part (e) to prove part (f). Combining parsimony and thrift and using $J \subset I$ we get

$$(J_k : a_{k+1}) \cap J = J_k.$$

For $0 \leq i \leq s-1$ we set $S_i = S/J_i$. For $0 \leq i \leq k \leq s-1$ we have

$$(J_k S_i : a_{k+1} S_i) \cap J S_i = J_k S_i;$$

because $J_i \subset J_k$ and $J_i \subset J$. This means that the images of a_{i+1}, \dots, a_s in S_i form a relative $*$ -regular sequence as defined in [HSV]. It follows by [HSV, Theorem 5.6] that the \mathcal{M} -complex of $J S_i$ is acyclic. Therefore by [HSV, Corollary 2.2 and Theorem 2.3] $J S_i$ is of linear type. (See also [H1, Theorem 3.1] or [V, Theorem 3.15].)

For part (e), we assume $j > 0$ and consider the diagram

$$\begin{array}{ccc} \frac{\text{Sym}_j(J)}{\text{Sym}_1(J_i) \text{Sym}_{j-1} J} & \longrightarrow & \frac{J^j}{J_i J^{j-1}} \\ \mu \downarrow & & \downarrow \beta \\ \text{Sym}_j\left(\frac{J}{J_i}\right) & \xrightarrow{\alpha} & \frac{J^j}{J_i \cap J^j}. \end{array}$$

Since μ is an isomorphism and α is an isomorphism by part (f), the map β is injective, and thus an isomorphism, as required. \square

We now return to the notation and assumptions in the first paragraph of this section.

Corollary 1.4. *In characteristic 0, I is weakly $(\ell - 2)$ -residually S_2 , and thus the conclusions of Proposition 1.3 hold for the ideal I with $s = 2n - 3$. In particular, the dimension of R_i is $2n - i$, as asserted in Theorem 1.1(a).*

Proof. Recall that $\ell := \ell(I) = 2n - 3$. By [RWW, Theorem 4.3], $\text{Ext}_S^{n+j-1}(S/I^j, S) = 0$ for $2 \leq j \leq n - 3 = (\ell - 2) - \text{codim } I + 1$ (this is where we require characteristic 0). The same formula holds for $j = 1$ because S/I is Cohen-Macaulay of codimension $n - 1$. By [CEU, Corollary 4.2], this implies that I is $(\ell - 2)$ -residually S_2 , verifying the hypotheses of Proposition 1.3. \square

Proof of Theorems 1.1 and 1.2. We begin by proving Theorem 1.2(b). The first statement of Theorem 1.1(a) is the case $j = 0$ of that statement.

The ideal I is a complete intersection on the punctured spectrum. The case $j = 0$ is the statement that R_i is Cohen-Macaulay on the punctured spectrum, proved in [H, Theorem 3.1]. The case $j > 0$ follows from [HU2, Corollary 3.10]. To handle $j = -1$ we can, on the punctured spectrum, write

$$(IR_i)^{-1} \cong \text{Hom}_{R_i}(IR_i, R_i) \cong \text{Hom}_{R_i}(IR_i \otimes \omega_{R_i}, \omega_{R_i}) \cong \text{Hom}_{R_i}((IR_i)^{i-g+2}, \omega_{R_i})$$

where the first isomorphism follows from Proposition 1.3(d), the second isomorphism uses the fact that R_i is Cohen-Macaulay on the punctured spectrum, and the third isomorphism uses the computation of the canonical module [HU1, Proposition 2.3] and the fact that the kernel of the surjection $IR_i \otimes \omega_{R_i} \rightarrow (IR_i)^{i-g+2}$ is torsion, because R_i is generically a complete intersection, completing the proof of Theorem 1.2(b).

Set

$$\begin{aligned} M_{i,j} &:= (I^j / J_i I^{j-1})(2j) \text{ if } j > 0, \\ M_{i,j} &:= (IR_i)^j(2j) \text{ if } j \leq 0. \end{aligned}$$

We will prove the statements in both Theorems together with all of the following items, by induction on i .

- (A) $H_m^p(M_{i,j}) = 0$ if $j \geq i - g$ and $5 \leq p \leq \min\{n - j + 2, \dim R_i - 1\}$.
- (B) The depth of each $M_{i,j}$ is at least what is given as the depth of $(IR_i)^j$ in Theorem 1.1(a) and Theorem 1.2(e).
- (C) The natural maps $M_{i,j} \rightarrow (IR_i)^j(2j)$ are isomorphisms.
- (D) The maps

$$M_{i,j} \xrightarrow{a_{i+1}} M_{i,j+1}$$

are monomorphisms, so that the natural sequences

$$(*_{i,j}) \quad 0 \longrightarrow M_{i,j} \xrightarrow{a_{i+1}} M_{i,j+1} \longrightarrow M_{i+1,j+1} \longrightarrow 0$$

are exact.

(E) Let $F^{i,j}$ be the minimal graded free resolution of $M_{i,j}$ over S .

(a) If $j \geq i$ then $F^{i,j}$ is linear.

