Chapter 7 Laplace Transform Methods

differential equation $\xrightarrow{\text{integral transform}}$ algebraic equation

$\Downarrow$

solution of DE $\xleftarrow{\text{inverse transform}}$ solution of AE

§7.1 Laplace Transforms and Inverse Transforms

• Laplace transform $\qquad F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

• Review of Improper Integral

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty} \int_a^A f(t)\,dt \begin{cases} \text{exists} & \text{convergence} \\ \text{DNE} & \text{divergence} \end{cases}$$

Examples

(1) $\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt = \lim_{A\to\infty} \int_0^A e^{-st}\,dt = \lim_{A\to\infty} \left[\frac{1}{(-s)} e^{-st}\right]_0^A$

$= \lim_{A\to\infty} \left(-\frac{1}{s}\right)\left[e^{-sA} - 1\right] = -\frac{1}{s}(-1) = \frac{1}{s}$ $\qquad (s>0)$

(2) $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty$

$= \frac{1}{a-s}[0-1] = \boxed{\frac{1}{s-a}}$ $\qquad \boxed{a-s<0}$

(3) $\boxed{\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}}$, where the Gamma function $(x>0)$

$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$

$\Gamma(1)=1,\ \Gamma(x+1) = x\Gamma(x)$

$\Gamma(n+1) = n!$

$LHS = \int_0^\infty e^{-st} t^a\,dt \overset{?}{=} RHS = \frac{1}{s^{a+1}} \int_0^\infty e^{-\gamma} \cdot \gamma^a\,d\gamma$

$= \int_0^\infty e^{-\gamma} \left(\frac{\gamma}{s}\right)^a \frac{d\gamma}{s}$ $\qquad \boxed{\gamma = st}$ $\quad d\gamma = s\,dt$ $\quad =$

$\mathcal{L}\{t^1\} = \frac{\Gamma(2)}{s^2} = \frac{1}{s^2} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad s>0$

- Linearity of Transforms

$$\mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Examples

(4) $\mathcal{L}\{3t^2 + 4t^{\frac{3}{2}}\} = 3\mathcal{L}\{t^2\} + 4\mathcal{L}\{t^{\frac{3}{2}}\}$

$= 3\cdot\frac{2!}{s^3} + 4\cdot\frac{\Gamma(\frac{3}{2}+1)}{s^{\frac{3}{2}+1}} = \frac{6}{s^3} + \frac{4\,\Gamma(\frac{3}{2}+1)}{s^{5/2}}$

(5) $\mathcal{L}\{\cosh(kt)\} = \mathcal{L}\{\frac{1}{2}(e^{kt}+e^{-kt})\}$

$= \frac{1}{2}\mathcal{L}\{e^{kt}\} + \frac{1}{2}\mathcal{L}\{e^{-kt}\} = \frac{1}{2}\frac{1}{s-k} + \frac{1}{2}\frac{1}{s-(-k)}$

$= \frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s+k}\right] = \frac{1}{2}\frac{(s+k)+(s-k)}{(s-k)(s+k)} = \frac{1}{\cancel{2}}\frac{\cancel{2}s}{s^2-k^2}$

- Inverse Laplace Transforms

$\mathcal{L}^{-1}\{F(s)\} = f(t)$ if $F(s) = \mathcal{L}\{f(t)\}$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	
t^n ($n \geq 0$ integer)	$n!/s^{n+1}$	$(s>0)$
t^a ($a>-1$ real)	$\Gamma(a+1)/s^{a+1}$	$(s>0)$
e^{at}	$\frac{1}{s-a}$	$(s>a)$
$\cos(kt)$	$s/(s^2+k^2)$	$(s>0)$
$\sin(kt)$	$k/(s^2+k^2)$	$(s>0)$

$f(t)$	$F(s)$	
$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$(s>\lvert k\rvert)$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$(s>\lvert k\rvert)$
$u(t-a)$	$\frac{1}{s}e^{-as}$	$(s>0)$

Examples

$\mathcal{L}^{-1}\{\frac{1}{s^3}\} = \mathcal{L}^{-1}\{\frac{1}{2!}\cdot\frac{2!}{s^3}\} = \frac{1}{2}\mathcal{L}^{-1}\{\frac{2!}{s^3}\} = \frac{1}{2}t^2$

$\mathcal{L}^{-1}\{\frac{1}{s+2}\} = \mathcal{L}^{-1}\{\frac{1}{s-(-2)}\} = e^{-2t}$

