Name: ______________________
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Midterm 1– Math 303 (10/07/21)
SHOW ALL RELEVANT WORK!!!

1. (10pts) The general solution of the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$ is given by

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find the solution satisfying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$.

$$\begin{pmatrix} 3 \\ -7 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$4 \left[\left(\begin{array}{cc|c} 1 & 1 & 3 \\ -4 & 1 & -7 \end{array} \right) \right. \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 5 & 5 \end{array} \right) \Rightarrow c_2 = 1$$

$$c_1 + c_2 = 3 \Longrightarrow c_1 = 3 - c_2 = 3 - 1 = 2$$

$$\vec{x}(t) = 2 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

97	89	77	71	68	56	46	34
96	84	75	70	67	54	46	
95	82	74	70	67	52	44	
94	82	74	70	66	50	41	
93	82	74		66			
93	82	74		65			
90		73		61			
		72					
7	6	12		7	4	4	1 = 41

2. (16pts) Find three linearly independent solutions in terms of **real-valued** (not complex-valued) functions of the system

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x}.$$

$$0 = \begin{vmatrix} -1-\lambda & 1 & 0 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = -(\lambda+1)\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = -(\lambda+1)\left[(\lambda+1)^2+1\right]$$

$$\Rightarrow \lambda_1 = -1, \quad \lambda_{2,3} = -1 \pm i \quad 4$$

$\underline{\lambda_1 = -1}$ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{cases} b=0 \\ -a=0 \end{cases} \Rightarrow \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = e^{-t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = -1+i}$ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & -i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{cases} -ia+b=0 \Rightarrow b=ai \\ -ic=0 \Rightarrow c=0 \end{cases}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$e^{-t}(\cos t + i\sin t)\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i\right]$$

$$= e^{-t}\left\{\begin{pmatrix} \cos t \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \sin t \\ 0 \end{pmatrix} + i\left[\begin{pmatrix} 0 \\ \cos t \\ 0 \end{pmatrix} + \begin{pmatrix} \sin t \\ 0 \\ 0 \end{pmatrix}\right]\right\}$$

$$\vec{x}_2(t) = e^{-t}\begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}, \quad \vec{x}_3(t) = e^{-t}\begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix}$$

$$\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix}$$

4 4 4

or $\begin{pmatrix} i \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$c_2 e^{-t}\begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} = \vec{x}_2(t)$

$c_3 e^{-t}\begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} = \vec{x}_3(t)$

3. For each of the following ~~three~~ two systems:

- Find the eigenvalues, eigenvectors, and general solution.
- Classify the critical point $(0, 0)$ as to type and determine whether it is stable, asymptotical stable, or unstable.

(a)(17pts) $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}.$

(1) $0 = \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = (\lambda-1)(\lambda+4)+6 = \lambda^2+3\lambda+2 = (\lambda+1)(\lambda+2) \Rightarrow \lambda=-1,\ \lambda=-2.$

$\underline{\lambda=-1}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a=b \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda=-2}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow b=\frac{3}{2}a \Rightarrow \vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \vec{x}_2(t) = e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

4.5 ~~4~~ 4.5 ~~4~~

(2) improper nodal sink 4

asymptotic stable

~~3~~ 4

(b)(17pts) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}.$

(1) $0 = \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda-3)(\lambda+1)+4 = \lambda^2-2\lambda+1 = (\lambda-1)^2 \Rightarrow \lambda_1=\lambda_2=1$

$\underline{\lambda_1=1}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a=2b \Rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}^2 \vec{v}_2 = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \vec{v}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$\Rightarrow \vec{x}(t) = c_1 e^t \begin{pmatrix} 2 \\ -4 \end{pmatrix} + c_2 \left[te^t \begin{pmatrix} 2 \\ -4 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$

4 8

or $= c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$

$e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(2) improper nodal source

unstable

$\lambda = 1, 1$ +2

$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ +2

5pt : each critical pt (4 critical pts)

(1) pt (a,b), (2) Jacobian $J(a,b)$, (3) eigenvalues, (4) classification

4. For the nonlinear systems in (a) and (b),
(1) find all critical points;
(2) find and classify the almost linear systems at all critical points.