(b) If $j \geq i - g$ then $F_k^{i,j}$ is generated in degree k for $k \geq i + 1$.

(c) If $j \leq i - g - 1$ then $F_k^{i,j}$ is generated in degree k for $k \geq i + 4$.

We first deal with the case $i = 0$, noting that $M_{0,j} = I^j(2j)$. Theorem 1.1 and parts (a) and (d) of Theorem 1.2 are obvious for $i = 0$. Theorem 1.2(c), and item (A) in the cases $-1 \leq j \leq 1$, follow because S and S/I are Cohen-Macaulay. The cases $j \geq 2$ of item (A) are a consequence of [RWW, Theorem 4.3]. Items (B) and (E) and Theorem 1.2(e) follow from [ABW, Theorem 5.4 and the beginning of its proof]. Items (C) and (D) are obvious for $i = 0$.

Now assume $i > 0$ and that the assertions all hold for $i - 1$.

To prove item (A) in case $j = -1$ note that i must be $\leq g - 1$, so R_i is a complete intersection of codimension i . If $i \leq g - 2$ then IR_i has codimension ≥ 2 , so $M_{i,-1} = (IR_i)^{-1}(-2) = R_i(-2)$. On the other hand if $i = g - 1$, then IR_i is a height 1 ideal and $R_i/(IR_i) = S/I$, is Cohen-Macaulay, so IR_i is a maximal Cohen-Macaulay module over the Gorenstein ring R_i . Thus its dual, which is $M_{i,-1}$ up to a twist, is again a maximal Cohen-Macaulay module. The local cohomology vanishing of part (A) follows. If $j \geq 0$ the statement of (A) follows from the long exact sequence in cohomology applied to the exact sequence $(*_{i-1,j-1})$ of (D) and item (A) for $i - 1$. This also proves Theorem 1.2(c) for $(IR_i)^j$ in case $i \leq g + j$.

We next consider item (B) for i , postponing the case $j = -1$.

First suppose $0 \leq j \leq 1$. By the induction on i , $\text{depth } M_{i-1,j-1}$ is greater than the claimed $\text{depth } M_{i,j}$ and $\text{depth } M_{i-1,j}$ is at least as large as the claimed depth , so the result follows from (D).

Next, suppose that $2 \leq j \leq i - g - 1$. The projective dimension of $M_{i-1,j}$ is $\leq 2n - \min\{2n - 3 - (i - 1), 4\} = \max\{i + 2, 2n - 4\}$ and by (E) the minimal resolution of $M_{i-1,j-1}$ is linear from step $(i - 1) + 4 = i + 3$ on. Using Lemma 1.8(a) we see that the projective dimension of $M_{i,j}$ is at most $\max\{2n - 4, i + 3\} = 2n - \min\{i - g, 4\}$, as required.

If $j \geq \max\{2, i - g\}$ then, by induction $\text{depth } M_{i-1,j} \geq 4$, so the projective dimension of $M_{i-1,j}$ is $\leq 2n - 4$. By (E) the minimal resolution of $M_{i-1,j-1}$ is linear from step $(i - 1) + 1 = i \leq 2n - 4$ on. Now again Lemma 1.8(a) implies that $\text{pd } M_{i,j} \leq 2n - 4$, as required.

We now prove items **(C)**, **(D)** for i and for all j .

For $j \leq 0$, item **(C)** is trivial. The case $j = -1$ of **(D)** follows because a_{i+1} is a nonzerodivisor on R_i by Proposition 1.3(d). The case $j = 0$ follows by Proposition 1.3(a), which implies that

$$S/(J_i : I) \xrightarrow{a_{i+1}} I/J_i$$

is a monomorphism.

Now assume that $j > 0$. We claim that

$$(**) \quad (J_i : a_{i+1}) \cap I^j = (J_i : I) \cap I^j = J_i I^{j-1}.$$

The first equality follows from Proposition 1.3(a). Since $j > 0$,

$$(J_i : I) \cap I^j = (J_i : I) \cap I \cap I^j = J_i \cap I^j,$$

where the last equality follows from Proposition 1.3(b).

Thus to complete the proof of **(**)** we must show that

$$(***) \quad J_i \cap I^j = J_i I^{j-1}.$$

We first prove equality locally on the punctured spectrum. Let $P \neq \mathfrak{m}$ be a homogeneous nonmaximal prime ideal containing I . Note that I_P is a complete intersection. Because $J := J_\ell$ is a reduction of I we see that $I_P = J_P$. Now Proposition 1.3(e) implies that $(J_i \cap I^j)_P = (J_i I^{j-1})_P$.

By definition $J_i \cap I^j / J_i I^{j-1} \subset M_{i,j}$ and $M_{i,j}$ has positive depth by **(B)** with $j \geq 0$. Since $J_i \cap I^j / J_i I^{j-1}$ is zero on the punctured spectrum, it must have finite length and thus is 0.

