Chapter 5 Linear Systems of Differential Equations

(sections 5.1, 5.2, 5.5, 5.3)

§5.1 Matrices and Linear Systems

- Review of Matrix Notation and Terminology

$$A_{m\times n} = (a_{ij})_{m\times n}\ ,\quad B_{m\times n} = (b_{ij})_{m\times n}$$

$$A_{m\times n} + B_{m\times n} = (a_{ij} + b_{ij})_{m\times n}\ ,\quad (A^T)_{n\times m} = (a_{ji})_{n\times m}$$

- Matrix Multiplication

$$A_{m\times p}\, B_{p\times n} = C_{m\times n} = (c_{ij})_{m\times n}\ ,\quad c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

in general $\quad A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$

- Inverse Matrices

$A_{n\times n}$ is nonsingular $\Longleftrightarrow$ $\exists\, B_{n\times n}$ such that $AB = BA = I$

$\Longleftrightarrow \det(A) = |A| \neq 0$

determinant $= ?$

Ex.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = ?$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\quad A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- First-Order Linear Systems

$$\frac{d\vec{x}(t)}{dt} = P(t)\vec{x}(t) + \vec{f}(t) \qquad (28)$$

associated homogeneous equation $$\frac{d\vec{x}(t)}{dt} = P(t)\vec{x}(t) \qquad (29)$$

Ex. 6 $\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 6x_1 - 7x_2 \end{cases}$ $\vec{x}(t) = ?$ $P(t) = ?$

$\vec{x}_1 = \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$ are solutions.

- Theorem 1 (Principle of Superposition)

Let $\vec{x}_1, \cdots, \vec{x}_n$ be n solutions of (29) on the open interval I.

$\Longrightarrow$ $\vec{x}(t) = \sum_{i=1}^{n} c_i \vec{x}_i$ is also a solution of (29).

Proof

- Independence and General Solutions

Def. $\vec{x}_1(t), \cdots, \vec{x}_n(t)$ are linearly dependent on the interval I.

$\Longleftrightarrow$ $\exists$ const. $c_1, \cdots, c_n$ not all zero such that

$$\sum_{i=1}^{n} c_i \vec{x}_i(t) = 0 \quad \forall\, t \in I.$$

Theorem 2 (Wronskians of Solutions)

Assume that $\vec{x}_1(t), \dots, \vec{x}_n(t)$ are n solutions of (29) on an open interval I and that $P(t)$ is continuous on I. Let

$$W = W(\vec{x}_1, \dots, \vec{x}_n) = \det(\vec{x}_1, \dots, \vec{x}_n).$$

$\Longrightarrow$ (1) "$\vec{x}_1, \dots, \vec{x}_n$ are l. dep. on I $\Longrightarrow$ $W=0$ at every pt in I"

(2) "$\vec{x}_1, \dots, \vec{x}_n$ are l. indep. on I $\Longrightarrow$ $W \neq 0$ at every pt in I".

Theorem 3 General Solutions of (29)

Let $\vec{x}_1, \dots, \vec{x}_n$ be n l. indep. solutions of (29) on an open interval I.
Assume that $P(t)$ is cont. on I.

$\Longrightarrow$ "if $\vec{x}(t)$ is a solution of (29) $\Longrightarrow$ $\exists\, c_i$ s.t. $\vec{x}(t) = \sum_{i=1}^{n} c_i \vec{x}_i(t)$".

- Initial Value Problem

$$\begin{cases} \dfrac{d\vec{x}(t)}{dt} = P(t)\vec{x}(t) \\ \vec{x}(a) = \vec{b} \end{cases} \Longrightarrow \vec{b} = \vec{x}(a) = \sum_{i=1}^{n} c_i \vec{x}_i(a) = \left(\vec{x}_1(a), \dots, \vec{x}_n(a)\right)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$\boxed{\vec{b} = X(a)\vec{c}}$$

- Nonhomogeneous Solutions

the general solution of (28)

$$\vec{x}(t) = \underbrace{\vec{x}_c(t)}_{\text{general solution of (28)}} + \underbrace{\vec{x}_p(t)}_{\text{a particular solution of (29)}}$$

Examples

#13 $\begin{cases} x' = 2x + 4y + 3e^t = (2, 4)\begin{pmatrix} x \\ y \end{pmatrix} + 3e^t \\ y' = 5x - y - t^2 = (5, -1)\begin{pmatrix} x \\ y \end{pmatrix} - t^2 \end{cases} = \begin{pmatrix} 2 & 4 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3e^t \\ -t^2 \end{pmatrix}$

write in the form $\vec{x}' = P\vec{x} + \vec{f}(t)$ $\qquad \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

#24 $\vec{x}' = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}\vec{x}$, $\vec{x}_1 = e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{x}_2 = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

- verify they are solutions LHS $= \vec{x}_1' = 3e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, RHS $= \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{3t}\begin{pmatrix} 4-1 \\ -2-1 \end{pmatrix}$ $= 3e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} =$ LHS

- l. indep. $|W| = \begin{vmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{vmatrix} = -2e^{5t} + e^{5t} = -e^{5t} \neq 0$

- general solution $\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$x_1(0) = 11$, $x_2(0) = -7$

- (#33) initial condition $\boxed{\vec{x}(0) = \begin{pmatrix} 11 \\ -7 \end{pmatrix}} = \begin{pmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{pmatrix}_{t=0}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$x_1(0) = 11$, $x_2(0) = -7$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 11 \\ -7 \end{pmatrix} = \frac{1}{(-1)}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 11 \\ -7 \end{pmatrix} = \begin{pmatrix} 15 \\ -4 \end{pmatrix}$$

the solution $\vec{x}(t) = 15\begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix} - 4\begin{pmatrix} e^{2t} \\ -2e^{2t} \end{pmatrix} = \begin{pmatrix} 15e^{3t} - 4e^{2t} \\ -15e^{3t} - 8e^{2t} \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

