Chapter 3 Solution of Nonlinear Equations

§3.0 Introduction

$f : R \to R$

Find $x \in R$ s.t. $f(x) = 0$.

methods: bisection, Newton, secant

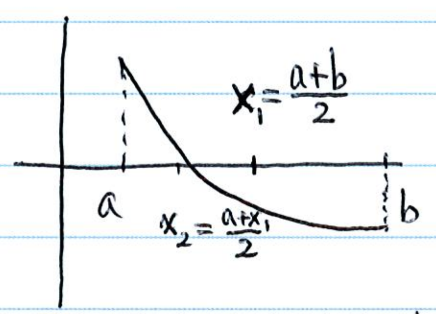
theory: fixed-pt, continuation

zeros of polynomials

§3.1 Bisection (Interval Having) Method

Assumption $f \in C[a,b]$ and $f(a)f(b) < 0$

$$\Longrightarrow \exists\ r \in (a,b) \text{ s.t. } f(r) = 0$$



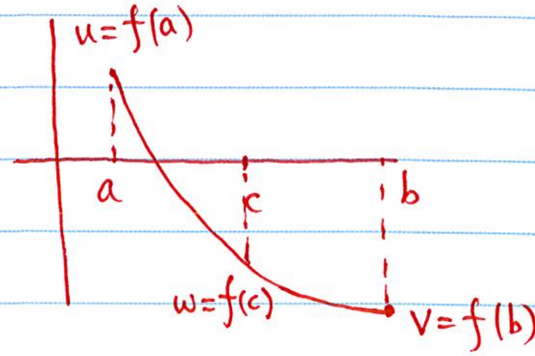
stopping criterion $|f(x_n)| < 10^{-5}$

on Marc-32

Example Find x s.t. $e^x = \sin x$ closest to 0.

$f(x) = e^x - \sin x$ $[a,b] = [-4,-3]$ (P76)

Bisection Algorithm



$e = b - a$

```
input   a, b, M, δ, ε
        u ← f(a)
        v ← f(b)
        e ← b - a
output  a, b, u, v

  if sign(u) = sign(v) then stop
  for k = 1 to M do
      e ← e/2
      c ← a + e
      w ← f(c)
      output k, c, w, e
      if |e| < δ or |w| < ε then stop
      if sign(w) ≠ sign(u) then
          b ← c
          v ← w
      else
          a ← c
          u ← w
      end if
  end do
```

$c = a + (b-a)/2$

$\frac{a+b}{2}$ could moves outside of $[a, b]$ on a limited precision machine

$uv < 0$
unnecessary multiplication
overflow or underflow

3 stopping criterion:
M, δ, ε

Error Analysis

$[a_0, b_0] = [a, b]$

$\Rightarrow$ $a_0 \le a_1 \le a_2 \cdots \le b_0 \Rightarrow \lim_{n\to\infty} a_n$ exists

$b_0 \ge b_1 \ge b_2 \cdots \ge a_0 \Rightarrow \lim_{n\to\infty} b_n$ exists.

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$$

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \frac{1}{2^n}(b_0 - a_0)$$

$$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = r$$

$$0 \ge f(a_n) f(b_n) \Rightarrow 0 \ge [f(r)]^2 \Rightarrow f(r) = 0$$

Let $c_n = \frac{1}{2}(a_n + b_n)$

$$|r - c_n| \le \frac{1}{2}(b_n - a_n) = \frac{1}{2^{n+1}}(b_0 - a_0)$$

Thm on Bisection Method

Let $[a_n, b_n]$ be the intervals in the bisection method

$\Rightarrow$ (1) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r$ and $f(r) = 0$

(2) $\left| r - \frac{1}{2}(a_n + b_n) \right| \le \frac{1}{2^{n+1}}(b_0 - a_0)$

example $[a, b] = [50, 63]$, find n s.t. $\left| \frac{r - c_n}{r} \right| \le 10^{-12}$

$$\text{!`} \frac{|r - c_n|}{50} \le \frac{1}{50} \cdot \frac{1}{2^{n+1}}(63 - 50)$$

$\Rightarrow n \ge 37$

§3.2 Newton's Method (Newton-Raphson Iteration)

Assumption $f''(x)$ exists and is cont. and $f(r)=0$

Let x be an approx to r $\Longrightarrow r = x+h$

$$0 = f(r) = f(x+h) = f(x) + hf'(x) + O(h^2)$$

$$\approx f(x) + hf'(x) \qquad \text{if } h = r-x \text{ is small}$$

$$\Longrightarrow h = -\frac{f(x)}{f'(x)}$$

$$\Longrightarrow x+h = x - \frac{f(x)}{f'(x)} \quad \text{is a better approx than } x.$$

