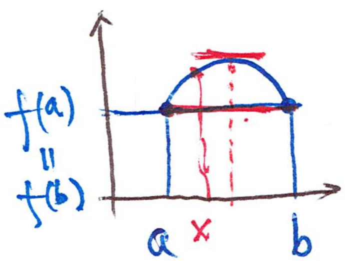
§4.2 The Mean Value Theorem

Rolle's Theorem

Assume that
$$\begin{cases} (1)\ f \text{ is continuous on } [a,b] \\ (2)\ f \text{ is differentiable on } (a,b) \\ (3)\ f(a)=f(b) \end{cases} \Longrightarrow \exists\, c \in (a,b) \text{ such that } f'(c)=0$$



Proof

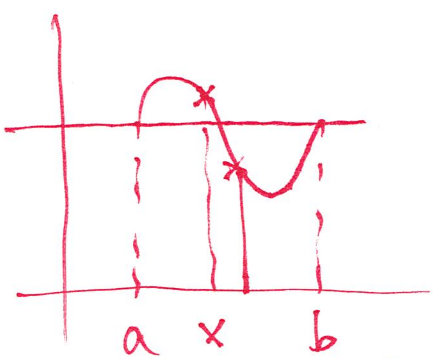
Case (a) $f(x)=k$ constant $\Longrightarrow f'(x)=0$

Case (b) For some $x\in(a,b)$, $f(x)>f(a)=f(b)$

f is continuous on $[a,b] \Longrightarrow \exists\, c\in[a,b]$ s.t. $f(c)$ is the abs. max. $\Longrightarrow f'(c)=0$

$\Longrightarrow f(c)\geq f(x)>f(a)=f(b)$

$\Longrightarrow c\neq a \text{ or } b \Longrightarrow c\in(a,b)$



Case (c) For some $x\in(a,b)$, $f(x)<f(a)$

Ex. 1 A ball is thrown directly upward.

$s = f(t)$ — position of a moving object.

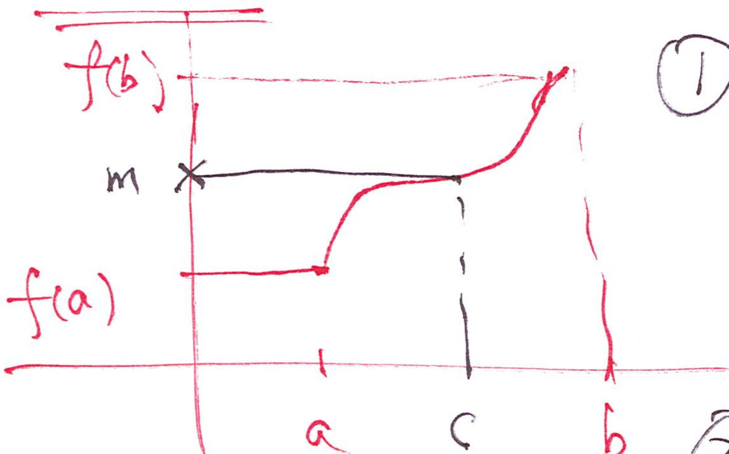
$f(a)=f(b) \Longrightarrow \exists\, c \in (a,b)$ s.t. $f'(c)=0$ — velocity.

Ex. 2 Prove that the equation $x^3+x-1=0$ has exactly one real root.

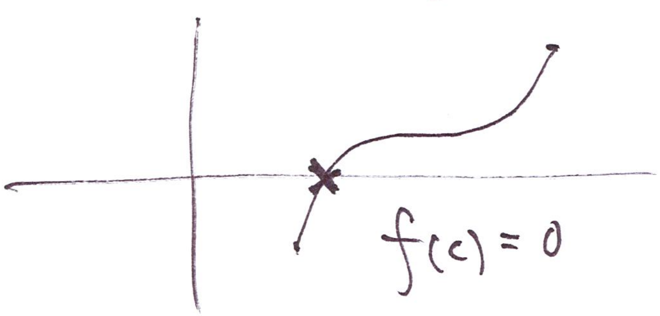
$\parallel$

$f(x)$

IVT



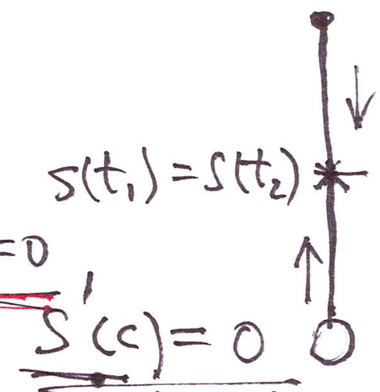
$f(c) = m$



① $f(0) = -1 < 0$

$f(1) = 1 > 0$

$\exists\, c \in (0,1)$ s.t. $f(c)=0$,



t_1, t_2

② $f'(x) = 3x^2+1 \geq 1 > 0$

Assume that it has at least 2 solutions:

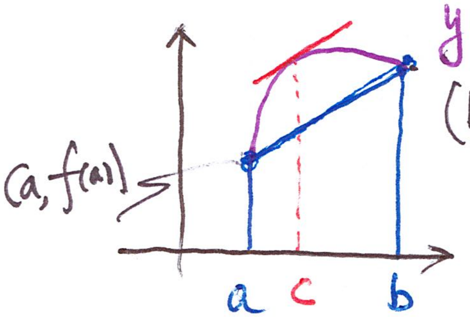
$\exists\, c_1$, and c_2 s.t. $f(c_1)=0$, $f(c_2)=0$

$\Rightarrow \exists\, c$ s.t. $f'(c)=0$

$f'(x) = 3x^2+1 > 0$ /

The Mean Value Theorem

Assume that $\begin{cases} (1)\ f \text{ is continuous on } [a,b] \\ (2)\ f \text{ is differentiable on } (a,b) \end{cases} \Longrightarrow \exists\, c \in (a,b)$ such that $\boxed{f'(c) = \dfrac{f(b)-f(a)}{b-a}}$



Proof (making use of Rolle's Thrm)

$(b, f(b))$ the equation of the secant line: $y = f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$

an auxiliary function: $h(x) = f(x) - \left[f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)\right]$

$$\frac{y-f(a)}{f(b)-f(a)} = \frac{x-a}{b-a}$$

$h(a) = 0 = h(b) = f(b) - [f(a) + (f(b)-f(a))] = 0$

$\Rightarrow \exists\, c \in (a,b)$ s.t. $0 = h'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$

Ex. 3 Use $f(x) = x^3 - x$ on $[0,2]$ to illustrate the Mean Value Thrm.

$f'(x) = 3x^2 - 1$, $\dfrac{f(2)-f(0)}{2-0} = \dfrac{2^3-2}{2} = \boxed{3 = f'(c) = 3c^2 - 1}$

$3c^2 = 4 \Longrightarrow c^2 = \dfrac{4}{3} \Rightarrow c = \pm\sqrt{\dfrac{4}{3}} = \pm\dfrac{2}{\sqrt{3}} \Longrightarrow c = \dfrac{2}{\sqrt{3}} \in (0,2)$

Ex. 4 Application of MVT to a moving object.

Ex. 5 Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x.

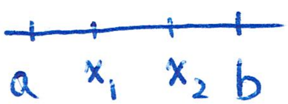
How large can $f(2)$ possibly be?

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \le 5 \Rightarrow f(2) \le f(0) + 10$$

$$= -3 + 10 = 7$$

Theorem $f'(x) = 0 \ \forall x \in (a,b) \Longrightarrow f$ is constant on (a,b)

Proof. $\forall x_1, x_2 \in (a,b)$ and $x_1 < x_2$



$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0 \Longrightarrow f(x_2) - f(x_1) = 0$$

Corollary $\boxed{f'(x) = g'(x)} \ \forall x \in (a,b) \Longrightarrow f(x) = g(x) + c$, where c is a constant.

$$0 = f'(x) - g'(x) = (f - g)' \Rightarrow f - g = c$$

Ex. 6 Prove the identity $\underbrace{\tan^{-1}x + \cot^{-1}x}_{= f(x)} = \frac{\pi}{2}$ by using calculus.

Want $0 \overset{?}{=} f'(x) \iff \underline{f(x) = C}$

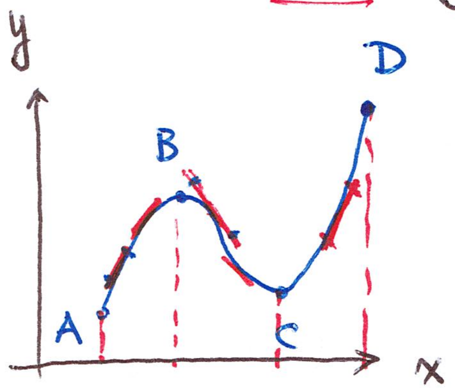
$$f'(x) = \left(\tan^{-1}x + \cot^{-1}x\right)' = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

$$\tan^{-1}x + \cot^{-1}x = \underline{C} = \frac{\pi}{2}$$

$$\tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

§4.3 How Derivatives Affect the Shape of a Graph

• what does $f'(x)$ say about f?