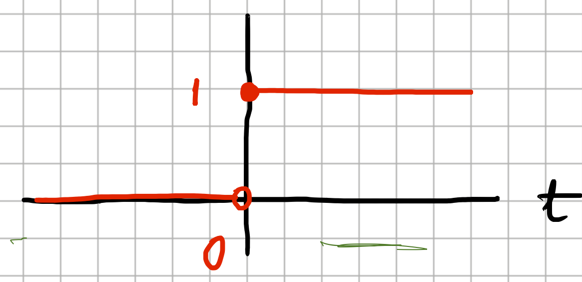
$\mathcal{L}^{-1}\{\frac{5}{s^2+9}\} = \mathcal{L}^{-1}\{\frac{3}{s^2+3^2}\cdot\frac{5}{3}\} = \frac{5}{3}\sin(3t)$

#28 $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{3s}{s^2+2^2} + \frac{1}{s^2+2^2}\right\}$

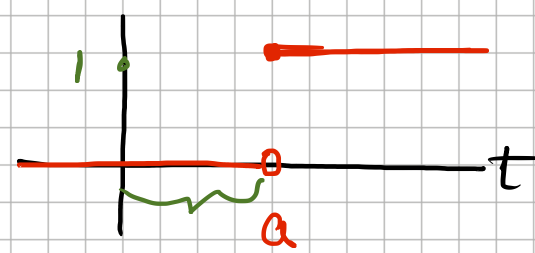
$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}\cdot 3 + \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\cdot\frac{1}{2}\right\} = 3\cdot\cos 2t + \frac{1}{2}\sin 2t$

- unit step function

$$u(t) = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t\geq 0 \end{cases}$$



$$u_a(t) = u(t-a) = \begin{cases} 0 & \text{for } t<a \\ 1 & \text{for } t\geq a \end{cases}$$



$$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st} u_a(t)\,dt = \int_0^a + \int_a^\infty e^{-st} u_a(t)\,dt$$

$$= \int_a^\infty e^{-st}\,dt = \frac{1}{(-s)} e^{-st}\Big|_a^\infty = -\frac{1}{s}\left[0 - e^{-as}\right] \quad \boxed{s>0}$$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)}_{v'}\,dt \qquad \frac{u' = -se^{-st}}{v = f} \qquad e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\overset{(s>0)}{=} \left[0 - f(0)\right] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\,\mathcal{L}\{f(t)\} - f(0)$$

$$= \mathcal{L}\{f'(t)\}$$

$\int t \sin t\,dt = -t\cos t + \int \cos t\,dt$

$u = t,\quad v' = \sin t$ $\qquad = -t\cos t + \sin t$

$u' = 1,\quad v = -\cos t$

§7.2 Transformation of Initial Value Problems

IVP
$$\begin{cases} a x''(t) + b x'(t) + c x(t) = f(t) \\ x(0) = x_0, \quad x'(0) = x_0' \end{cases}$$

- $\mathcal{L}$-transformation of derivatives

$$\boxed{\mathcal{L}\left\{ f^{(n)}(t) \right\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}$$

IVP $\quad a\mathcal{L}\{x''\} + b\mathcal{L}\{x'\} + c\mathcal{L}\{x\} = \mathcal{L}\{f\}$

Example 1

$$\begin{cases} x'' - x' - 6x = 0 \\ x(0) = 2, \quad x'(0) = -1 \end{cases}$$

Example 2

$$\begin{cases} x'' + 4x = \sin 3t \\ x(0) = x'(0) = 0 \end{cases}$$

- Linear System

Example 3

$$\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40 \sin 3t \\ x(0) = x'(0) = y(0) = y'(0) = 0 \end{cases}$$

- Transform Perspective

$$\begin{cases} m x'' + c x' + k x = f(t) \\ x(0) = x_0, \quad x'(0) = x_0' \end{cases}$$

- additional transform techniques

Example 4 Show that $\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$. (using $\mathcal{L}\{f'(t)\}$)

$f(t) = te^{at}$

$f'(t) = e^{at} + t \cdot a e^{at}$

$= e^{at} + a(te^{at})$

$= e^{at} + a f(t)$

$f(0) = 0$

$\mathcal{L}\{f'(t)\} = \mathcal{L}\{e^{at}\} + a\mathcal{L}\{f(t)\}$

$\parallel$

$s\mathcal{L}\{f(t)\} - f(0)$

$(s-a)\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

Example 5 Find $\mathcal{L}\{t\sin kt\}$. (using $\mathcal{L}\{f''(t)\}$).