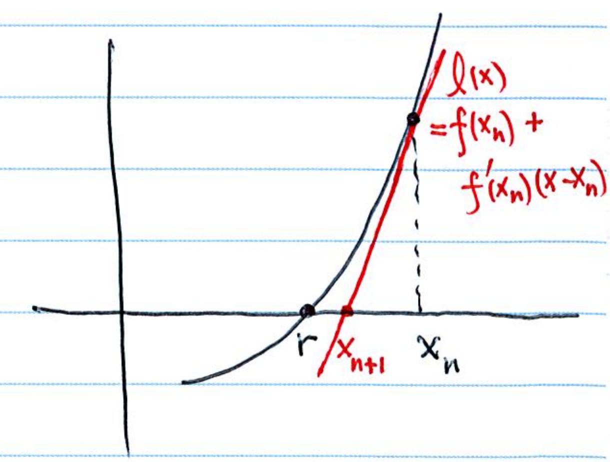
or

$$0 = f(r) = f(x) + f'(x)(r-x) + O\left((r-x)^2\right)$$

$$\approx f(x) + f'(x)(r-x)$$

$$\Longrightarrow r \approx x - f(x)/f'(x)$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$



Newton's Algorithm

```
input  x, M
   y ← f(x)
output  0, x, y

   for k = 1 to M do
        x ← x − y/f'(x)
        y ← f(x)
        output  k, x, y
   end do
```

Or

```
input  x0, M, δ, ε
   v ← f(x0)
   output  0, x0, v
   if |v| < ε then stop
   for k = 1 to M do
      x1 ← x0 − v/f'(x0)
      v ← f(x1)
      output  k, x1, v
      if |x1 − x0| < δ or |v| < ε then stop
      x0 ← x1
   end do
```

example $f(x) = e^x - 1.5 - \tan^{-1} x$, find the negative zero.

Error Analysis

Assumptions f'' is cont., $f(r)=0$, and $f'(r)\neq 0$

error $e_n = x_n - r$

$$e_{n+1} = x_{n+1} - r = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)}$$

$$0 = f(r) = f(x_n - e_n) = \boxed{f(x_n) - e_n f'(x_n)} + \frac{1}{2} e_n^2 f''(\xi_n)$$

$$\Longrightarrow \quad e_{n+1} = \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} e_n^2 \approx \frac{1}{2} \frac{f''(r)}{f'(r)} e_n^2$$

Let $c(\delta) = \frac{1}{2} \max\limits_{|x-r|\leq\delta} |f''(x)| \Big/ \min\limits_{|x-r|\leq\delta} |f'(x)|$ for $\delta > 0$

$$\Longrightarrow \quad |e_1| = \frac{1}{2} \frac{|f''(\xi_0)|}{|f'(x_0)|} e_0^2 \leq c(\delta)\, e_0^2$$

choose small enough δ s.t. $\min\limits_{|x-r|\leq\delta} |f'(x)| > 0$ and $\delta c(\delta) < 1$

$\boxed{\text{if } |e_0| \leq \delta}$

$$\leq c(\delta)\, \delta\, |e_0|$$

$$= \rho |e_0| < |e_0| \leq \delta \qquad \rho = c(\delta)\,\delta$$

$$\overset{|e_i| \leq \delta}{\Longrightarrow} \quad |e_n| \leq \rho^n |e_0|$$

Thrm 1 Assume that f'' is cont. and let r be a simple zero of f, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\Longrightarrow \exists\, O(r,\delta)$, if $x_0 \in O(r,\delta)$, then $\lim\limits_{n\to\infty} x_n = r$.

moreover, $\exists$ const $c > 0$ s.t.

$$|x_{n+1} - r| \leq c\,(x_n - r)^2.$$

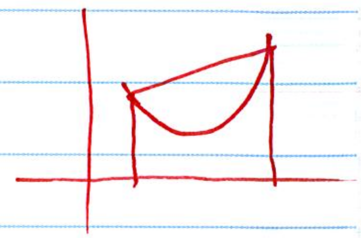
Thrm 2 (convex function)

Assume that $f \in C^2(R)$, is increasing, is convex, and has a zero,

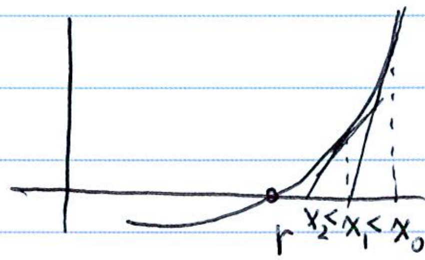
$f'' > 0$ or $f(tx_1 + (1-t)x_2) \le t f(x_1) + (1-t) f(x_2)$

$\Longrightarrow$ (1) the zero is unique

(2) the Newton iteration converges globally.