§5.2 The Eigenvalue Method for Homogeneous System

$$\underset{n\times 1}{\vec{x}'} = \underset{n\times n}{A}\, \vec{x}_{n\times 1}$$

A is a const. matrix

$$\boxed{\text{find } n \text{ l. indep. solutions } \vec{x}_1(t), \cdots, \vec{x}_n(t)}$$

where to start ?

solutions of $a y''(x) + b y'(x) + c y(x) = 0$

$$\boxed{\vec{x}(t) \overset{?}{=} e^{\lambda t} \vec{v}}$$

$$\vec{x}'(t) = \lambda e^{\lambda t} \vec{v} \xrightarrow{\vec{x}' = A\vec{x}} \underline{\lambda e^{\lambda t}} \vec{v} = A e^{\lambda t} \vec{v} = \underline{e^{\lambda t} A \vec{v}}$$

$$\Longrightarrow \boxed{A\vec{v} = \lambda \vec{v}}$$

λ — eigenvalue of A

$\vec{0} \neq \vec{v}$ — eigenvector of A

$\boxed{\text{How to compute } \lambda \text{ and } \vec{v} \text{ ?}}$

(1) find λ such that $|A - \lambda I| = 0$.

(2) find $\vec{v}$ such that $(A - \lambda I)\vec{v} = 0$.

- Distinct Real Eigenvalues

example 1

$$\begin{cases} x_1' = 4x_1 + 2x_2 \\ x_2' = 3x_1 - x_2 \end{cases} \Longrightarrow \vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$\vec{x}_1 = ?$, $\vec{x}_2 = ?$, $W(\vec{x}_1, \vec{x}_2) = ?$, directional field,

(1) Eigenvalues

$$0 = |A - \lambda I|$$
$$= \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix}$$
$$= (\lambda-4)(\lambda+1) - 6$$
$$= \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5)$$
$$\Rightarrow \boxed{\lambda_1 = -2, \ \lambda_2 = 5}$$

(2) Eigenvectors

$\lambda_1 = -2$ $\qquad$ $\lambda_2 = 5$ $\ \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$0 = (A + 2I)\vec{v}_1$$
$$= \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$
$$0 = 3a + b$$
$$\Rightarrow \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$$

(3) general solution

$$\Rightarrow \vec{x}(t) = e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$|W| = |\vec{x}_1(t), \vec{x}_2(t)|$$
$$= \begin{vmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{vmatrix}$$
$$= e^{3t} + 6e^{3t} = 7e^{3t} \neq 0$$

example 2

$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad\quad k_2 x_2 - k_3 x_3 \end{cases}$$

initial condition

$x_1(0) = 15$

$x_2(0) = x_3(0) = 0$

$k_i = \dfrac{r}{V_i}$

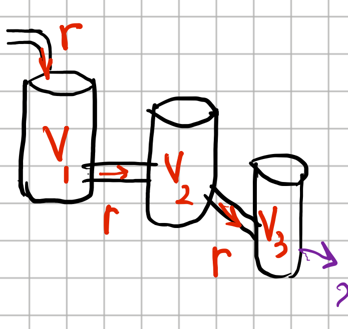
$r = 10$ gal/min

$V_1 = 20$, $V_2 = 40$, $V_3 = 50$

Find the amount of salt in each tank, $x_i(t) = ?$

$k_1 = 0.5$, $k_2 = 0.25$, $k_3 = 0.2$

$$\vec{x}'(t) = \begin{vmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{vmatrix} \vec{x}(t)$$



- eigenvalues

$$0 = |A - \lambda I| = \begin{vmatrix} -k_1-\lambda & 0 & 0 \\ k_1 & -k_2-\lambda & 0 \\ 0 & k_2 & -k_3-\lambda \end{vmatrix} = -(\lambda+k_1)\begin{vmatrix} -k_2-\lambda & 0 \\ k_2 & -k_3-\lambda \end{vmatrix}$$

$$= -(\lambda+k_1)(\lambda+k_2)(\lambda+k_3)$$

$$\Rightarrow \lambda_1 = -k_1, \ \lambda_2 = -k_2, \ \lambda_3 = -k_3$$

• eigenvector

$$\boxed{\lambda_1 = -k_1 \qquad \vec{v}_1 = ?}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ k_1 & -k_2+k_1 & 0 \\ 0 & k_2 & -k_3+k_1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} k_1 a + (k_1 - k_2) b = 0 \\ k_2 b + (k_1 - k_3) c = 0 \end{cases}$$

$$b = 1 \Longrightarrow a = \frac{k_1}{k_2 - k_1}, \quad c = \frac{k_2}{k_3 - k_1}$$

$$\vec{v}_1 = \begin{pmatrix} \frac{k_1}{k_2-k_1} \\ 1 \\ \frac{k_2}{k_3-k_1} \end{pmatrix}, \quad \vec{x}_1 = e^{-k_1 t}\begin{pmatrix} \frac{k_1}{k_2-k_1} \\ 1 \\ \frac{k_2}{k_3-k_1} \end{pmatrix}$$