$$(a)(20\text{pts}) \quad \begin{cases} \frac{dx}{dt} = 30x - 2x^2 - xy, \\ \frac{dy}{dt} = 20y - 4y^2 + 2xy \end{cases}$$

(1) $\begin{cases} 0 = x(30-2x-y) \\ 0 = y(20-4y+2x) \end{cases}$ $(0,0)$, $(0,5)$, $(15,0)$, $(10,10)$

$\begin{cases} 2x+y=30 \\ -2x+4y=20 \end{cases} \Rightarrow 5y=50 \Rightarrow y=10$

$x = \frac{1}{2}(30-y) = \frac{1}{2}\cdot 20 = 10$

(2) $J(x,y) = \begin{pmatrix} 30-4x-y & -x \\ 2y & 20-8y+2x \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 30 & 0 \\ 0 & 20 \end{pmatrix} \Rightarrow \lambda_1 = 30 > 0,\ \lambda_2 = 20 > 0 \Rightarrow$ improper nodal source unstable

$J(0,5) = \begin{pmatrix} 25 & 0 \\ 10 & -20 \end{pmatrix} \Rightarrow \lambda_1 = -20 < 0,\ \lambda_2 = 25 > 0 \Rightarrow$ unstable saddle pt.

$J(15,0) = \begin{pmatrix} -30 & -15 \\ 0 & 50 \end{pmatrix} \Rightarrow \lambda_1 = -30 < 0,\ \lambda_2 = 50 > 0 \Rightarrow$ unstable saddle pt.

$J(10,10) = \begin{pmatrix} -20 & -10 \\ 20 & -40 \end{pmatrix}$ $0 = \begin{vmatrix} -20-\lambda & -10 \\ 20 & -40-\lambda \end{vmatrix} = (\lambda+20)(\lambda+40)+200$

$= \lambda^2 + 60\lambda + 1000$

$= \lambda^2 + 2\cdot 30\lambda + 30^2 + 100$

$= (\lambda+30)^2 + 10^2$

$\Rightarrow \lambda_{1,2} = -30 \pm 10i$

asy. stable spiral sink.

$$(b)(20\text{pts}) \quad \begin{cases} \frac{dx}{dt} = 3\sin x + y, \\ \frac{dy}{dt} = \sin x + 2y \end{cases}$$

(1) $\begin{cases} 0 = 3\sin x + y \\ 0 = \sin x + 2y \end{cases} \Longrightarrow 5\sin x = 0 \Longrightarrow \sin x = 0 \Longrightarrow x = 0, \pm\pi, \pm 2\pi, \cdots$

$y = 0$ 2 $\boxed{(\pm n\pi, 0) \text{ for } n = 0, 1, \cdots}$

(2) $J(x,y) = \begin{pmatrix} 3\cos x & 1 \\ \cos x & 2 \end{pmatrix}$

(i) $n = 2m+1$ (odd) $\quad J(\pm n\pi, 0) = \begin{pmatrix} 3\cos(2m+1)\pi & 1 \\ \cos(2m+1)\pi & 2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -1 & 2 \end{pmatrix}$ 3

(9)

$$0 = \begin{vmatrix} -3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = (\lambda+3)(\lambda-2) + 1 = \lambda^2 + \lambda - 5$$

$$\Longrightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{1+20}}{2}$$

$\lambda_1 = (-1-\sqrt{21})/2 < 0$

3

$\lambda_2 = \frac{1}{2}(-1+\sqrt{21}) > 0$

unstable saddle pt. 3

(ii) $n = 2m$ (even) $\quad J(\pm(2m)\pi, 0) = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ 3

(9)

$$0 = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-3)(\lambda-2) - 1 = \lambda^2 - 5\lambda + 5$$

$$\lambda_{1,2} = \frac{1}{2}\left[5 \pm \sqrt{25 - 4 \cdot 5}\right] = \frac{1}{2}\left[5 \pm \sqrt{5}\right]$$

$\lambda_1 = \frac{1}{2}(5-\sqrt{5}) > 0$

3

$\lambda_2 = \frac{1}{2}(5+\sqrt{5}) > 0$

unstable improper nodal source 3