$f(t) = t\sin kt$

$f'(t) = \sin kt + t \cdot k\cos kt$

$f'' = k\cos kt + k[\cos kt - kt\sin kt]$

$= 2k\cos kt - k^2 f(t)$

$s^2\mathcal{L}\{f(t)\} - s\cancel{f(0)} - \cancel{f'(0)} = \mathcal{L}\{f''\} = 2k\mathcal{L}\{\cos kt\} - k^2\mathcal{L}\{f(t)\}$

$\Longrightarrow (s^2+k^2)\mathcal{L}\{f(t)\} = 2k\frac{s}{s^2+k^2}$

- Transforms of Integrals

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s} \iff \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$$

Example 6 $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} = ?$

§7.3 Translation and Partial Fractions

- Partial Fraction

$R(s) = \frac{P(s)}{Q(s)}$, where $P(s)$, $Q(s)$ are polynomials $\deg P < \deg Q$

$$\frac{P(s)}{(s-a)^n} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$

$$\frac{P(s)}{[(s-a)^2+b^2]^n} = \frac{A_1 s + B_1}{(s-a)^2+b^2} + \cdots + \frac{A_n s + B_n}{[(s-a)^2+b^2]^n}$$

• Translation

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \iff \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

Proof $$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\left[e^{at}f(t)\right]dt \qquad \#$$

(1) $$\mathcal{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = e^{at}t^n$$

(2) $$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+k^2}\right\} = e^{at}\cos kt$$

(3) $$\mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2+k^2}\right\} = e^{at}\sin kt$$

Examples

(1) Spring-mass system : $\begin{cases} mx'' + cx' + kx = 0 \\ x(0)=3,\ x'(0)=1 \end{cases}$ $\left(m=\frac{1}{2},\ k=17,\ c=3\right)$

$x(t) = ?$

$$(ms^2+cs+k)\,\mathcal{L}\{x(t)\} - \left[(ms+c)\cdot 3 + m\cdot 1\right] = 0$$

$$X(s) = \frac{3ms + (3c+m)}{ms^2+cs+k}$$

$$= \frac{s^{\cancel{3}}}{\cancel{s}(s-4)(s+2)} + \frac{1}{s(s-4)(s+2)}$$

$$\frac{s^4+1}{s^3-2s^2-8s} = \frac{s^4+1}{s(s-4)(s+2)} = ?$$

$$\frac{s^2+1}{s^3-2s^2-8s} = \frac{s^2+1}{s(s^2-2s-8)}$$

(3) $\mathcal{L}^{-1}\left\{\frac{s^2+1}{s^3-2s^2-8s}\right\}$

$$= \frac{s^2+1}{s(s-4)(s+2)} = \frac{A_1}{s} + \frac{A_2}{s-4} + \frac{A_3}{s+2}$$

$$= A_1\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + A_2\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + A_3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{A_3 s(s-4) + A_1(s-4)(s+2) + A_2 s(s+2)}{s(s-4)(s+2)}$$

$$= A_1 t + A_2 e^{4t} + A_3 e^{-2t}$$

$$s^2+1 = A_1(s-4)(s+2) + A_2 s(s+2) + A_3 s(s-4)$$

$s=0$: $1 = A_1(-8) \Rightarrow A_1 = -\frac{1}{8}$

$s=4$: $9 = A_2 \cdot 4 \cdot 6 \Rightarrow A_2 = \frac{8}{3}$

$s=-2$: $5 = A_3(-2)\cdot(-6) \Rightarrow A_3 = \frac{5}{12}$

(4) $\begin{cases} y'' + 4y' + 4y = t^2 \\ y(0) = y'(0) = 0 \end{cases}$ $\quad \mathcal{L}\{t^2\} = \frac{2}{s^3}$

$$(s^2+4s+4)Y(s) = \frac{2}{s^3} \Rightarrow Y(s) = \frac{2}{\boxed{s^3(s+2)^2}}$$

$$Y(s) \overset{?}{=} \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s^3} + \frac{A_4}{s+2} + \frac{A_5}{(s+2)^2}$$

$$2 = \underbrace{A_1 s^2(s+2)^2}_{A_1(s^4+\cdots)} + \underbrace{A_2 s(s+2)^2}_{A_2(s^3+\cdots)} + A_3(s+2)^2 + \underbrace{A_4 s^3(s+2)}_{A_4(s^4+\cdots)} + A_5 s^3$$