$\vec{x}_2(t) = ?$, $\vec{x}_3(t) = ?$

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2 + c_3 \vec{x}_3$$

• Complex Eigenvalues

eigenvalues $\lambda = p \pm q i$

eigenvectors $\vec{v} = \vec{a} \pm \vec{b} i$

solution
$$\vec{x}(t) = e^{(p+qi)t}(\vec{a} + i\vec{b})$$
$$= e^{pt}(\cos qt + i \sin qt)(\vec{a} + i\vec{b})$$
$$= e^{pt}\left\{(\vec{a}\cos qt - \vec{b}\sin qt) + i(\vec{b}\cos qt + \vec{a}\sin qt)\right\}$$

$$\Rightarrow \vec{x}_1(t) = R_e[\vec{x}(t)] = e^{pt}(\vec{a}\cos qt - \vec{b}\sin qt)$$
$$\vec{x}_2(t) = I_m[\vec{x}(t)] = e^{pt}(\vec{a}\sin qt + \vec{b}\cos qt)$$

example 3

$$\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 3x_1 + 4x_2 \end{cases}$$

$$0 = |A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (\lambda - 4)^2 + 3^2$$

$$(\lambda - 4)^2 = -3^2$$

$$\lambda - 4 = \pm\sqrt{-3^2} = \pm 3i$$

$$\Rightarrow \lambda = 4 \pm 3i$$

$\lambda = 4 + 3i$

$4 - \lambda = -3i$

$$\begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-ia - b = 0 \Rightarrow b = -ai$$

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -ai \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -ai \end{pmatrix}$$

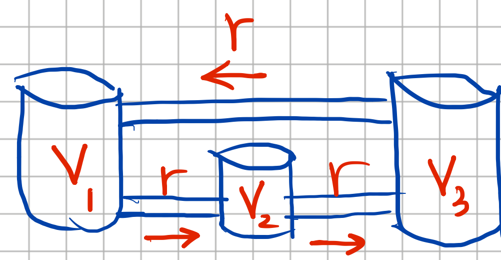
$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i \Rightarrow$$

$$\vec{x}_1(t) = e^{4t}\left\{\vec{a}\cos 3t - \vec{b}\sin 3t\right\} = e^{4t}\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$$

$$\vec{x}_2(t) = e^{4t}\left\{\vec{a}\sin 3t + \vec{b}\cos 3t\right\} = e^{4t}\begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

• **example 4** (example 2 with $k_1=0.2,\ k_2=0.4,\ k_3=0.2$)

$$\vec{x}\,' = \begin{pmatrix} -0.2 & 0 & 0.2 \\ 0.2 & -0.4 & 0 \\ 0 & 0.4 & -0.2 \end{pmatrix} \vec{x}$$



$$0 = |A - \lambda I| \overset{?}{=} -\lambda \left\{ (\lambda+0.4)^2 + (0.2)^2 \right\} \Rightarrow \lambda_1 = 0,\quad \lambda_{2,3} = -0.4 \pm 0.2i$$

Case 1 $\lambda_1 = 0$ $\quad \vec{v}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} -0.2 & 0 & 0.2 \\ 0.2 & -0.4 & 0 \\ 0 & 0.4 & -0.2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.2 & 0 & 0.2 \\ 0 & -0.4 & 0.2 \\ \cancel{0} & \cancel{0.4} & \cancel{-0.2} \end{pmatrix} \to \begin{array}{l} -a + c = 0 \\ -2b + c = 0 \end{array}$$

$$a = c,\quad b = \tfrac{1}{2}c \to \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\vec{x}_1(t) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Case 2 $(\lambda+0.4)^2 + 0.2^2 = 0 \Rightarrow \boxed{\lambda = -0.4 + 0.2i}$

$$\begin{pmatrix} 0.2-0.2i & 0 & 0.2 \\ 0.2 & -0.2i & 0 \\ 0 & 0.4 & 0.2-0.2i \end{pmatrix}$$

(2)
$$\begin{pmatrix} 1-i & 0 & 1 \\ 1 & -i & 0 \\ 0 & 2 & 1-i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (1-i)a + c = 0 \\ a - ib = 0 \\ 2b + (1-i)c = 0 \end{cases} \qquad \boxed{\begin{array}{l} c = -(1-i)a \\ b = -ia \end{array}}$$

$$-2ai \mp (1-i)\cdot(1-i)a \overset{?}{=} 0$$

$$-2i - (1-i)^2 \overset{?}{=} 0$$
$$(1-i)^2 = (1-2i-1)$$

$$\vec{v} = \begin{pmatrix} 1 \\ -i \\ -1+i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \vec{a} + i\vec{b}$$

$$\vec{x}_2(t) = e^{-0.4t}\left\{ \vec{a}\cos 0.2t - \vec{b}\sin 0.2t \right\} = e^{-0.4t} \begin{pmatrix} \cos 0.2t \\ +\sin 0.2t \\ -\cos 0.2t - \sin 0.2t \end{pmatrix}$$

$$\vec{x}_3(t) = e^{-0.4t}\left\{ \vec{a}\sin 0.2t + \vec{b}\cos 0.2t \right\} = e^{-0.4t} \begin{pmatrix} \sin 0.2t \\ -\cos 0.2t \\ -\sin 0.2t + \cos 0.2t \end{pmatrix}$$

§5.5 Multiple Eigenvalue Solutions

example 1

$$\vec{x}' = \begin{pmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{pmatrix} \vec{x}$$

(a) eigenvalues

(b) eigenvector

Case 1 $\lambda_1 = 5$

Case 2 $\lambda_2 = 3$

(c) general solution

Defective Eigenvalues

example 2 $\vec{x}' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix} \vec{x}$

(a) eigenvalue $0 = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = (\lambda-1)(\lambda-7)+9 = \lambda^2 - 8\lambda + 16$

$= (\lambda-4)^2 \Longrightarrow \lambda = 4$

(b) eigenvector

① $A-4I = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}$ $\quad (A-4I)\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a+b=0, \ \vec{v}_1 = a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

② $(A-4I)^2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\vec{v}_2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} = (-3)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{x}_1(t) = e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{x}_2(t) = e^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Multiplicity 2 Eigenvalues (defective)

the 1st solution $\vec{x}_1(t) = e^{\lambda t}\vec{v}_1$.

how to find the 2nd solution $\vec{x}_2(t)$?

