Chapter 9 Fourier Series Methods (4) and Partial Differential Equations (3

§9.1 Periodic Functions and Trigonometric Series

• Periodic Function with period p

$$f(t+p) = f(t) \quad \forall t$$

find the smallest period of $f(t)$

$f(t) = 1, \quad \sin nt, \quad \cos nt, \quad 3 + \cos t + 5\sin 3t$

• Fourier Series of Period 2π functions

integrals
$$\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt \, dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mt \sin nt \, dt = 0 \qquad \forall m, n$$

the Fourier series of $f(t)$ with period 2π is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt \right)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt.$$

Example (1)

$$f(t) = \begin{cases} -1, & t \in (-\pi, 0) \\ 1, & t \in (0, \pi) \\ 0, & t = -\pi, 0, \pi \end{cases}$$

the square-wave function

find Fourier series. (Fig. 9.1.3)

(2)

$$f(t) = \begin{cases} 0, & t \in (-\pi, 0] \\ t, & t \in [0, \pi) \\ \pi/2, & t = \pm\pi \end{cases}, \quad f(t) = f(t + 2\pi)$$

find F-series.

§9.2 General F-series and Convergence

$f(t)$ is a periodic function with period $P = 2L$

$$f(t+2L) = f(t)$$

$\Longrightarrow$ $g(u) = f\left(\frac{L}{\pi}u\right)$ is a periodic function with period $P = 2\pi$

$$g(u+2\pi) = f\left(\frac{L}{\pi}(u+2\pi)\right) = f\left(\frac{L}{\pi}u + 2L\right) = f\left(\frac{L}{\pi}u\right) = g(u)$$

$t = \frac{L}{\pi}u \Rightarrow u = \frac{\pi}{L}t$

$$\Longrightarrow \quad f(t) = g\left(\frac{\pi}{L}t\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos nu\, du \;\underset{du=\frac{\pi}{L}dt}{\overset{u=\frac{\pi}{L}t}{=\!=\!=}}\; \frac{1}{L}\int_{-L}^{L} g\left(\frac{\pi t}{L}\right)\cos\frac{n\pi t}{L}\,dt$$

$$= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

- Convergence of F-series

Assume that (1) $f(t)$ is a periodic function with period $2L$

(2) f is piecewise smooth.

$$\Longrightarrow \quad \hat{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

where

$$\hat{f}(t) = \begin{cases} f(t), & f \text{ is cont. at } t \\ \dfrac{f(t^-)+f(t^+)}{2}, & f \text{ is discont. at } t \end{cases}$$

Examples

(1) $f(t) = \begin{cases} -1, & -2<t<0 \\ 1, & 0<t<2 \end{cases}$ $\qquad f(t+4)=f(t)$

Find F-series and $\hat{f}(t)$. $\qquad \hat{f}(2n) = ?$

(2) $f(t) = t^2$ for $t \in (0,2)$ and $f(t+2)=f(t)$

$f(2n) = 2$.

Find F-series.

(3) Use the F-series in (2) to show

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ $\qquad (f(0)=2)$

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ $\qquad (f(1)=1)$

- $\displaystyle\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$

§9.3 Fourier Sine and Cosine Series

- Even and Odd functions

f is even: $f(-t) = f(t)$ examples: t^2, t^{10}, $\cos t$

$$\int_{-a}^{a} f(t)\,dt = 2\int_0^a f(t)\,dt$$

f is odd: $f(-t) = -f(t)$ examples: t, t^5, $\sin t$

$$\int_{-a}^{a} f(t)\,dt = 0$$

f is even $\quad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt$

$$b_n = 0$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$ Fourier Cosine series

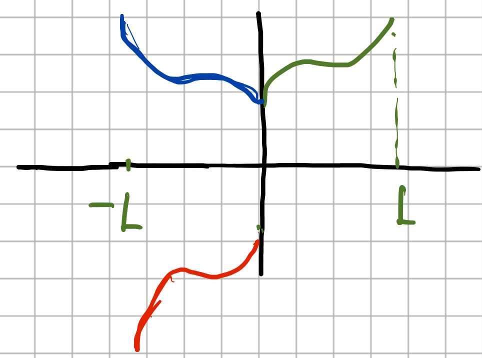
f is odd $\quad a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}\,dt$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$ Fourier Sine Series