$s=0$ $\quad 2 = A_3 \cdot 4 \Rightarrow A_3 = \frac{1}{2}$

$s=-2$ $\quad 2 = A_3 \cdot (-8) \Rightarrow A_5 = -\frac{1}{4}$

(4) $\begin{cases} x'' + 6x' + 34x = 30\sin 2t \\ x(0) = 0 = x'(0) \end{cases}$

- Resonance and Repeated Quadratic Factors

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+k^2)^2}\right\} = \frac{1}{2k} t \sin kt$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^3}\left(\sin kt - kt\cos kt\right)$$

x(t)

Example 5

$$\begin{cases} \boxed{x''(t) + \omega_0^2 x(t)} = F_0 \sin \omega t = f(t) \\ x(0) = 0 = x'(0) \end{cases}$$

$$(s^2+\omega_0^2) X(s) = F_0 \frac{\omega}{s^2+\omega^2} \Rightarrow X(s) = F_0\omega \boxed{\frac{1}{(s^2+\omega_0^2)(s^2+\omega^2)}}$$

$\omega = \omega_0$ $\quad X(s) = F_0\omega_0 \frac{1}{(s^2+\omega_0^2)^2}$

$$x(t) = F_0\omega_0 \mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega_0^2)^2}\right\} = \frac{F_0\omega_0}{2\omega_0^3}\left[\sin\omega_0 t - \omega_0 t\cos\omega_0 t\right]$$

$\boxed{\omega_0, \omega > 0}$

$$\Rightarrow 1 = (A_1 s + A_2)(s^2+\omega^2) + (B_1 s + B_2)(s^2+\omega_0^2)$$

$s = \omega i$ $\quad 1 = (B_1\omega i + B_2)(-\omega^2+\omega_0^2)$ $\quad B_1 = 0, \quad B_2 = \frac{1}{\omega_0^2-\omega^2}$

$s = \omega_0 i$ $\quad 1 = (A_1\omega_0 i + A_2)(\omega^2-\omega_0^2)$ $\quad A_1 = 0, \quad A_2 = \frac{1}{\omega^2-\omega_0^2}$

$$\frac{1}{(s^2+\omega_0^2)(s^2+\omega^2)} = \left[\frac{1}{s^2+\omega_0^2} - \frac{1}{s^2+\omega^2}\right]\left(\frac{1}{\omega^2-\omega_0}\right)$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = F_0\omega \frac{1}{\omega^2-\omega_0^2}\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\}\right]$$

$$= \frac{F_0\omega}{\omega^2-\omega_0^2}\left[\frac{1}{\omega_0}\sin(\omega_0 t) - \frac{1}{\omega}\sin(\omega t)\right]$$

$\omega \neq \omega_0$

Example 6 $\begin{cases} y^{(4)} + 2y'' + y = 4te^{t} \\ y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0 \end{cases}$

§7.4 Derivative, Integral, and Product of Inverse Transform

- Convolution

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

Examples

$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

(1) $f(t) = \cos t$

$g(t) = \sin t$

#2 $f(t) = t$

$g(t) = e^{at}$

$$\mathcal{L}\{f*g\}=F(s)G(s) \iff \mathcal{L}^{-1}\{F(s)G(s)\}=(f*g)(t)$$

Examples

$$\int_0^t e^{-\tau}\sin 2\tau \, d\tau = \left[\frac{e^{-\tau}}{5}(-\sin 2\tau - 2\cos 2\tau)\right]_0^t$$

(2) $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-1)(s^2+4)}\right\}$

$$\mathcal{L}\{-t f(t)\} = F'(s) \iff \boxed{\mathcal{L}^{-1}\{(-1)F'(s)\} = t f(t)}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \iff \boxed{\mathcal{L}^{-1}\{(-1)^n F^{(n)}(s)\} = t^n f(t)}$$

Examples

(3) $\mathcal{L}\{t^2 \sin kt\} =$

$$\boxed{\left(\tan^{-1} s\right)' = \frac{1}{1+s^2}}$$

$$-t\,\mathcal{L}^{-1}\{F(s)\} = -t\,f(t) = \mathcal{L}^{-1}\{F'(s)\}$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

(4) $\mathcal{L}^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\}$

$=$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma \iff f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}$$

