Name: ____________________

PUID#: ____________________

Midterm 2– Math 303 (11/09/21)

SHOW ALL RELEVANT WORK!!!

1. (14pts) Use the Laplace transform to solve the initial value problem

$$x'' - 5x' + 6x = 0, \quad x(0) = 1, \quad x'(0) = +2.$$

$$0 = s^2 X(s) - s\,x(0) - x'(0) - 5\left[s X(s) - x(0)\right] + 6X(s)$$

$$= (s^2 - 5s + 6)X(s) - s - 2 + 5$$

7 $\cancel{8}$ $$= (s-2)(s-3)X(s) - (s-3)$$

$$\Rightarrow X(s) = \frac{1}{s-2}$$

$\cancel{8}$ 7 $$x(t) = e^{2t}$$

2. (~~12~~ 10pts) Find the Laplace transform of $f(t) = \int_0^t e^{-(t-\tau)} \cos\tau \, d\tau$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\}\,\mathcal{L}\{\cos t\}$$

$$= \frac{1}{s+1} \cdot \frac{s}{s^2+1}$$

100 99 89 85 80 78 69 60 58 44

97 89 85 71 69 56 26

95 88 85 70 69 51

95 88 84 70 67 50

94 86 84 63

93 86 84 62

93 85 80 62

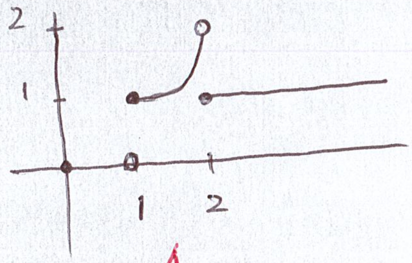
1 7 15 4 8 4 2 = 41

wrong $f(t)$ -9

right $\mathcal{L}\{f(t)\} = F(s)$ 5

18

3. (~~16~~pts) Sketch the graph and find the Laplace transform (using step function) of



$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2 - 2t + 2, & 1 \le t < 2 \\ 1, & 2 \le t < \infty. \end{cases}$$

4

$t^2 - 2t + 1$

$= t(t-2) + 1$

$= (t-2)^2 + 2(t-2) + 1$

4 $= [u_1(t) - u_2(t)](t^2 - 2t + 2) + u_2(t)$

$= -u_2(t)(t^2 - 2t + 1) + u_1(t)[(t-1)^2 + 1]$

$= -u_2(t)[(t-2)^2 + 2(t-2) + 1] + u_1(t)[(t-1)^2 + 1]$

6 $= -u_2(t)\, g(t) + u_1(t)\, h(t)$

where $g(t) = t^2 + 2t + 1$, $h(t) = t^2 + 1$

$\mathcal{L}\{f(t)\} \overset{4}{=} -e^{-2s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right] + e^{-s}\left[\frac{2}{s^3} + \frac{1}{s}\right]$

4. (12pts) Find the Laplace transform of $f(t) = \cancel{t}\, t^2 \cos(2t)$.

$\mathcal{L}\{f(t)\} = (-1)^2 F''(s)$, $F(s) = \mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$

10 $= +\left(\frac{s}{s^2+4}\right)'' = \left[+\frac{(s^2+4) - s \cdot 2s}{(s^2+4)^2}\right]' = +\left[\frac{4 - s^2}{(s^2+4)^2}\right]'$

$\left[s(s^2+4)^{-1}\right]''$

$= \left[(s^2+4)^{-1} - s(s^2+4)^{-2} \cdot 2s\right]'$

$= \left[(s^2+4)^{-1} - 2s^2(s^2+4)^{-2}\right]'$

$= -2s(s^2+4)^{-2} - 4s(s^2+4)^{-2} + 4s^2(s^2+4)^{-3} \cdot 2s$

$= -2s(s^2+4)^{-2} - 4s(s^2+4)^{-2} + 8s^3(s^2+4)^{-3}$ 2

$= -6s(s^2+4)^{-2} + 8s^3(s^2+4)^{-3}$

$= -\left(\frac{s^2-4}{(s^2+4)^2}\right)' = \cancel{\left(1 - \frac{8}{s^2+4}\right)' = +8(-1)(s^2+4)^{-2} \cdot 2s}$

2 $= \cancel{\frac{16s}{(s^2+4)^2}} - \frac{2s(s^2+4)^2 - (s^2-4) \cdot 2(s^2+4) \cdot 2s}{(s^2+4)^4}$

$= -2s\frac{(s^2+4) - 2(s^2-4)}{(s^2+4)^3} = \frac{2s(s^2-8)}{(s^2+4)^3}$

$-s^2+8$

5. (14pts) Find the inverse Laplace transform of $F(s) = \frac{1}{s(s^2+4s+5)}$. $= \dfrac{1}{s[(s+2)^2+1]}$

3

$$= \frac{A}{s} + \frac{Bs+C}{s^2+4s+5} = \frac{A(s^2+4s+5) + s(Bs+C)}{s(s^2+4s+5)}$$

$$1 = (A+B)s^2 + (4A+C)s + 5A \Rightarrow A = \frac{1}{5} \quad C = -4A = -\frac{4}{5}$$

$$B = -A = -\frac{1}{5}$$

6

$$F(s) = \frac{1}{5}\left[\frac{1}{s} - \frac{s+4}{(s+2)^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{1}{s} - \frac{s+2}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1}\right]$$

5

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{5}\left[1 - e^{-2t}\cos t - 2e^{-2t}\sin t\right]$$

6. (12pts) Find the inverse Laplace transform of $F(s) = \frac{e^{-s} - e^{2-2s}}{s^2}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$F(s) = e^{-s}\frac{1}{s^2} - e^2 \cdot e^{-2s}\frac{1}{s^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = u_1(t)(t-1) - e^2 \cdot u_2(t)(t-2)$$

$u_1(t)\,t - e^2 t\, u_2(t)$ $\quad -4$

$u(t-1)(t-1) - u(2t-2)(2t-2)$ $\quad -4$

7. (20pts) Use the Laplace transform to solve the initial value problems

$$y'' + y = u_{\frac{\pi}{2}}(t) + 3\delta(t - \frac{3\pi}{2}) - u_{2\pi}(t), \quad y(0) = 1, \quad y'(0) = 2.$$

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{1}{s}e^{-\frac{\pi}{2}s} + 3e^{-\frac{3\pi}{2}s} - \frac{1}{s}e^{-2\pi s}$$

$$(s^2+1)Y(s) = s+2 + \frac{1}{s}e^{-\frac{\pi}{2}s} + 3e^{-\frac{3\pi}{2}s} - \frac{1}{s}e^{-2\pi s}$$

8

$$Y(s) = \frac{s+2}{s^2+1} + \frac{1}{s(s^2+1)}\left(e^{-\frac{\pi}{2}s} - e^{-2\pi s}\right) + \frac{3}{s^2+1}e^{-\frac{3\pi}{2}s}$$

$$\frac{s+2}{s^2+1} = \left(\frac{s}{s^2+1} + 2\frac{1}{s^2+1}\right) \qquad \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+1}\right\} = \cos t + 2\sin t, \quad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$$

4   2

$$\Rightarrow y(t) = \left(\cos t + 2\sin t\right) + u_{\frac{\pi}{2}}(t)\left(1 - \cos\left(t - \frac{\pi}{2}\right)\right)$$

2   4

$$- u_{2\pi}(t)\left(1 - \cos(t - 2\pi)\right) + u_{\frac{3\pi}{2}}(t)\cdot 3\cdot \sin\left(t - \frac{3\pi}{2}\right)$$

no shift -5