• Even and Odd Extensions

$f(t)$ is defined on $(0, L)$



$$f_E(t) = \begin{cases} f(t), & t \in (0, L) \\ f(-t), & t \in (-L, 0) \end{cases}$$

$$f_0(t) = \begin{cases} f(t), & t \in (0, L) \\ -f(-t), & t \in (-L, 0) \end{cases}$$

<u>Example 1</u> $f(t) = t$ on $(0, L)$

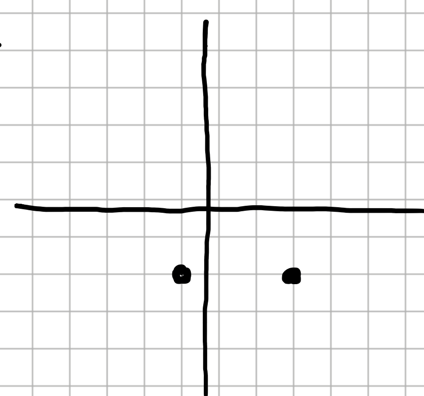
Find both F-sine and F-cosine series.

• Differentiation of F-series.

$$f'(t) = \sum_{n=1}^{\infty} \left\{ -\frac{n\pi}{L} a_n \sin \frac{n\pi t}{L} + \frac{n\pi}{L} b_n \cos \frac{n\pi t}{L} \right\}$$

<u>Example 2</u> Find F-series solution of

$$(BVP) \begin{cases} x'' + 4x = 4t \\ x(0) = x(1) = 0. \end{cases}$$



§9.4 Applications of Fourier Series

- Undamped Forced Oscillations

$$m\,x''(t) + k\,x(t) = F(t)$$

general solution $\quad x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + x_p(t)$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency

$x_p(t)$ is a particular solution.

use F-series to find a periodic particular solution $x_{sp}(t)$

$\left(\frac{n\pi}{L} \neq \omega_0\right)$

(1) compute F-series of $F(t)$, e.g., $F(t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi t}{L}$

(2) set $x_{sp}(t)$ having the same form: $x_{sp}(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$

(3) find b_n by substituting $x_{sp}(t)$ into $m x'' + k x = F$.

Example 1 $\quad 2x'' + 32x = F(t) = \begin{cases} 10, & 0<t<1 \\ -10, & 1<t<2 \end{cases}$

where $F(t)$ is an odd periodic force with a period 2.

Find $x_{sp}(t)$ and general solution

Solution (1) $F(t) = \frac{40}{\pi} \sum_{n \text{ odd}} \frac{\sin n\pi t}{2}$

(2) $x_{sp}(t) = \sum_{n \text{ odd}} b_n \sin n\pi t$

(3) $b_n = ?$

(4) general solution $x(t) = A\cos 4t + B\sin 4t + x_{sp}(t)$.

Pure Resonance will occur

$\Longleftrightarrow$ there exists an integer N such that $\frac{N\pi}{L} = \omega_0$ and $B_N \neq 0$.

Example 2 $\quad 2x'' + 32x = F(t)$, $F(t)$ is an odd periodic function given

(a) $F(t) = \begin{cases} 10, & 0 < t < \pi \\ -10, & \pi < t < 2\pi \end{cases}$; (b) $F(t) = 10t$, $-\pi < t < \pi$

Solution $\quad \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{32}{2}} = 4$

(a) $F(t) = \frac{40}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$

(b) $F(t) = 20 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt$.

- Damped Forced Oscillations

$$m x'' + c x' + k x = F(t)$$

(1) $F(t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi t}{L}$ when $F(t)$ is an odd periodic function with period $2L$.

(2) $x_{sp}(t) = \sum_{n=1}^{\infty} \frac{B_n \sin(w_n t - \alpha_n)}{\sqrt{(k - m w_n^2)^2 + (c w_n)^2}}$

where $w_n = \frac{n\pi}{L}$ and $\alpha_n = \tan^{-1} \frac{c w_n}{k - m w_n^2}$

<u>Example 4</u> $3x'' + 0.02x' + 27x = F(t) = \pi t - t^2$ on $[0, \pi)$.