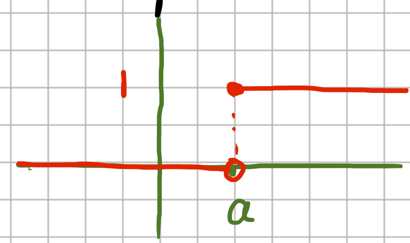
Examples

(6) $\mathcal{L}\left\{\frac{\sinh t}{t}\right\} =$

(7) $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s^2-1)^2}\right\} =$

§7.5 Periodic and Piecewise Continuous Input Functions

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

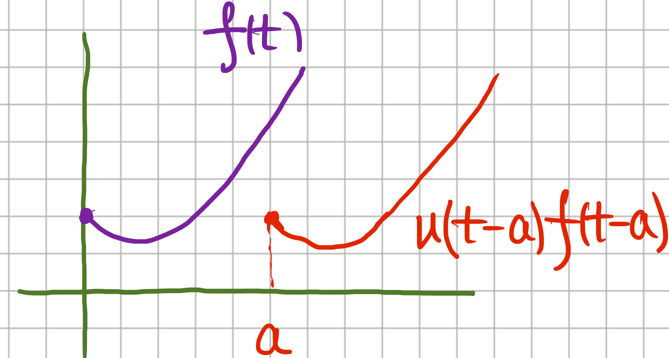


$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$\mathcal{L}\{u_0(t)\} = \mathcal{L}\{u(t)\} = \frac{1}{s}$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

$$u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$



Ex. (1)

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s^3}\right\} =$$

(2) $g(t) = \begin{cases} 0, & t < 3 \\ t^2, & t \geq 3 \end{cases}$

(3) $f(t) = \begin{cases} \cos 2t, & 0 \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$

$\mathcal{L}\{g(t)\} =$

$\mathcal{L}\{f(t)\} =$

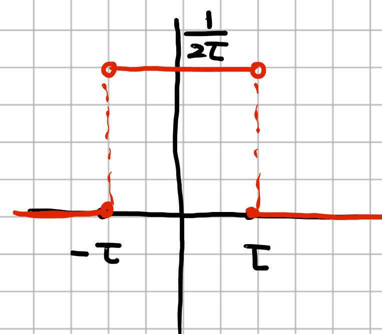
(4) $\begin{cases} x'' + 4x = f(t) \quad \text{in (3)} \\ x(0) = x'(0) = 0 \end{cases}$

(5) $\begin{cases} i'(t) + 110\, i(t) + 1000 \int_0^t i(\tau)\, d\tau = e(t) = \begin{cases} 90, & 0 < t < 1 \\ 0, & t > 1 \end{cases} \\ i(0) = i'(0) = 0 \end{cases}$

§7.6 Impulse and Delta Functions

- Impulse Function

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & \text{otherwise} \end{cases}$$



Properties

(1) $I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)\, dt = 1 \quad \forall \tau \neq 0$

(2) $\lim\limits_{\tau \to 0^+} d_\tau(t) = 0 \quad \forall t \neq 0$

(3) $\lim\limits_{\tau \to 0} I(\tau) = 1$

- ## Delta Function

$$\delta(t) = \lim_{\tau \to 0^+} d_\tau(t)$$

$\Rightarrow$ (1) $\delta(t) = 0 \quad \forall t \neq 0$

(2) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

(3) $\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$

at a

(1) $\delta_a(t) = \delta(t-a) = 0 \quad \forall t \neq a$

(2) $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$

(3) $\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}} \Rightarrow \mathcal{L}\{\delta(t)\} = 1$$

Examples

(1) $\begin{cases} x'' + 4x = 8\,\delta_{2\pi}(t) \\ x(0) = 3, \quad x'(0) = 0 \end{cases}$

(2) $\begin{cases} i'' + 110\,i' + 1000\,i = -90\,\delta(t-1) \\ i(0) = 0, \quad i'(0) = 90 \end{cases}$

- ## System Analysis and Duhamel's Principle

$$\begin{cases} a x'' + b x' + c x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$$

$\Rightarrow X(s) = W(s)\,F(s)$, where $W(s) = \dfrac{1}{as^2 + bs + c} \Rightarrow w(t) = \mathcal{L}^{-1}\{W(s)\}$

$$\Rightarrow \quad x(t) = \int_0^t w(\tau)\, f(t-\tau)\, d\tau .$$ Duhamel's principle

<u>example 4</u>

$$\begin{cases} x'' + 6x' + 10x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$$

$$w(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s+3)^2+1} \right\} = e^{-3t} \sin t$$

$$x(t) = \int_0^t e^{-3\tau} \sin\tau \; f(t-\tau)\, d\tau$$