The second equality of **(**)** implies **(C)** for $j > 0$. The equality of the first and third terms in **(**)** implies

$$(J_i I^j : a_{i+1}) \cap I^j = J_i I^{j-1},$$

which gives **(D)** for $j > 0$. This concludes the proof of **(C)** and **(D)** for i , and with it Theorem 1.2(a).

We next prove Theorem 1.1(c) and Theorem 1.2(d). If $i \leq g - 1$ the ring R_i is a complete intersection, and the results are easy, so we may suppose $i \geq g$. By [EU1, Theorem 4.1] there is a map $\mu : M_{i,i-g+1}(-4) \rightarrow \omega_{R_i}$, which is an isomorphism on the punctured spectrum because I is a complete intersection there. Notice that $i - g + 1 \geq \max\{1, i - g\}$, and by **(B)** with $j > 0$, the depth of $M_{i,i-g+1}$ is ≥ 2 . Thus μ is an isomorphism.

Furthermore, the multiplication maps $(IR_i)^j \otimes (IR_i)^{(i-g+1)-j} \rightarrow (IR_i)^{i-g+1}$ induce maps

$$(IR_i)^j(-4) \rightarrow \text{Hom}((IR_i)^{(i-g+1)-j}, \omega_{R_i}).$$

These maps become isomorphisms locally at each prime P on the punctured spectrum: Since $\omega_{R_i} = (IR_i)^{i-g+1}$ up to shift, the map is the natural isomorphism

at P as long as $(IR_i)_P$ is principal and generated by a nonzerodivisor. By Theorem 1.2(b) it suffices, for the proof on the punctured spectrum, to prove the isomorphism for primes P containing IR_i and of codimension 1 in R_i . But $I + K_{i+1} = I + (J_{i+1} : I)$ has codimension $\geq i + 2$. Thus locally at P , IR_i is generated by a_{i+1} , which is a nonzerodivisor by Proposition 1.3(d). This completes the proof of the fact that the maps above are isomorphisms locally on the punctured spectrum.

Hence, to prove that these maps are isomorphisms globally it suffices to show that $\text{depth}(IR_i)^j \geq 2$. Suppose that $i \leq \ell - 2$. If $j \geq 0$ then the assertion follows from (B) for $j \geq 0$. As, in particular, $\text{depth} R_i \geq 2$ by (B) and IR_i contains a nonzerodivisor on R_i by Proposition 1.3(d), we conclude that the inverse ideal $(IR_i)^{-1}$ has depth at least 2, proving the assertion for $j = -1$. Finally, if $i = \ell - 1$ and $j \geq i - g$, we may again apply (B) since $j \geq 0$.

Theorem 1.2(d)(2) follows from Lemma 1.10 because R_i is equidimensional of dimension $d - i$ by Proposition 1.3(c), $(IR_i)^j$ contains a nonzerodivisor on R_i by Proposition 1.3(d), and this module is Cohen-Macaulay locally on the punctured spectrum by Theorem 1.2(b).

We can now prove (B) for $j = -1$ and Theorem 1.2(c). If $i \leq g - 1$ and $j = -1$ then $M_{i,j}$ is a maximal Cohen-Macaulay R_i -module by the argument at the beginning of the proof of (A) above. If $i \leq g - 1$ and $0 \leq j \leq 1$, Theorem 1.2(c) follows from the exact sequence

$$0 \rightarrow I/J_i \rightarrow S/J_i \rightarrow S/I \rightarrow 0.$$

Now suppose that $i \geq g$. As $i - g + 1 - j \geq i - g \geq 0$, we may use (A) and (B) in the case of exponent $i - g + 1 - j$ to say that $H_m^p((IR_i)^{i-g+1-j}) = 0$ for $5 \leq p \leq 2n - i - 1$ and that $\text{depth}(IR_i)^{i-g+1-j} \geq 4$. Thus $(IR_i)^{i-g+1-j}$ has at most one nonvanishing intermediate local cohomology module, $H_m^4((IR_i)^{i-g+1-j})$. Therefore by the duality statement, Theorem 1.2(d), equation (2), the only possible nonvanishing intermediate local cohomology modules of $(IR_i)^j$ are $H_m^q((IR_i)^j)$ for q equal to 0, 1, $\dim R_i - 3$. The first two possibilities, $q = 0$ and $q = 1$, are ruled out by Theorem 1.2(d) when $i \leq \ell - 2 = 2n - 5$. If $i = \ell - 1 = 2n - 4$, then $1 = \dim R_i - 3$ and $\text{depth}(IR_i)^j \geq 1$. The last inequality follows from (B) if $j \geq 0$ and holds for $j = -1$ because R_i has depth ≥ 1 . This concludes the proof of (B) for $j = -1$ and of Theorem 1.2(c), and this also proves the last statement of Theorem 1.1(a), for i .