Find $x_{sp}(t)$, i.e., coefficients and phase angles.

<u>solution</u> $F(t) = \frac{8}{\pi}\left(\sin t + \frac{1}{3^3}\sin 3t + \frac{1}{5^3}\sin 5t + \cdots\right)$

<u>§9.5 Heat Conduction and Separation of Variables</u>

- <u>one dimensional heat equation</u>

$$\frac{\partial u(x,t)}{\partial t} = k \frac{\partial^2 u(x,t)}{\partial x^2},$$ k is the thermal diffusivity.

- <u>boundary and initial conditions</u>

<u>initial condition</u> $u(x,0) = f(x)$

<u>boundary condition</u> $u(0,t) = u(L,t) = 0$

or $u_x(0,t) = u_x(L,t) = 0$

- Separation of Variables

(1) set $u(x,t) = X(x)\,T(t)$

(2) substitute it into $u_t = k\,u_{xx} \Longrightarrow X \cdot T' = k X'' T$

$$\Longrightarrow \frac{X''}{X} = \frac{T'}{kT} = 0$$

(3) two ODEs $\begin{cases} X'' + \lambda X = 0 \\ T' + \lambda k T = 0 \end{cases}$

(4) use boundary conditions $u(0,t) = u(L,t) = 0$

$$X(0) = X(L) = 0$$

(5) eigenfunctions of $\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$

- Case $\lambda > 0$ $(\lambda = \alpha^2)$

$0 = r^2 + \alpha^2 \Longrightarrow r = \pm \alpha i \Longrightarrow X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

$\begin{cases} 0 = X(0) = c_1 \\ 0 = X(L) = c_2 \sin(\alpha L) \end{cases} \Longrightarrow \alpha_n = \frac{n\pi}{L}$ for $n = 1, 2, \cdots$

eigenvalues $\lambda_n = \alpha_n^{\alpha} = \left(\frac{n\pi}{L}\right)^2$

eigenfunctions $X_n(x) = \sin \frac{n\pi x}{L}$

for $n = 1, 2, \cdots$

- Case $\lambda < 0$
- Case $\lambda = 0$

$\Big\}$ not eigenvalue

(6) $T_n' + \frac{n^2\pi^2 k}{L^2} T_n = 0$

$$T_n(t) = e^{-\frac{kn^2\pi^2}{L^2}t}$$

(7) $u_n(x,t) = X_n(x)T_n(t) = e^{-\frac{kn^2\pi^2}{L^2}t} \sin\frac{n\pi x}{L}$

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{kn^2\pi^2}{L^2}t} \sin\frac{n\pi x}{L}$$

(8) $c_n = ?$ $\quad f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$

$$\Longrightarrow c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

<u>Examples</u> 2. $L = 50$ cm, $u_0 = 100°C$, $u(0,t) = u(50,t) = 0°C$

Find $u(x,t)$ for $k = 0.15$ (iron) and 0.005 (concrete)

- <u>Insulated Endpt Conditions</u>

$$u_x(0,t) = u_x(L,t) = 0$$

(5) eigenvalues/eigenfunctions $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, \ X'(L) = 0 \end{cases}$

$\lambda_0 = 1, \quad X_0(x) = 1$

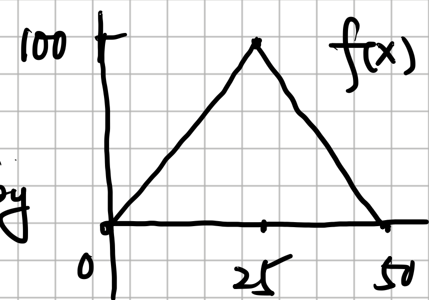
$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \cos\frac{n\pi x}{L} \qquad$ for $n = 1, 2, \cdots$

(7) $u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\frac{kn^2\pi^2}{L^2}t} \cos\frac{n\pi x}{L}$