$\vec{x}(t) = c_1 e^{4t}(-3)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$

$\vec{x}_2(t) = te^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_2$

$\vec{x}_2'(t) = (1+\lambda t)e^{\lambda t}\vec{v}_1 + \lambda e^{\lambda t}\vec{v}_2 = e^{\lambda t}\{\vec{v}_1 + \lambda t\vec{v}_1 + \lambda\vec{v}_2\}$

$A\vec{x}_2 = te^{\lambda t}A\vec{v}_1 + e^{\lambda t}A\vec{v}_2 = e^{\lambda t}\{\lambda t\vec{v}_1 + A\vec{v}_2\}$

$\Longrightarrow \boxed{(A-\lambda I)\vec{v}_2 = \vec{v}_1} \Longrightarrow \boxed{(A-\lambda I)^2\vec{v}_2 = \vec{0}}$

(c) $\vec{v}_2 = ?$ (rank 2 generalized eigenvector)

(d) general solution

#3 $\vec{x}' = \begin{pmatrix}1 & -2\\ 2 & 5\end{pmatrix}\vec{x}$

$$0 = |A-\lambda I| \underset{(1+\lambda)}{=} \begin{vmatrix}-1-\lambda & 0 & 1\\ 0 & -1-\lambda & 1\\ 1 & -1 & -1-\lambda\end{vmatrix} = \begin{vmatrix}0 & -(1+\lambda) & 1-(1+\lambda)^2\\ 0 & -1-\lambda & 1\\ 1 & -1 & -1-\lambda\end{vmatrix}$$

$$= \begin{vmatrix}-(1+\lambda) & 1-(1+\lambda)^2\\ -(1+\lambda) & 1\end{vmatrix} = -(1+\lambda)\begin{vmatrix}1 & 1-(1+\lambda)^2\\ 1 & 1\end{vmatrix} = -(1+\lambda)\left[1-\left(1-(1+\lambda)^2\right)\right]$$

$$= -(1+\lambda)^3 \Longrightarrow \boxed{\lambda = -1}$$

#12 $\vec{x}' = \begin{pmatrix}-1 & 0 & 1\\ 0 & -1 & 1\\ 1 & -1 & -1\end{pmatrix}\vec{x}$

$$= A\vec{x}$$

rank 3 generalized eigenvector

$(A-\lambda I)^3\vec{v} = 0$ but $(A-\lambda I)^2\vec{v} \neq 0$

$(A-\lambda I)^3\vec{v}_3 = 0$ $\quad \vec{v}_2 = (A-\lambda I)\vec{v}_3$

$\vec{v}_1 = (A-\lambda I)\vec{v}_2$

$\vec{x}_1(t) = e^{\lambda t}\vec{v}_1$, $\quad \vec{x}_2(t) = (t\vec{v}_1 + \vec{v}_2)e^{\lambda t}$

$\vec{x}_3(t) = \left(\frac{1}{2}t^2\vec{v}_3 + t\vec{v}_2 + \vec{v}_1\right)e^{\lambda t}$

$$A+I = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}$$

$$\vec{0} = (A+I)\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}c\\ c\\ a-b\end{pmatrix} \Longrightarrow \begin{matrix}c=0\\ a=b\end{matrix} \quad \vec{v}_1 = \begin{pmatrix}a\\ a\\ 0\end{pmatrix} = a\begin{pmatrix}1\\ 1\\ 0\end{pmatrix} \checkmark$$

$$(A+I)^2 = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{pmatrix} = \begin{pmatrix}1 & -1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$(A+I)^3 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \Longrightarrow \vec{v}_3 = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$$

$\vec{x}_1 = e^{-t}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$, $\quad \vec{x}_2(t) = e^{-t}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + te^{-t}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$

$$\vec{x}_3 = e^{-t}\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} + te^{-t}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + \frac{1}{2}t^2e^{-t}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$$

§5.3 A Gallery of Solution Curves of Linear Systems

geometric understanding of the role on eigenvalues and eigenvectors

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}' = A_{2\times 2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- $\underline{\lambda_1 \neq \lambda_2 \text{ (real)}}$ $\quad \vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

  $\{\vec{v}_1, \vec{v}_2\}$ l. indep.

  $e^{pt + qti} \quad (\vec{a} + i\vec{b})$

- $\lambda = p + iq$ $\quad \vec{x}(t) = c_1 e^{pt}\underbrace{(\vec{a}\cos qt - \vec{b}\sin qt)} + c_2 e^{pt}\underbrace{(\vec{a}\sin qt + \vec{b}\cos qt)}$

  $\vec{v} = \vec{a} + i\vec{b}$

- $\underline{\lambda_1 = \lambda_2 = \lambda}$ $\quad \{\vec{v}_1, \vec{v}_2\}$ l. indep. $\quad \vec{x}(t) = (c_1\vec{v}_1 + c_2\vec{v}_2)e^{\lambda t}$

  defective $\begin{cases} (A - \lambda I)\vec{v}_1 = \vec{0} \\ (A-\lambda I)\vec{v}_2 = \vec{v}_1 \end{cases}$ $\quad \vec{x}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$

  $\boxed{0 = (A - \lambda I)^2 \vec{v}_2}$

  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

  $\begin{cases} (A-\lambda I)\vec{v}_1 = \vec{0} \\ \boxed{(A-\lambda I)\vec{v}_2 = \vec{v}_1} \\ \boxed{(A-\lambda I)\vec{v}_3 = \vec{v}_2} \end{cases}$ $\quad \boxed{(A-\lambda I)^3\vec{v}_3 = 0}$