Proof



increasing $f'(x) > 0$

convex $f''(x) > 0$

$$x_{n+1} - r = \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} e_n^2 > 0$$

$$\Rightarrow x_n > r \quad \forall n \ge 1 \xrightarrow{f' > 0} f(x_n) > f(r) = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n \quad \text{and} \quad x_n > r \Longrightarrow \lim_{n\to\infty} x_n = x^*$$

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} < e_n \quad \text{and} \quad e_n > 0 \Longrightarrow \lim_{n\to\infty} e_n = e^*$$

$$\Longrightarrow e^* = e^* - \frac{f(x^*)}{f'(x^*)} \Longrightarrow f(x^*) = 0$$

and $x^* = r$

#

Example Newton's method for $f(x) \equiv x^2 - R = 0$ for $R > 0$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

Implicit Functions (an application of Newton's method)

$$G(x, y(x)) = 0$$

for a prescribed x_1, $\begin{cases} \text{given } y_0 \\ y_{k+1} = y_k - G(x_1, y_k) \Big/ \frac{\partial G}{\partial y}(x_1, y_k) \end{cases}$

find ~~f(y)~~ y s.t. $f(y) = G(x_1, y) = 0$

$y_{k+1} = y_k - \dfrac{f(y_k)}{f'(y_k)}$

compute an approx. to y_1 in which $G(x_1, y_1) = 0$

$\Longrightarrow$ generate a table for $y(x)$. (see p87)

Systems of Nonlinear Equations

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases} \quad \text{or} \quad \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \vec{0}$$

Let (x_1, x_2) be an approximation, and let (h_1, h_2) be the correction to be computed, then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ new approximation

$$0 = f_1(x_1 + h_1, x_2 + h_2) \approx f_1(x_1, x_2) + h_1 \frac{\partial f_1}{\partial x_1}(x_1, x_2) + h_2 \frac{\partial f_1}{\partial x_2}(x_1, x_2)$$

$$0 = f_2(x_1 + h_1, x_2 + h_2) \approx f_2(x_1, x_2) + h_1 \frac{\partial f_2}{\partial x_1}(x_1, x_2) + h_2 \frac{\partial f_2}{\partial x_2}(x_1, x_2)$$

$$\Longleftrightarrow \vec{0} = \vec{f}(\vec{x} + \vec{h}) \approx \vec{f}(\vec{x}) + \underbrace{\begin{bmatrix} \partial_1 f_1(\vec{x}) & \partial_2 f_1(\vec{x}) \\ \partial_1 f_2(\vec{x}) & \partial_2 f_2(\vec{x}) \end{bmatrix}_{2\times 2}}_{= J \text{ Jacobian matrix}} \vec{h}$$

$$\Longrightarrow \vec{h} = -J^{-1} \vec{f}(\vec{x})$$

Newton's Iteration

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} - J^{-1}(\vec{x}^{(k)})\,\vec{f}(\vec{x}^{(k)}) \qquad \text{for } k=0,1,\cdots$$

<u>example 4</u>

$$\begin{cases} xy = z^2+1 \\ xyz + y^2 = x^2+2 \\ e^x + z = e^y + 3 \end{cases} \qquad \vec{f}(x) = \begin{pmatrix} x_1x_2 - x_3^2 - 1 \\ x_1x_2x_3 + x_2^2 - x_1^2 - 2 \\ e^{x_1} + x_3 - e^{x_2} - 3 \end{pmatrix} = \vec{0}$$

$$J = \begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 & \partial_3 f_1 \\ \partial_1 f_2 & \partial_2 f_2 & \partial_3 f_2 \\ \partial_1 f_3 & \partial_2 f_3 & \partial_3 f_3 \end{pmatrix} = \begin{pmatrix} x_2 & x_1 & -2x_3 \\ x_2x_3 - 2x_1 & x_1x_3 + 2x_2 & x_1x_2 \\ e^{x_1} & -e^{x_1} & 1 \end{pmatrix}$$

$$\vec{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

3.3 Secant Method

to avoid computing derivative in Newton's method several methods have been studied.