(8) $c_n = ?$

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{L}$$

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$



Example 3 similar to Example 2 with $f(x)$ given by

§9.6 Vibrating Strings and the One-dimensional Wave Equation

vibrating string

$u(x,t)$ displacement at (x,t)

- one-dimensional wave equation

PDE $\quad \frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}$

BC $\quad u(0,t) = u(L,t)$

IC $\quad u(x,0) = f(x)$ and $u_t(x,0) = g(x)$

Problem A $\quad u(x,0) = f(x), \quad u_t(x,0) = 0$

(Problem B $\quad u(x,0) = 0, \quad u_t(x,0) = g(x)$)

Separation of Variables

$$u(x,t) = X(x)\,T(t)$$

$$u_{tt}=a^2u_{xx} \Longrightarrow X\cdot T''=a^2X''T \Longrightarrow \frac{X''}{X}=\frac{T''}{aT}$$

$$\Longrightarrow \begin{cases} X''+\lambda X=0 \\ X(0)=X(L)=0 \end{cases} \Longrightarrow \begin{array}{l} \lambda_n=\left(\frac{n\pi}{L}\right)^2 \\ X_n(x)=\sin\frac{n\pi x}{L} \end{array} \quad \text{for } n=1,2,\cdots$$

$$\begin{cases} T_n''+a^2\lambda_nT_n=0 \\ T_n'(0)=0 \end{cases} \Longrightarrow T_n(t)=\cos\frac{n\pi at}{L}$$

$$\Longrightarrow u(x,t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$$

$$f(x)=u(x,0)=\sum c_n\sin\frac{n\pi x}{L} \Longrightarrow c_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

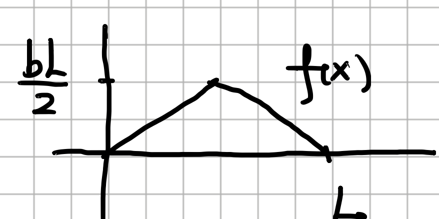
Example 1 (Problem A)

$$\begin{cases} u_{tt}=4u_{xx}, \quad x\in(0,\pi) \\ u(0,t)=u(\pi,t)=0, \\ u(x,0)=\frac{1}{10}\sin^3x=\frac{3}{40}\sin x-\frac{1}{40}\sin 3x, \quad u_t(x,0)=0 \end{cases}$$

$$\Rightarrow u(x,t)=\frac{3}{40}\cos 2t\sin x-\frac{1}{40}\cos 6t\sin 3x.$$

Example 2

$$\begin{cases} u_{tt}=a^2u_{xx}, \quad x\in(0,L) \\ u(0,t)=u(L,t)=0 \\ u(x,0)=f(x), \quad u_t(x,0)=0 \end{cases}$$



$$f(x)=\sum A_n\sin\frac{n\pi x}{L}, \quad A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx=\frac{4bL}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$u(x,t)=\frac{4bL}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}$$

Initial Conditions

$$u(x,0)=f(x) \quad \text{and} \quad u_t(x,0)=g(x)$$

$$0=T_n'' + a^2\lambda_n T_n \implies T_n(t) = a_n \cos\frac{an\pi t}{L} + b_n \sin\frac{an\pi t}{L}$$

$$\implies \boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\cos\frac{an\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\sin\frac{an\pi t}{L}}$$

$$f(x)=u(x,0)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \implies \boxed{a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$$

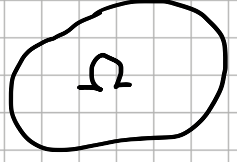
$$g(x)=u_t(x,0)=\sum_{n=1}^{\infty} b_n\cdot\frac{an\pi}{L}\sin\frac{n\pi x}{L} \implies \boxed{b_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx}$$

§9.7 Steady-State Temperature and Laplace's Equation

2D heat equation
$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = k\,\Delta u$$

2D wave equation
$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = a^2\,\Delta u$$