  $\begin{cases} x(t) = c_1 e^{-2t}(-1) \\ y(t) = c_1 e^{-2t}\,6 \end{cases}$

- Real Eigenvalues

  - distinct eigenvalues $(\lambda_1 \neq \lambda_2)$

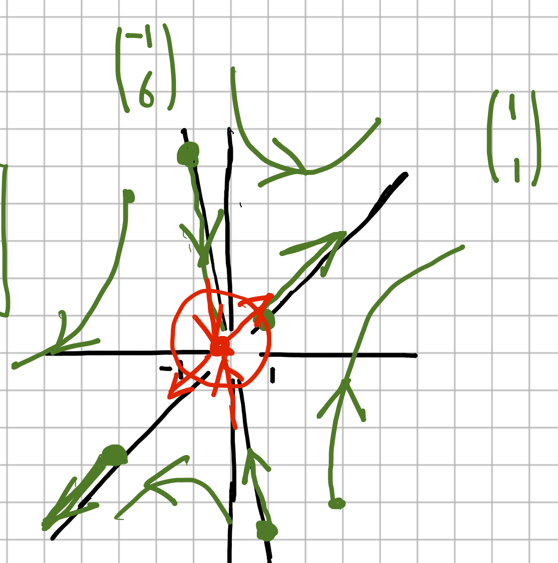
(1) $\lambda_1 < 0 < \lambda_2$ (saddle pt)

$$A = \begin{pmatrix} 4 & 1 \\ 6 & -1 \end{pmatrix}$$

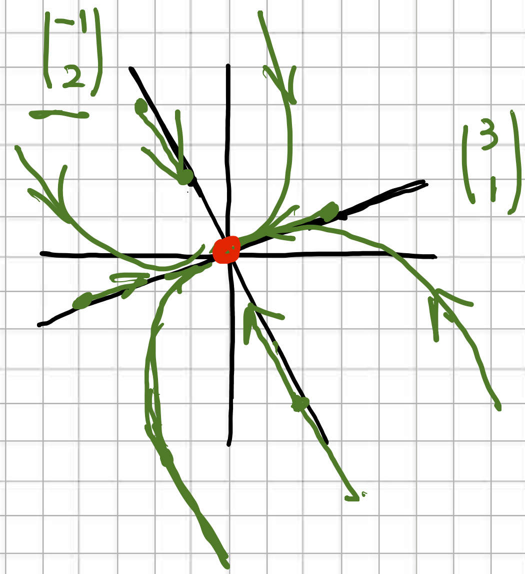
$\lambda_1 = -2, \ \vec{v}_1 = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

$\lambda_2 = 5, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{x}(t) = \boxed{c_1 e^{-2t}\begin{pmatrix} -1 \\ 6 \end{pmatrix}} + c_2 e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



(2) $\begin{cases} \lambda_1 < \lambda_2 < 0 \\ 0 < \lambda_1 < \lambda_2 \end{cases}$ improper nodal $\begin{cases} \text{sink} \\ \text{source} \end{cases}$



- $A = \begin{pmatrix} -8 & 3 \\ 2 & -13 \end{pmatrix}$, $\lambda_1 = -14$, $\vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\lambda_2 = -7$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

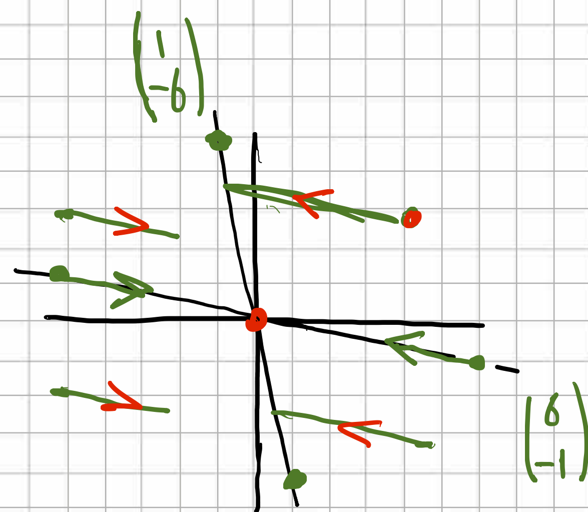
sink

$\vec{x}(t) = c_1 e^{-14t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{-7t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

- $A = -\begin{pmatrix} -8 & 3 \\ 2 & -13 \end{pmatrix}$, $\lambda_1 = 14$, $\vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\lambda_2 = 7$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Source

(3) $\begin{cases} \lambda_1 < \lambda_2 = 0 \\ 0 = \lambda_2 < \lambda_1 \end{cases}$ straight-line solution



$A = \begin{pmatrix} -36 & -6 \\ 6 & 1 \end{pmatrix}$, $\lambda_1 = -35$, $\vec{v}_1 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$, $\lambda_2 = 0$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

$\vec{x}(t) = c_1 e^{-35t}\begin{pmatrix} 6 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

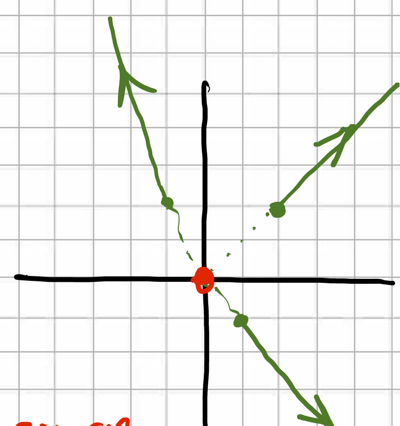
- repeated eigenvalue $(\lambda_1 = \lambda_2)$

(4) $\begin{cases} \lambda_1 = \lambda_2 < 0 \\ \lambda_1 = \lambda_2 > 0 \end{cases}$

