Numerical Analysis

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Chapter One Mathematical Preliminaries

§1.1 Basic Concepts and Taylor's Thrm

limit	$\lim_{x \to c} f(x) = L$
continuity	$\lim_{x \to c} f(x) = f(c)$
derivative	$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

Intermediate-Value Thrm for Continuous Functions

$$f(x) \in C[a,b] \Longrightarrow \forall c \text{ between } f(a) \text{ and } f(b)$$

$$\exists \xi \in [a,b] \text{ s.t. } f(\xi) = c$$

Taylor's Thrm with Lagrange Remainder

$f \in C^n[a,b]$ and $f^{(n+1)}$ exists on (a,b)

$$\Longrightarrow \forall x, c \in [a,b] \qquad f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (x-c)^k + E_n(x)$$

where, for some ξ between x and c,

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x-c)^{n+1} .$$

Mean-Value Thrm

$f \in C[a,b]$ and f' exists on (a,b)

$\Rightarrow \forall x, c \in [a,b]$, $\exists \xi$ between x and c

s.t. $$\frac{f(x)-f(c)}{x-c} = f'(\xi)$$

Rolle's Thrm

$f \in C[a,b]$, f' exists on (a,b), $f(a)=f(b)$

$\Rightarrow \exists \xi \in (a,b)$ s.t. $f'(\xi)=0$

Taylor's Thrm with Integral Remainder

$f \in C^{n+1}[a,b]$, $\forall x, c \in [a,b]$

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k + R_n(x)$$

where $$R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n dt$$

Proof

$$= \int_c^x \frac{(x-t)^n}{n!} df^{(n)}(t)$$

$$= \frac{(x-t)^n}{n!} f^{(n)}(t)\Big|_{t=c}^{t=x} - \int_c^x f^{(n)}(t)\, d\left(\frac{(x-t)^n}{n!}\right)$$

$$= -\frac{1}{n!} f^{(n)}(c)(x-c)^n + R_{n-1}$$

Alternative Form of Taylor's Thrm

$$f(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

Taylor's Thrm in 2-Variables

Let $R = [a,b] \times [c,d]$, $f \in C^{n+1}(R)$,

(x,y), $(x+h, y+k) \in R$

$$\Rightarrow f(x+h, y+k) = \sum_{i=0}^{n} \frac{1}{i!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{i} f(x,y)$$

$$+ \frac{1}{(n+1)!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{n+1} f(x+\theta h, y+\theta k)$$

§1.2 Orders of Convergence and Additional Basic Concepts

Convergent Sequences $\{x_n\}$

$$f(x)=x^2-2=0$$

$$0 \approx f(x) \approx f(x_n)+f'(x_n)\cdot(x-x_n)$$

$$\Rightarrow x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$\lim_{n\to\infty} x_n = L$$

examples $x_n=\left(1+\frac{1}{n}\right)^n \to e$, $\begin{cases} x_0=20,\ x_1=15 \\ x_{n+1}=x_n-\dfrac{x_n^2}{x_n^2+x_{n-1}^2} \end{cases}$, $\begin{cases} x_1=2 \\ x_{n+1}=\frac{1}{2}x_n+\frac{1}{x_n} \end{cases}$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad \sqrt{2}$$

$$\frac{|x_{n+1}-e|}{|x_n-e|}\to 1, \qquad \frac{|x_{n+1}|}{|x_n|}\to 0, \qquad \frac{|x_{n+1}-\sqrt{2}|}{|x_n-\sqrt{2}|^2}\le 0.36$$

worse than linear, superlinear, quadratic

Order of Convergence $x_n \to x^*$ as $n\to\infty$

linear: $|x_{n+1}-x^*| \le c\,|x_n-x^*|$ with $c<1$ $\forall\, n\ge N$

superlinear: $|x_{n+1}-x^*| \le \varepsilon_n\,|x_n-x^*|$, $\varepsilon_n\to 0$, $\forall\, n\ge N$

quadratic: $|x_{n+1}-x^*| \le C\,|x_n-x^*|$, $\forall\, n\ge N$

order α: $|x_{n+1}-x^*| \le C\,|x_n-x^*|^\alpha$, $\forall\, n\ge N$.

Big O and Little o Notation

$\{x_n\}, \{\alpha_n\}$ sequences

- $x_n = O(\alpha_n) \iff \exists\ C > 0$ and n_0 s.t.

$$|x_n| \leq C|\alpha_n| \quad \forall n \geq n_0$$

- $x_n = o(\alpha_n) \iff \exists \overset{\text{positive}}{\ } \varepsilon_n \to 0$ and n_0 s.t.