Continuing the induction, we next prove (E). First suppose that $j = -1$. If $i \leq g - 1$ then $\text{depth} M_{i,-1} \geq 2n - i$, so $\text{pd} M_{i,-1} \leq i$. Thus the assertion of linearity from step $i + 1$ on is trivial. On the other hand, for $i \geq g$ we have $\text{depth} M_{i,-1} \geq 2n - 3 - i$, hence $\text{pd} M_{i,-1} \leq i + 3$ and again linearity is trivial. Finally, if $j \geq 0$ then by (E) the resolutions of $M_{i-1,j-1}$ and $M_{i-1,j}$ become linear at

least one step earlier than what is asserted for the resolution of $M_{i,j}$. Now the exact sequence $(*_{i-1,j-1})$ of (D) and the long exact sequence for Tor prove (E) for $M_{i,j}$.

This completes the proof by induction on i .

Having proven that $\text{depth } M_{i,j} = \text{depth}(IR_i)^j$ is at least the value given in Theorems 1.1(a) and 1.2(e), we now prove equality. We will handle five regions with different methods of proof.

- (A) $j \geq 2$ and $i \leq \ell - 5$ or $j \geq i - g$ and $\ell - 4 \leq i \leq \ell - 2$.
- (B) $j \leq \min\{1, i - g - 1\}$ and $i \leq \ell - 5$.
- (C) $i - g \leq j \leq 1$.
- (D) $j \leq i - g - 1$ and $i \geq \ell - 4$.
- (E) $j \geq i - g$ and $i = \ell - 1$.

Let h be the Hilbert function of the coordinate ring of the Grassmannian of 2-dimensional subspaces of k^n , and write Δ for the difference operator, so that $\Delta(h)(t) = h(t) - h(t - 1)$.

To deal with region (A) we need a result that holds in a slightly larger region:

Lemma 1.5. *With hypotheses as above, for (i, j) such that $j \geq -1$ and $i \leq \ell - 5$ or $j \geq i - g$ and $\ell - 4 \leq i \leq \ell - 2$, the socle of $H_m^4(M_{i,j})$ is concentrated in degree -4 and has k -dimension $\Delta^i(h)(j - 2)$.*

Proof of Lemma 1.5. The concentration statement follows from the fact that the module $M_{i,j}$ has $\text{depth} \geq 4$ by (B) and its resolution is linear in position $2n - 4$ by (E). For the same reason, the dimension of the socle of $H_m^4(M_{i,j})$ is equal to the $(2n - 4)$ -th Betti number of $M_{i,j}$.

We prove by induction on i that this Betti number is $\Delta^i(h)(j - 2)$. Suppose that $i = 0$. In this case $M_{i,j} = I^j$ and the assertion follows from Lemma 1.7 if $j \geq 2$ and from the inequality $\text{depth } M_{i,j} \geq 5$ if $j \leq 1$.

Next let $i > 0$. If $j = -1$ then $M_{i,j}$ has depth at least 5 by (B) and so the $(2n - 4)$ -th Betti number is 0, as claimed. If $j \geq 0$ we use the exact sequence from (D),

$$0 \longrightarrow M_{i-1,j-1} \longrightarrow M_{i-1,j} \longrightarrow M_{i,j} \longrightarrow 0.$$

The resolutions of the modules in this sequence have length at most $2n - 4$ and are linear in position $2n - 4$. Moreover by (E), the resolution of the leftmost module is linear in position $2n - 5$. Now it follows from the long exact sequence of Tor that $(2n - 4)$ -th Betti numbers are additive on this sequence, as claimed. \square

To deal with region (B) we will use

Lemma 1.6. *For (i, j) in the region (B), the module $H_m^{2n-3-i}(M_{i,j})$ is generated in degree 0 and has minimal number of generators $\Delta^i(h)(i - g - 1 - j)$.*

Proof of Lemma 1.6. We first observe that $i \geq g - 1$ and so $j \leq i - g + 2$. Hence the duality isomorphism of Theorem 1.2(d)(2) applies and gives

$$H_{\mathfrak{m}}^{2n-3-i}(M_{i,j}) \cong H_{\mathfrak{m}}^4(M_{i,i-g+1-j})^{\vee}(4).$$

Note that $(i, i - g + 1 - j)$ is in region (A), giving the desired result through Lemma 1.5. \square

We can now prove the depth equalities in all the regions. For region (A) we must show that $H_{\mathfrak{m}}^4(M_{i,j}) \neq 0$. Because the coordinate ring of the Grassmannian is Cohen-Macaulay of dimension $\ell = 2n - 3$, the value of $\Delta^i(h)(t)$ for $i \leq \ell - 1$ is the value of the Hilbert function of a ring of dimension $\ell - i > 0$ at t , which is nonzero for all $t \geq 0$. In region (A) it is clear that $i \leq \ell - 2$ and $j - 2 \geq 0$, so Lemma 1.5 concludes the proof.

For region (B) we must show that $H_{\mathfrak{m}}^{2n-3-i}(M_{i,j}) \neq 0$. Because $i - g - 1 - j \geq 0$, we may argue as above using Lemma 1.6.