(i) two l. indep. eigenvectors

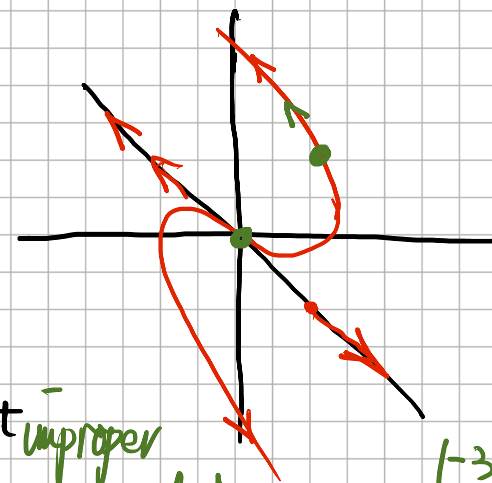
$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\lambda_1 = \lambda_2 = 2$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{x}(t) = e^{2t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$



proper nodal source

(ii) without two l. indep. eigenvectors

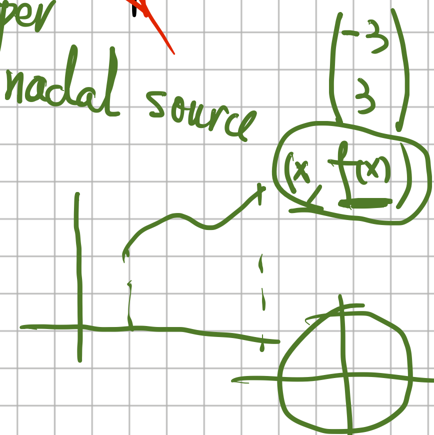


$A=\begin{pmatrix}1 & -3\\ 3 & 7\end{pmatrix}$, eigenvector $\vec{v}_1=\begin{pmatrix}-3\\3\end{pmatrix}$

$\lambda_1=\lambda_2=4$, generalized eigenvector $\vec{v}_2=\begin{pmatrix}1\\0\end{pmatrix}$

$$\vec{x}(t)=c_1e^{4t}\begin{pmatrix}-3\\3\end{pmatrix}+c_2\left\{\begin{pmatrix}1\\0\end{pmatrix}+t\begin{pmatrix}-3\\3\end{pmatrix}\right\}e^{4t}$$ improper nodal source

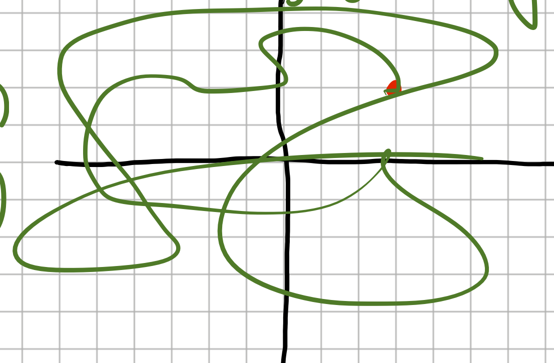
$$=(c_1+c_2t)e^{4t}\begin{pmatrix}-3\\3\end{pmatrix}+c_2e^{4t}\begin{pmatrix}1\\0\end{pmatrix}$$



- Complex Conjugate Eigenvalues ($\lambda_1,\lambda_2=p\pm qi$)

(1) $p=0$ and $q\neq 0$ centers and elliptical orbits

$f(x,y)=x^2+y^2=1$



$A=\begin{pmatrix}6 & -17\\ 8 & -6\end{pmatrix}$, $\lambda_1,\lambda_2=\pm 10i$ $\quad \begin{cases}x=x(t)\\ y=y(t)\end{cases}$

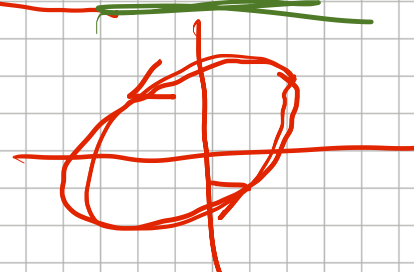
$\vec{v}=\vec{a}+i\vec{b}$

$\vec{a}=\begin{pmatrix}3\\4\end{pmatrix}$, $\vec{b}=\begin{pmatrix}5\\0\end{pmatrix}$

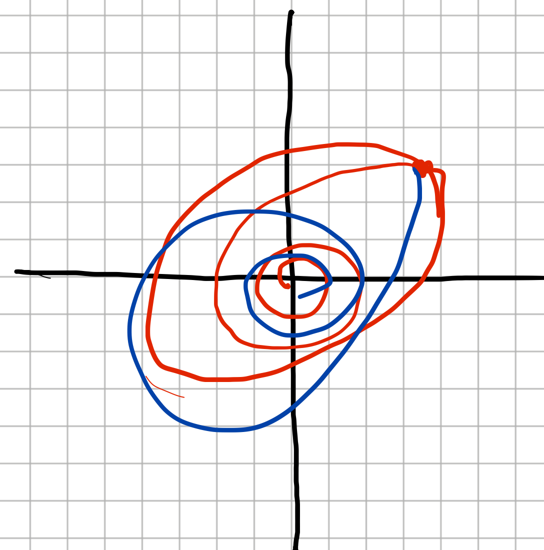
$$\vec{x}(t)=\begin{pmatrix}(3c_1+5c_2)\cos 10t+(-5c_1+3c_2)\sin 10t\\ 4c_1\cos 10t+4c_2\sin 10t\end{pmatrix}$$

$$\vec{x}(0)=\begin{pmatrix}4\\2\end{pmatrix}\Rightarrow \vec{x}(t)=\begin{pmatrix}4\cos 10t-\sin 10t\\ 2\cos 10t+2\sin 10t\end{pmatrix}=\begin{pmatrix}x(t)\\y(t)\end{pmatrix}$$

$$\begin{pmatrix}\cos 10t\\ \sin 10t\end{pmatrix}=\begin{pmatrix}4 & -1\\ 2 & 2\end{pmatrix}^{-1}\begin{pmatrix}x\\y\end{pmatrix}$$