$$|x_n| \leq \varepsilon_n |\alpha_n| \quad \forall n \geq n_0$$

examples

$$\frac{n+1}{n^2} = O\left(\frac{1}{n}\right), \quad \frac{1}{n \ln n} = o\left(\frac{1}{n}\right), \quad \text{as } n \to \infty$$

$$\frac{5}{n} + e^{-n} = O\left(\frac{1}{n}\right), \quad e^{-n} = o\left(\frac{1}{n^2}\right)$$

$$\ln 2 - \sum_{k=1}^{n-1} (-1)^{k-1} \frac{1}{k} = O\left(\frac{1}{n}\right), \quad e^x - \sum_{k=0}^{n-1} \frac{1}{k!} x^k = O\left(\frac{1}{n!}\right)$$

$$|x| \leq 1$$

- $f(x) = \boxed{\cancel{g(x)+}}\, O(g(x))$ as $x \to 0 \iff |f(x)| \leq C|g(x)|$

$$\sin x = x - \frac{x^3}{6} + O(x^5)$$

- as $x \to x^*$

$$f(x) = O(g(x)) \iff |f(x)| \leq C|g(x)| \quad \text{in } O(x^*;\varepsilon)$$

$$f(x) = o(g(x)) \iff \lim_{x \to x^*} \frac{f(x)}{g(x)} = 0$$

▨ Mean-Value Thrm for Integrals

$$u, v \in C[a,b], \quad v \geq 0 \text{ on } [a,b]$$

$$\Longrightarrow \quad \exists\, \xi \in [a,b] \text{ s.t. } \int_a^b u(x)v(x)\,dx = u(\xi)\int_a^b v(x)\,dx$$

Proof $u \in C[a,b] \Longrightarrow \alpha \leq u(x) \leq \beta \quad \forall x \in [a,b]$

$$\xrightarrow{v \geq 0} \alpha v \leq uv \leq \beta v$$

$$\Longrightarrow \alpha\int_a^b v \leq \int_a^b uv \leq \beta\int_a^b v \Rightarrow \alpha \leq \frac{\int_a^b uv}{\int_a^b v} \leq \beta$$

$$\Longrightarrow \exists\, \xi \in [a,b] \text{ s.t. } u(\xi) = \frac{\int_a^b uv}{\int_a^b v} \quad \#$$

Upper and Lower Bounds

S — a nonempty, bounded set of real numbers

$$\forall x \in S, \quad \underset{\text{lower bound}}{a} \leq x \leq \underset{\text{upper bound}}{b}$$

~~Axiom~~ Def. of Supremum (least upper bound)

$$v = \sup S \iff \begin{cases} (1)\ x \leq v & \forall x \in S \\ (2)\ \forall \varepsilon > 0,\ \exists\, x \in S, \text{ s.t. } & x > v - \varepsilon. \end{cases}$$

Axiom 1 S is nonempty and has upper bounds $\Longrightarrow \sup S$ exists.

example $\sup\{x \mid x^2 < 2\} = \sqrt{2}$

Def. of Infimum (greatest lower bound)

$$u = \inf S \iff \begin{cases} (1) & u \leq x, \ \forall x \in S \\ (2) & \forall \varepsilon > 0, \ \exists x \in S, \ \text{s.t.}, \quad u + \varepsilon \geq x . \end{cases}$$

Explicit and Implicit Functions

$$f(x) = x^2 \qquad \qquad G(x,y)$$

$$y = f(x) \text{ satisfies } \quad \underset{}{y^2 + 3xy - 7} \overset{\text{\scriptsize $\parallel$}}{} = 0$$

Implicit Function Thrm

$O(x_0, y_0)$ — a neighborhood of (x_0, y_0)

$G(x,y) \in C^1\big(O(x_0,y_0)\big)$, $G(x_0,y_0) = 0$, and $\frac{\partial G}{\partial y}(x_0,y_0) \neq 0$

$\Longrightarrow \exists \, \delta > 0$, and $\exists$ a cont. diff. $f(x)$ defined on $|x - x_0| < \delta$

s.t. $\quad f(x_0) = y_0$ and $G(x, f(x)) = 0$ on $|x - x_0| < \delta$.

§1.3 Difference Equations

$$x=[x_1,x_2,\cdots,x_n,\cdots] \quad \text{— sequence of complex numbers}$$

a diff. eq. $\quad x_{n+2}-3x_{n+1}+2x_n=0, \quad n\geq 1$

Find solution $[x_1,\cdots,x_n,\cdots]$?

setting $\quad x_n=\lambda^n$

$$\Longrightarrow \quad 0=\lambda^{n+2}-3\lambda^{n+1}+2\lambda^n$$
$$=\lambda^n(\lambda^2-3\lambda+2)$$
$$=\lambda^n(\lambda-1)(\lambda-2) \quad \Longrightarrow \lambda=0,\ 1,\ 2$$