In region (C) the asserted depths are equal to the dimensions, except when $j = 1$ and $i \leq g - 2$. In this case R_i has depth $2n - i \geq n + 3$ and $R_i/IR_i = R/I$ has depth $n + 1$, showing that $M_{i,1} = (IR_i)(2)$ has depth $n + 2$, as claimed.

For region (D) we do decreasing induction on i . For $i = \ell - 1$ we have, by (D) and (C), an inclusion

$$(IR_{\ell-1})^j(2j) \cong M_{\ell-1,j} \xrightarrow{a_{\ell}} M_{\ell-1,j+1} \cong (IR_{\ell-1})^{j+1}(2(j+1)).$$

Because $J = (a_1, \dots, a_{\ell})$ is a reduction of I , and I is locally a complete intersection on the punctured spectrum, it follows that $I = J$ on the punctured spectrum, and thus the cokernel of this inclusion has finite length. We know that $M_{\ell-1,j+1}$ has depth ≥ 1 by (B). Suppose that $M_{\ell-1,j}$ has depth ≥ 2 . It would follow that the inclusion above is an isomorphism. If $j = -1$ then

$$M_{\ell-1,-1}(2) = (IR_{\ell-1})^{-1} = (a_{\ell}R_{\ell-1})^{-1} \cong R_{\ell-1}(4),$$

so $R_{\ell-1} = M_{\ell-1,0}$ would also have depth ≥ 2 , and it suffices to prove the depth equalities for $j \geq 0$.

In this case and still assuming depth $M_{\ell-1,j} \geq 2$, we would have, by the isomorphism above,

$$\begin{aligned} I^{j+1} &= a_{\ell}I^j + (J_{\ell-1} : I) \cap I^{j+1} \\ &= a_{\ell}I^j + J_{\ell-1}I^j \quad \text{by (**)} \\ &= JI^j. \end{aligned}$$

This implies that the reduction number of I is $\leq (\ell - 1) - g - 1 = n - 4$; however the reduction number of I is $n - 3$ ([EHU, Proposition 4.2]), a contradiction. Thus depth $M_{\ell-1,j} = 1$ in region (D).

For the induction step we use the exact sequence $(*_i, j)$. Since $\text{depth } M_{i, j+1} \geq 2n - 3 - i$ by **(B)** and $\text{depth } M_{i+1, j+1} = 2n - 3 - i - 1$ by the induction hypothesis, we see that $\text{depth } M_{i, j} = 2n - 3 - i$ as required.

For (i, j) in region **(E)** we remark that $\dim M_{i, j} = 4$ so certainly $\text{depth } M_{i, j} \leq 4$, concluding the proof of equality.

We now show that for $n \geq 5$ the ring R_{g+1} is Buchsbaum, which is a statement in Theorem 1.1(a). Although our proof does not apply, we have verified by computation that it is also Buchsbaum in the case $n = 4$ in an example.

By Theorem 1.2(c) this ring has only one nonzero intermediate cohomology module, which is $H_m^{n-3}(R_{g+1})$ by the parts of Theorem 1.1(a) that we have already proven. We must show that this module is annihilated by \mathfrak{m} .

By the duality statement of Theorem 1.2(d) equation (2) it suffices to prove that $H_m^4((IR_{g+1})^2) \cong H_m^{n-3}(R_{g+1})^\vee$ is annihilated by \mathfrak{m} . We will show that this module is $k(1)$.

By Theorem 1.2(e) we have $\text{depth } IR_i \geq n + 1 \geq 6$ for all $i \leq g$ and thus from the exact sequences $(*_0, 1), \dots, (*_{g, 1})$ of **(D)** we get $H_m^4(I^2) \cong H_m^4((IR_{g+1})^2)$. By **(B)** and **(E)** the resolution of I^2 is linear of length $2n - 4$, and by Lemma 1.7 the last Betti number of this resolution is 1. So $\text{Ext}_S^{2n-4}(I^2, \omega_S)$ is linearly presented and generated by one element of degree one. Since I is a complete intersection on the punctured spectrum, this module has finite length, and thus must be $k(-1)$ as claimed. This completes the proof of the Buchsbaum property and hence the proof of Theorem 1.1(a).

For the regularity statement Theorem 1.1(d), first note that for $i \leq g - 1$ the ring R_i is a complete intersection of i quadrics, and thus has regularity i . If $i \geq g - 1$, then by Theorem 1.1(c) the a -invariant satisfies $a(R_i) = -4$. For $i = g$ the ring R_i is Cohen-Macaulay and so its regularity is $a(R_i) + \dim R_g = i - 2$. If $i = g + 1 \geq 5$, then by Theorem 1.1(b) the ring R_i has only one intermediate local cohomology, which has been computed above as $H_m^{n-3}(R_{g+1}) = k(1)$. Thus the regularity is $\max\{-1 + n - 3, -4 + \dim R_{g+1}\} = n - 4 = i - 4$. This completes the proof of Theorem 1.1(d).

Now we prove Theorem 1.1(b). The ideal IR_i contains a nonzerodivisor on R_i by Proposition 1.3(d) and is S_2 as a module by Theorem 1.2(d). From this it follows that IR_i is unmixed of codimension 1.