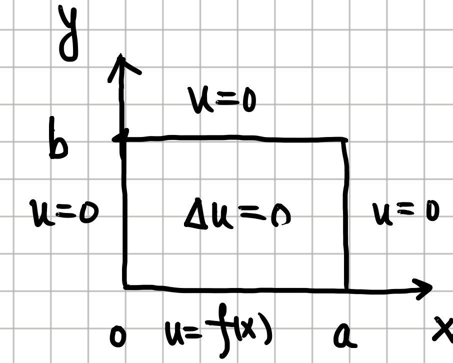
2D Laplace equation
$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = 0 & \text{in } \Omega \subset \mathbb{R}^2 \\[2ex] u\big|_{\partial\Omega} = f(x,y) \end{cases}$$



Dirichlet Boundary Condition

Example 1

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega = (0,a)\times(0,b) \\ u(0,y) = u(a,y) = u(x,b) = 0 \\ u(x,0) = f(x) \end{cases}$$



Solution $\quad u(x,y) = X(x)\,Y(y)$

$$0 = \Delta u \implies 0 = X''Y + XY'' \implies \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$$

$$0 = u(0,y) \Rightarrow X(0) = 0 \quad \text{and} \quad 0 = u(a,y) \Rightarrow X(a) = 0$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(a) = 0 \end{cases} \implies \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{a}$$

$$0 = u(x,b) \Rightarrow Y(b) = 0$$

$$\begin{cases} Y_n'' - \lambda_n Y_n = 0 \\ Y_n(b) = 0 \end{cases} \implies Y_n(y) = A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a}$$

$$0 = Y_n(b) = A_n \cosh\frac{n\pi b}{a} + B_n \sinh\frac{n\pi b}{a} \implies B_n = -A_n \coth\left(\frac{n\pi b}{a}\right)$$

$$Y_n(y) = \frac{A_n}{\sinh\left(\frac{n\pi b}{a}\right)}\left[\sinh\left(\frac{n\pi b}{a}\right)\cosh\frac{n\pi y}{a} - \cosh\frac{n\pi b}{a}\sinh\frac{n\pi y}{a}\right]$$

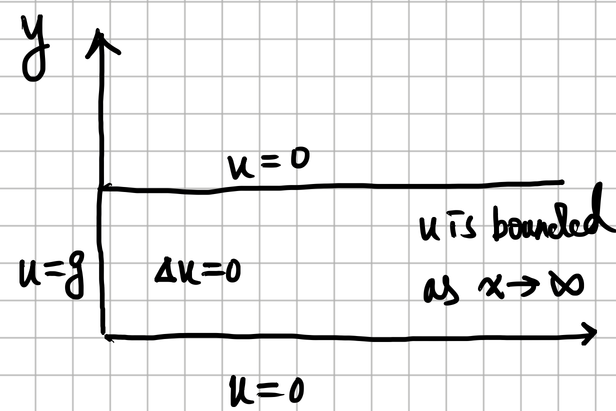
$$= c_n \sinh\frac{n\pi(b-y)}{a}$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} X_n(x)\,Y_n(y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi(b-y)}{a}}$$

$$0 = f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi b}{a}\,\sin\frac{n\pi x}{a} \Longrightarrow \boxed{c_n = \frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx}$$

<u>Example 2</u>

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = (0,\infty)\times(0,b) \\ u(x,0) = u(x,b) = 0 & \text{in } (0,\infty) \\ u(x,y) \text{ is bounded as } x\to\infty \\ u(0,y) = g(y) \end{cases}$$



<u>Solution</u>

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = Y(b) = 0 \end{cases} \Longrightarrow \lambda_n = \left(\frac{n\pi}{b}\right)^2, \quad Y_n(y) = \sin\frac{n\pi y}{b}$$

$$X_n'' - \left(\frac{n\pi}{b}\right)^2 X_n = 0 \Longrightarrow X_n(x) = e^{-\frac{n\pi x}{b}}$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n\, e^{-\frac{n\pi x}{b}} \sin\frac{n\pi y}{b}}$$

$$g(y) = u(0,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{b} \Longrightarrow \boxed{b_n = \frac{2}{b}\int_0^b g(y)\sin\frac{n\pi y}{b}\,dy}$$

§10.1 Sturm-Liouville Problems and Eigenfunction Expansions

- Sturm-Liouville problems

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)\,y + \lambda\, r(x)\, y = 0$$