$\lambda=0$ $\quad x=[0,0,\cdots]$ — trivial solution

$\lambda=1$ $\quad u=[1,1,\cdots]=\{1^n\}$

$\lambda=2$ $\quad v=[2,4,8,\cdots,2^n,\cdots]=\{2^n\}$

general solution $\quad x=\alpha u+\beta v \quad \forall \alpha,\beta.$

$$\begin{cases} x_1=\alpha+2\beta \\ x_2=\alpha+4\beta \end{cases} \xrightarrow{\det\begin{bmatrix}1&2\\1&4\end{bmatrix}\neq 0} \exists 1\ \alpha \text{ and } \beta$$

for $n\geq 3$, $x_n=\alpha u_n+\beta v_n$ by induction

shift operator $\quad E$

$$\forall x=[x_1,\cdots,x_n,\cdots] \quad \text{define} \quad Ex=[x_2,x_3,\cdots]$$

$$\Longleftrightarrow \quad (Ex)_n=x_{n+1}$$

$E^0 x = x$ by convention

$(E^2 x)_n = x_{n+2}$, . . .

diff. eq.

$$0 = x_{n+2} - 3x_{n+1} + 2x_n$$
$$= (E^2 x)_n - (3E x)_n + (2E^0 x)_n \Longrightarrow \underset{\underset{p(E)}{\parallel}}{\left(E^2 - 3E + 2I\right)} x = 0$$

where $p(\lambda) = \lambda^2 - 3\lambda + 2$

characteristic polynomial $\qquad p(E)x = 0$

<u>general diff. eq.</u> $\quad p(E)\, x = 0 \quad$ with $L = p(E) = c_0 I + c_1 E^1 + \cdots + c_m E^m$
$$= \sum_{i=0}^{m} c_i E^i$$

$$p(\lambda) = \sum_{i=0}^{m} c_i \lambda^i \text{ — characteristic eq.}$$

- Simple Root

<u>Thrm on Null Space $L = p(E)$</u> (solutions of $Lx = 0$)

(1) $\lambda \neq 0$ is a simple root of $p \Longrightarrow u = [\lambda, \lambda^2, \lambda^3, \cdots]$ is one solution of $p(E)x = 0$

(2) $\lambda_k \neq 0$ for $k = 1, \cdots, m$ are roots of p and $\lambda_i \neq \lambda_j$ for $i \neq j$ $\Longrightarrow$ general solution of $p(E)x = 0$ is $\sum_{k=1}^{m} a_k u^{(k)}$

where $u^{(k)} = [\lambda_k, \lambda_k^2, \cdots]$.

<u>Proof</u> (1) $Eu = \lambda u$ and $E^i u = \lambda^i u \Longrightarrow p(E)u = p(\lambda)u = 0$

(2) Let $x = [x_1, \cdots, x_m, x_{m+1}, \cdots]$ be any solution of $p(E)x = 0$

want to prove that $\exists \{a_k\}_{k=1}^{m}$ s.t. $x = \sum_{k=1}^{m} a_k u^{(k)}$

- Let $\{a_k\}_{k=1}^{m}$ satisfy

$$[x_1, \cdots, x_m] = \sum_{k=1}^{m} a_k [\lambda_k, \lambda_k^2, \cdots, \lambda_k^m]$$

which has a unique solution because

$$\begin{vmatrix} \lambda_1 & \cdots & \lambda_1^m \\ \vdots & & \vdots \\ \lambda_m & \cdots & \lambda_m^m \end{vmatrix} = \prod_{i=1}^{m} \lambda_i \prod_{0 \le k < j \le m-1} (\lambda_j - \lambda_k) \ne 0$$

- Let $z = x - \sum_{k=1}^{m} a_k u^{(k)}$

$$\xrightarrow[p(E)u^{(k)}=0]{p(E)x=0} p(E)z = 0 \iff 0 = \sum_{i=0}^{m} c_i E^i z_n = \sum_{i=0}^{m} c_i z_{n+i} \quad \forall n$$

$$\xrightarrow{c_m \ne 0 \ (\deg p(\lambda) = m)} z_{n+m} = -\frac{1}{c_m}\left(c_0 z_n + \cdots + c_{m-1} z_{n+m-1}\right)$$

$$\xrightarrow{z_1 = \cdots = z_m = 0} 0 = z_{m+1} = z_{m+2} = \cdots$$

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- Multiple Roots

Let $\lambda \ne 0$ be a root of $p(\lambda)$ having multiplicity k

$\Longrightarrow$ k basic solutions of $p(E)x = 0$ are

$$x(\lambda), \ x'(\lambda), \ \cdots, \ x^{(k-1)}(\lambda)$$

where $x(\lambda) = [\lambda, \lambda^2, \lambda^3, \cdots]$

Stable Difference Equations

- $x = [x_1, x_2, \cdots]$ is bounded $\Longleftrightarrow$ $|x_n| \le c \ \forall n$ $\Longleftrightarrow$ $\sup_n |x_n| < \infty$.
- $p(E)x = 0$ is stable $\Longleftrightarrow$ all of its solutions are bounded.

Thrm on Stable Diff. Eq. assume that a poly. p satisfies $p(0) \neq 0$ then these properties are equivalent

(1) $p(E)x = 0$ is stable

(2) All roots of p satisfy $|z| \le 1$ and all multiple roots satisfy $|z| < 1$.