To prove the two depth inequalities, first note that by Theorem 1.2(e) the depth of the modules IR_i and $(IR_i)^{-1}$ is greater or equal to the depth of R_i . Thus it suffices to show that the natural maps

$$H_m^{\text{depth } R_i}(IR_i) \rightarrow H_m^{\text{depth } R_i}(R_i) \rightarrow H_m^{\text{depth } R_i}((IR_i)^{-1})$$

are both injective. Recall that $2 \leq \text{depth } R_i = 2n - 3 - i \leq \dim R_i - 1$ by Theorem 1.1(a). Using the duality isomorphisms in Theorem 1.2(d)(2) we see that

it is enough to prove the surjectivity of the natural maps in the bottom row of the following commutative diagram:

$$\begin{array}{ccccc}
 H_m^4(I^{i-g}) & \longleftarrow & H_m^4(I^{i-g+1}) & \longleftarrow & H_m^4(I^{i-g+2}) \\
 \downarrow & & \downarrow & & \downarrow \\
 H_m^4((IR_i)^{i-g}) & \longleftarrow & H_m^4((IR_i)^{i-g+1}) & \longleftarrow & H_m^4((IR_i)^{i-g+2}).
 \end{array}$$

By local duality the surjectivity of the maps in the top row follows from the injectivity of the natural maps $\text{Ext}^{2n-4}(I^j, S) \rightarrow \text{Ext}^{2n-4}(I^{j+1}, S)$, which is stated in [RWW, Display (4.8) in the proof of Theorem 4.5] for $j \geq 2$ and is obvious for $j \leq 1$.

We claim that the two left hand vertical maps are both surjective, which will complete the argument. This follows using the sequences $(*_{k,j})$ of (D) for $0 \leq k \leq i-1$ and $i-g-1 \leq j \leq i-g$, together with the vanishing of $H_m^5(M_{k,j})$ in that range of k, j from (A), which applies because

$$j \geq i-g-1 \geq k-g$$

and

$$\min\{n-j+2, 2n-k-1\} \geq \min\{n-i+g+2, 2n-i\} = 2n-i \geq 5.$$

This completes the proof of Theorem 1.1(b), and with it all of Theorems 1.2 and 1.1, as well as items (A)–(E). \square

We are indebted to Claudiu Raicu for providing a proof of the next result.

Lemma 1.7. *If $j \geq 2$ then the last Betti number of I^j is the minimal number of generators of I^{j-2} .*

Proof. Minimal free resolutions of I^j are computed in [ABW, Theorem 5.4] and a description of the modules in the resolutions can be obtained from [ABW, Corollary 4.13]. For a more explicit version of these resolutions and their modules see also [RW, Theorem 3.1] with $a = 2$ and [RW, Formulas (1.6), (1.4), (1.5)]. \square

Lemma 1.8. *Let $N \subset M$ be finitely generated graded modules over a Noetherian positively graded ring S with S_0 a field. Write G_\bullet, F_\bullet for the minimal graded free resolutions of N, M respectively. Suppose that M is generated in degrees ≥ 0 .*

- (a) *If, for some integer $q \geq \text{pd } M$, the module G_q is generated in degree q , then $\text{pd}(M/N) \leq q$.*
- (b) *If in addition G_{q-1} is generated in degree $q-1$, then*

$$\text{Ext}^q(M, S)_{-q} \cong \text{Ext}^q(N, S)_{-q} \oplus \text{Ext}^q(M/N, S)_{-q}.$$

Proof. Consider the diagram of a map of complexes ϕ corresponding to the inclusion $N \subset M$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & G_q & \longrightarrow & G_{q-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_q & \longrightarrow & F_{q-1} & \longrightarrow & \cdots \end{array}$$

For item (a), note that the mapping cone of ϕ is a possibly nonminimal resolution of M/N , which is generated in non-negative degrees. The module G_q is the $(q+1)$ -st term in this non-minimal resolution, but is generated in degree q ; however the minimal generators of the $(q+1)$ -st module in the minimal resolution of M/N must have degrees $\geq q+1$, so $G_q = 0$. For item (b) we note in addition that the map $G_{q-1} \rightarrow F_{q-1}$ must be the inclusion of a direct summand. Thus the q -th (last) term in the minimal resolution of M/N is F_q/G_q . \square

Remark 1.9. If G_{q-1} is generated in degrees $q-1, q$ and F_q is generated in degree q , then the q -th module in the minimal graded free resolution of M/N is generated in degree q .

Lemma 1.10. *Let R be a positively graded Noetherian ring over a field k , with maximal homogeneous ideal \mathfrak{m} , and let M be a finitely generated graded R -module of dimension d that is Cohen-Macaulay locally on the punctured spectrum and equidimensional. For $2 \leq p \leq d-1$ there are isomorphisms of graded R -modules*

$$H_{\mathfrak{m}}^p(\mathrm{Hom}(M, \omega_R)) \cong H_{\mathfrak{m}}^{d+1-p}(M)^{\vee},$$

where $-^{\vee}$ denotes k -dual.