(2) $q \neq 0$, $\begin{cases} p < 0 \\ 0 > p \end{cases}$ spiral $\begin{cases} \text{sinks} \\ \text{sources} \end{cases}$



$$\vec{x}' = \begin{pmatrix} 5 & -17 \\ 8 & -7 \end{pmatrix} \vec{x}$$

$$\lambda = -1 + 10i$$

$$\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + i \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\vec{x}(t) = e^{-t} \begin{pmatrix} (3c_1 + 5c_2)\cos 10t + (-5c_1 + 3c_2)\sin 10t \\ 4c_1 \cos 10t + 4c_2 \sin 10t \end{pmatrix}$$

$$\vec{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \Longrightarrow \vec{x}(t) = e^{-t} \begin{pmatrix} 4\cos 10t - \sin 10t \\ 2\cos 10t + 2\sin 10t \end{pmatrix}$$

Special Case of a Repeated Zero Eigenvalues ($\lambda_1 = \lambda_2 = 0$)

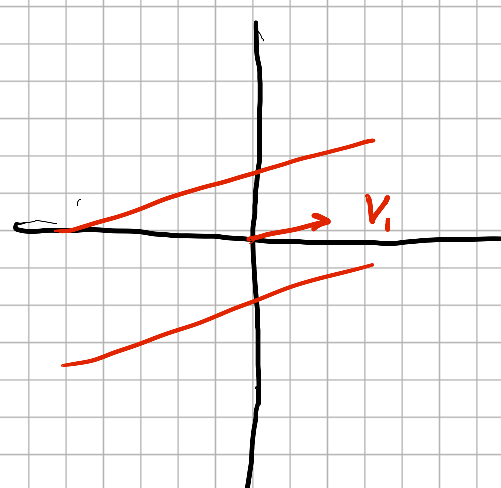
$A\vec{v} = \vec{0}$ $\quad \begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \begin{matrix} x = c_1 \\ y = c_2 \end{matrix}$

(i) two l. indep vectors

$$\Longrightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \vec{x}(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

(ii) $\vec{v}_1$ — eigenvector

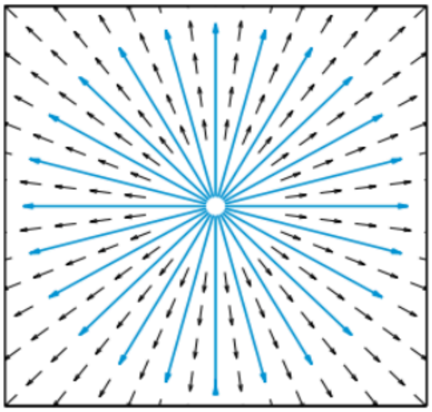
$\vec{v}_2$ — generalized eigenvector



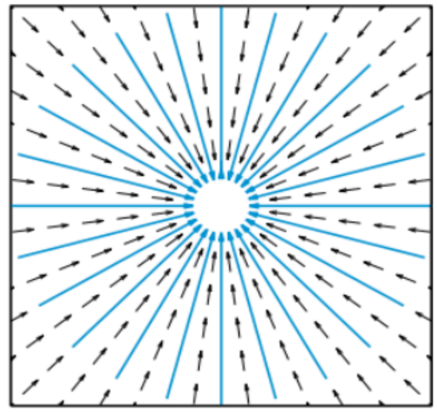
$$\vec{x}(t) = c_1\vec{v}_1 + c_2(t\vec{v}_1 + \vec{v}_2)$$

$$= (c_1\vec{v}_1 + c_2\vec{v}_2) + c_2 t \vec{v}_1$$

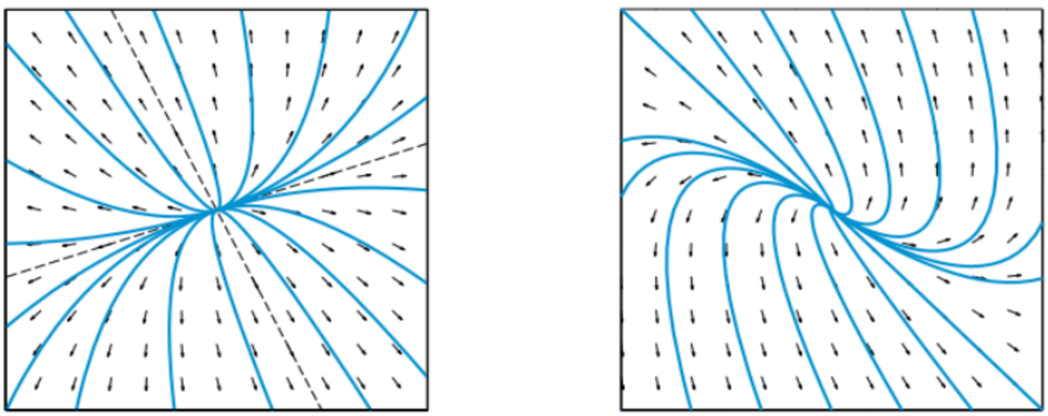
Gallery of Typical Phase Portraits for the System $\mathbf{x}' = \mathbf{A}\mathbf{x}$: Nodes