Increasing / Decreasing Test

(a) $f'(x) > 0$ on an interval $I \Longrightarrow f(x) \nearrow$

(b) $f'(x) < 0$ on an interval $I \Longrightarrow f(x) \searrow$

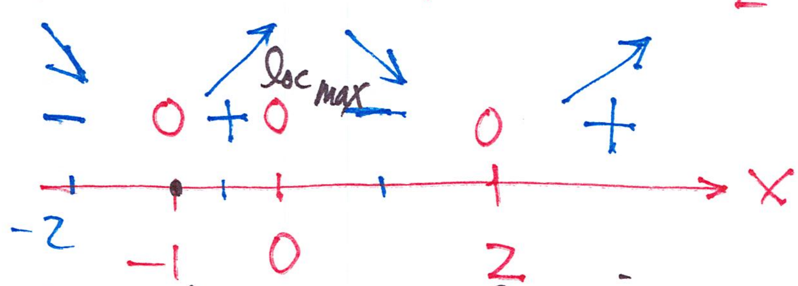
Proof of (a) $\forall\, x_1, x_2 \in I$ and $x_1 < x_2$

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0 \Rightarrow f(x_2) > f(x_1)$$

Ex. 1 Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

$$\boxed{0 = f'(x)} = 12x^3 - 12x^2 - 24x = 12x\left[x^2 - x - 2\right] = 12x(x-2)(x+1)$$

$\Longrightarrow x = 0,\ 2,\ -1$



$f(-1)$ loc. min   loc min

$f(-1) = 3 + 4 - 12 + 5 = 0.$ l. min

$f(0) = 5$ l. max

$f(2) = 2 \cdot 8 - 12 \cdot 4 + 5$
$= -27.$ l. min

The First Derivative Test

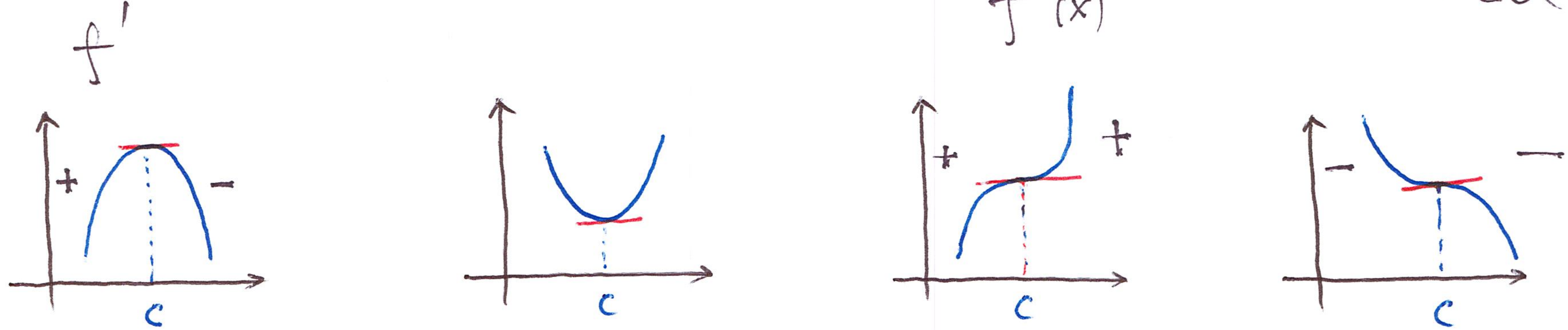
Assume that c is a critical number of a continuous function f.

(a) f' changes from positive to negative at $c \Longrightarrow f(c)$ is a local max

(b) f' changes from negative to positive at $c \Longrightarrow f(c)$ is a local min

(c) f' does not change sign at $c \Longrightarrow f(c)$ is neither a local max nor a local min

$f'(x)$



Ex. 2 Find the local minimum and maximum values of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Ex. 3 Find the local maximum and minimum values of $g(x) = x + 2\sin x$ on $[0, 2\pi]$.

$$\boxed{0 = g'(x)} = 1 + 2\cos x \Rightarrow \cos x = -\frac{1}{2}$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

x	0		$\frac{2\pi}{3}$		$\frac{4\pi}{3}$		2π
$g'(x)$		$+$	0	$-$	0	$+$	
			l. max		l. min		

$$g\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3}$$