Proof. Mapping a graded polynomial ring over k onto R and factoring out an ideal generated by a homogeneous regular sequence of maximal length contained in the annihilator of M , we may assume that R is Cohen-Macaulay of dimension d . Note that the local cohomology modules in question have finite length by our assumption on M , so the k -dual coincides with the graded k -dual. Thus by graded local duality, it suffices to show that

$$H_{\mathfrak{m}}^p(\mathrm{Hom}(M, \omega_R)) \cong \mathrm{Ext}_R^{p-1}(M, \omega_R).$$

Let \mathbb{F}_{\bullet} be a graded free resolution of M over R . Using the fact that $\mathrm{Hom}(\mathbb{F}_{\bullet}, \omega_R)$ has finite length cohomology except in cohomological degree 0, we see from the exact sequences in local cohomology of the boundaries and cycles that

$$H_{\mathfrak{m}}^p(\mathrm{Hom}(M, \omega_R)) \cong H_{\mathfrak{m}}^0(\mathrm{Ext}_R^{p-1}(M, \omega_R)).$$

Since $H_{\mathfrak{m}}^0(\mathrm{Ext}_R^{p-1}(M, \omega_R)) \cong \mathrm{Ext}_R^{p-1}(M, \omega_R)$, the assertion follows. \square

2. THE CLASS GROUP OF R_i

One reason for interest in the powers $(IR_i)^j$ is that, at least in the generic case, they constitute the class group of R_i . This is the content of the next theorem, a version of which was proved in [HU2, Theorem 3.4] under more stringent depth hypotheses on the Koszul homology of I .

Theorem 2.1. *Let*

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \end{pmatrix}$$

be a generic $2 \times n$ matrix over the ring $S = k(\{z_{p,q,r}\})[x_{1,1}, \dots, x_{2,n}]$ with $n \geq 4$, where $1 \leq p \leq \ell := 2n - 3$, and $1 \leq q < r \leq n$. Let $\delta_{q,r}$ be the minor of X involving columns q, r , and let $I = I_2(X)S$ be the ideal generated by these minors. Let $a_p = \sum_{q,r} z_{p,q,r} \delta_{q,r}$, be a generic linear combination of the minors and set $J_i = (a_1, \dots, a_i) \subset I$, $K_i = J_i : I$, and $R_i = S/K_i$, which is a geometric i -residual intersection of I .

(a) *If $g - 1 = n - 2 \leq i \leq \ell - 2$ then R_i is a normal domain, nonsingular in codimension*

$$\min\{\dim R_i - 1, i - n + 4\}.$$

(b) *The divisor class group of R_i is generated by the class of IR_i .*

(c) *For positive j the power $(IR_i)^j$ is equal to the symbolic power $(IR_i)^{(j)}$ and $(IR_i)^{-j} = (K_{i+1}R_i)^{(j)}(2j)$.*

Proof. Recall that R_i is equidimensional of codimension i by Corollary 1.4, and notice that for every $q \in V(K_i) \setminus V(IS)$ the localization $(R_i)_q = S_q/(a_1, \dots, a_i)$ is regular by the generic choice of a_1, \dots, a_i . The ring S/I has an isolated singularity, so after localizing at any prime of S other than the maximal homogeneous ideal, item (a) (except for the easy case $i = g - 1$) follows from [HU2, Lemma 2.3] and item (b) as well as (c) for $j > 0$ are a consequence of [HU2, Theorem 3.4].

It follows from (the statement or the proof of) Theorem 1.2(e) that the rings R_i and the powers $(IR_i)^j$ have depth ≥ 2 , and thus the maximal ideal of R_i is not an associated prime of $R_i/(IR_i)^j$, proving that R_i is normal and $(IR_i)^j = (IR_i)^{(j)}$ for $j > 0$.

To complete the prove of (c) we notice that $(K_{i+1}R_i)^{(j)} = a_{i+1}^j (IR_i)^{-j} \cong (IR_i)^{-j}(-2j)$. The equality holds because a_{i+1} is a nonzerodivisor on the domain R_i , the ideals on both sides are unmixed and have codimension 1 by Proposition 1.3(b), and $IR_i + K_{i+1}R_i$ has codimension ≥ 2 as the residual intersection K_{i+1} is geometric.

To prove item (b) we first show that $K_{i+1}R_i$ and IR_i are prime and IR_i has codimension 1. As to the ideal IR_i , note that it is prime of codimension 1 locally on the punctured spectrum by [HU2, Theorem 3.4(a)]. So it suffices to prove that the punctured spectrum of R/IR_i is connected and the maximal homogeneous ideal

is not an associated prime of this ring. For both these, we only need to show that $R_i/(IR_i)$ as $\text{depth} \geq 2$, which follows from Theorem 1.1(b).