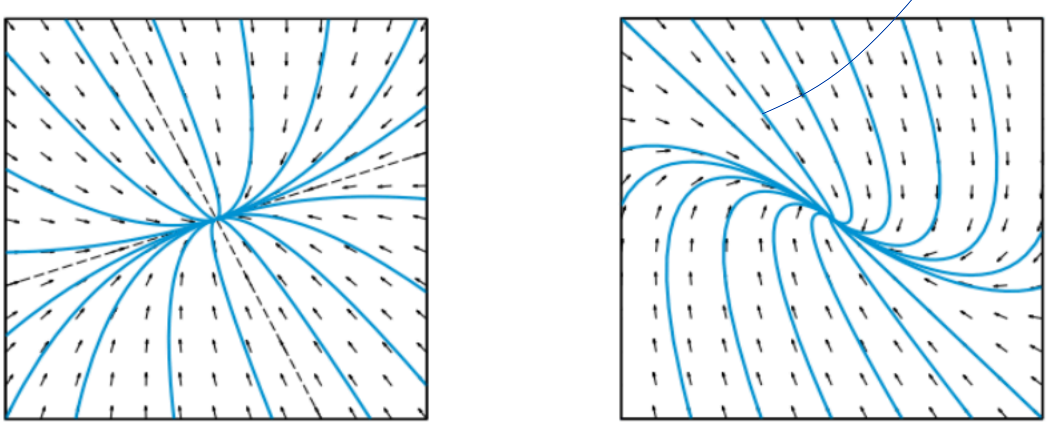
Proper Nodal Source: A repeated positive real eigenvalue with two linearly independent eigenvectors.



Proper Nodal Sink: A repeated negative real eigenvalue with two linearly independent eigenvectors.

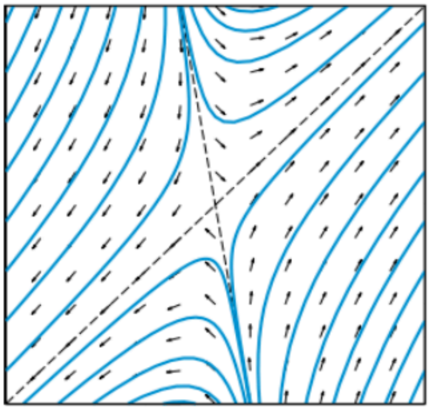


Improper Nodal Source: Distinct positive real eigenvalues (left) or a repeated positive real eigenvalue without two linearly independent eigenvectors (right).

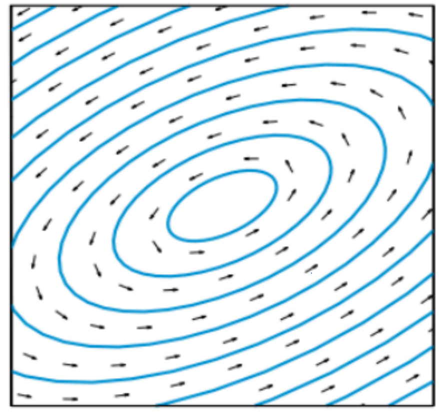


Improper Nodal Sink: Distinct negative real eigenvalues (left) or a repeated negative real eigenvalue without two linearly independent eigenvectors (right).

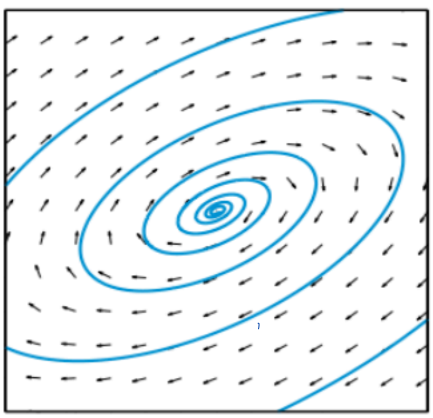
Saddles, Centers, Spirals, and Parallel Lines



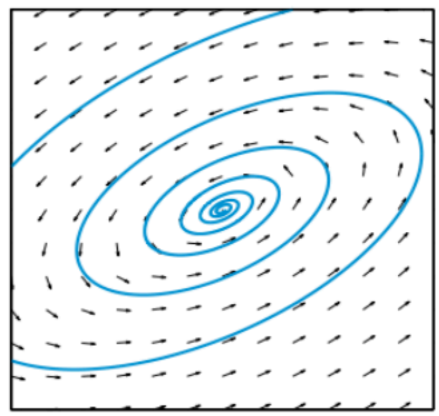
Saddle Point: Real eigenvalues of opposite sign.



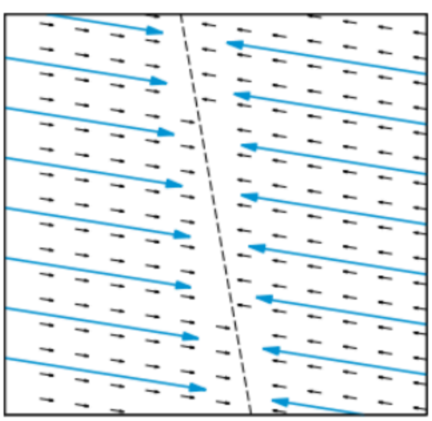
Center: Pure imaginary eigenvalues.



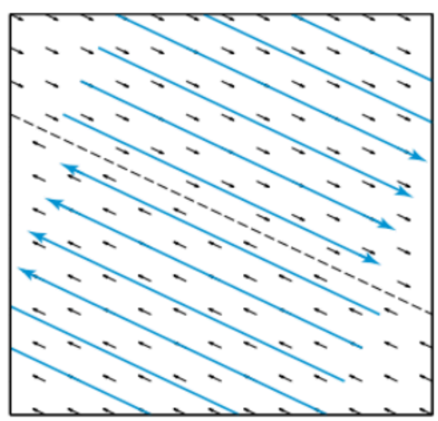
Spiral Source: Complex conjugate eigenvalues with positive real part.



Spiral Sink: Complex conjugate eigenvalues with negative real part.



Parallel Lines: One zero and one negative real eigenvalue. (If the nonzero eigenvalue is positive, then the trajectories flow *away* from the dotted line.)



Parallel Lines: A repeated zero eigenvalue without two linearly independent eigenvectors.