For the case of $K_{i+1}R_i$, we wish to show that $R_{i+1} \cong R_i/K_{i+1}R_i$ is a domain. Since $a := a_{i+2}$ is a nonzerodivisor on R_{i+1} by Proposition 1.3(d), it suffices to show that R_{i+1} becomes a domain after inverting a . This holds because $(R_{i+1})_a = S_a/(a_1, \dots, a_{i+1})$ is a ring of fractions of a polynomial ring over $k[x_{1,1}, \dots, x_{2,n}]$ (for details see [HU2, Lemma 2.2 and the proof of Theorem 3.4(a)]).

Also, we know that $K_{i+1} \cap IR_i = (a_{i+1}R_i)$ because R_i is a normal domain and $IR_i + K_{i+1}R_i$ has codimension ≥ 2 . Now item (b) follows from Nagata's Lemma since both IR_i and $K_{i+1}R_i$ are prime ideals, IR_i has codimension 1, and the ring $(R_i)_{a_{i+1}}$ is factorial as a ring of fractions of a polynomial ring over k . \square

3. THE ASYMPTOTIC DEPTH OF $(IR_i)^j$

We will show that the maximal Cohen-Macaulay property when $i = \ell - 1$ mentioned after Theorem 1.2 is not possible for s -residual intersections with $s \neq \ell - 1$.

Proposition 3.1. *If A is a standard graded Noetherian ring over a local ring S , and M is a finitely generated graded A -module, then $\text{depth } M_j$ is constant for $j \gg 0$.*

This proposition is part of [CJR, Theorem 3.3]; we give an elementary proof.

Proof. We may assume that the residue field of S is infinite, and we proceed by induction on the minimal number of generators $\mu_S(A_1)$. If $\mu_S(A_1) = 0$ then $M_j = 0$ for $j \gg 0$.

Now suppose that $\mu_S(A_1) > 0$, and that the result fails for M . Choose $x \in A_1$ general. Truncating M , we may assume that $0 :_M x = 0$, and we let $N = M/xM$, so that we have an exact sequence

$$0 \longrightarrow M(-1) \xrightarrow{x} M \longrightarrow N \longrightarrow 0.$$

By induction, the proposition holds for N , so $\text{depth } N_j$ is constant for $j \geq n \gg 0$.

Let t be minimal so that $\text{depth } M_j = t$ for infinitely many j . It follows that there exists $j > n$ with $\text{depth } M_j = t < \text{depth } M_{j-1}$. From the exact sequence above we see that $\text{depth } N_j = t$. Thus the stable value of the depths of the components of N is t .

Let r be the second smallest value of $\text{depth } M_j$ that is taken on infinitely many times, and observe that there are then infinitely many values $j \geq n$ such that $\text{depth } M_j = r \neq \text{depth } M_{j-1}$. Since $\text{depth } N_j = t < r$, it follows that $\text{depth } M_{j-1} = t + 1$. From the definition of r , we get $r = t + 1$, contradicting the assumption that $\text{depth } M_{j-1} \neq r$. \square

Corollary 3.2. *If R is a Noetherian local ring and $I \subset R$ an ideal of positive codimension, then*

$$\ell(I) \leq \dim S - \limsup \text{depth } I^j + 1.$$

Proof. Applying the proposition to the Rees algebra \mathcal{R} of I we see that $\text{depth } I^j$ is constant for $j \gg 0$, and replacing I with a power of I we may assume that $t := \text{depth } I = \text{depth } I^j$ for all $j \geq 1$.

Let $\mathfrak{m} \subset S$ be the maximal ideal. It follows that $\text{grade}(\mathfrak{m}, \mathcal{R}) = t$, so $\text{codim } \mathfrak{m}\mathcal{R} \geq t$ and therefore

$$\ell(I) = \dim \mathcal{R}/\mathfrak{m}\mathcal{R} \leq \dim \mathcal{R} - \text{codim } \mathfrak{m}\mathcal{R} \leq (\dim S + 1) - t.$$

□

Corollary 3.3. *Let S be a Noetherian local ring with infinite residue field, and let $J \subset I \subset S$ be ideals. Suppose that J is generated by s general elements of I and set $R := S/(J : I^\infty)$. If $R \neq 0$ and $(IR)^j$ is a maximal Cohen-Macaulay R -module for infinitely many j , then $s = \ell(I) - 1$.*

Proof. Notice that $s \leq \ell(I) - 1$ because otherwise $R = 0$. Since $\text{grade } IR > 0$ it follows from Corollary 3.2 that $\ell(IR) \leq 1$. By the Artin-Rees Lemma $(J : I^\infty) \cap I^j = J \cap I^j$ for $j \gg 0$ and so the large powers of IR and of $I(S/J)$ coincide. This shows that $\ell(IR) = \ell(I(S/J))$. On the other hand, since J is generated by s general elements of I , [SH, Proposition 8.6.1] gives $\ell(I) \leq s + \ell(I(S/J))$, which shows that $\ell(I) \leq s + 1$ as required. □

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Author Addresses:

David Eisenbud

Mathematical Sciences Research Institute, Berkeley, CA 94720, USA

de@msri.org

Bernd Ulrich

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

bulrich@purdue.edu