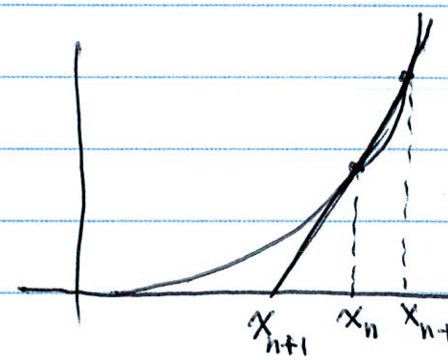
(1) Steffensen's iteration

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}$$ quadratically convergent

• $f'(x) \approx g(x) = \dfrac{f(x+f(x)) - f(x)}{f(x)}$

(2) the secant method

$$x_{n+1} = x_n - f(x_n)\,\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$



• $f'(x_n) \approx \dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$

example $f(x) = x^3 - \sinh x + 4x^2 + 6x + 9$

there is a zero between $x_0 = 7$ and $x_1 = 8$

Secant Algorithm

```
input a, b, M, δ, ε
    fa ← f(a) ; fb ← f(b)
output 0, a, fa
output 1, b, fb

for k=2 to M do
    if |fa| > |fb| then
        a ↔ b ; fa ↔ fb
    endif
    s ← (b-a)/(fb-fa)
    b ← a
    fb ← fa
    a ← a - fa*s
    fa ← f(a)

    output k, a, fa
    if |fa| < ε or |b-a| < δ then stop
enddo
```

$\{\underset{a}{x_n}, \underset{b}{x_{n-1}}\}$

$|\underset{fa}{f(x_n)}| \le |\underset{fb}{f(x_{n-1})}|$

$x_n = x_n - f(x_n) * s$

Error Analysis

$$x_{n+1} = x_n - f(x_n)\,\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} = \frac{f(x_n)\,x_{n-1} - f(x_{n-1})\,x_n}{f(x_n) - f(x_{n-1})}$$

Let $e_n = x_n - r$

$$\Rightarrow \quad e_{n+1} = x_{n+1} - r = \frac{f(x_n)\,e_{n-1} - f(x_{n-1})\,e_n}{f(x_n) - f(x_{n-1})}$$

$$= \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot \frac{\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}}{x_n - x_{n-1}} \cdot e_n e_{n-1}$$

Taylor's expansion

$$f(x_n) = f(e_n + r) = \overset{0}{\overset{\shortparallel}{f(r)}} + e_n f'(r) + \tfrac{1}{2} e_n^2 f''(r) + O(e_n^3)$$

$$\Rightarrow \quad \frac{f(x_n)}{e_n} = f'(r) + \tfrac{1}{2} e_n f''(r) + O(e_n^2)$$

$$- \quad \frac{f(x_{n-1})}{e_{n-1}} = f'(r) + \tfrac{1}{2} e_{n-1} f''(r) + O(e_{n-1}^2)$$

$$\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}} = \tfrac{1}{2}(e_n - e_{n-1}) f''(r) + O(e_{n-1}^2)$$

$$\underbrace{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}_{\approx f'(r)} = \frac{(e_n - e_{n-1}) f'(r) + O(e_{n-1}^2)}{x_n - x_{n-1}} \qquad = \tfrac{1}{2}(x_n - x_{n-1}) f''(r) + O(e_{n-1}^2)$$

$$\Longrightarrow \quad e_{n+1} \approx \frac{1}{f'(r)} \cdot \tfrac{1}{2} f''(r)\, e_n e_{n-1} = C e_n e_{n-1}$$

Assume that $\quad |e_{n+1}| \sim A|e_n|^{\alpha} \qquad$ i.e., $\dfrac{|e_{n+1}|}{A|e_n|^{\alpha}} \to 1$ as $n \to \infty$

$$\Rightarrow \quad |e_n| \sim A|e_{n-1}|^{\alpha} \quad \text{or} \quad |e_{n-1}| \sim \left(A^{-1}|e_n|\right)^{1/\alpha}$$

$$\underset{\underset{A\,|e_n|^{\alpha}}{\wr}}{|e_{n+1}|} \sim |C|\,|e_n|\,\underset{\underset{|C|\,|e_n|\,A^{-\frac{1}{\alpha}}\,|e_n|^{\frac{1}{\alpha}}}{\wr}}{|e_{n-1}|}$$

$$\Longrightarrow\ |e_n|^{1-\alpha+\frac{1}{\alpha}} \sim A^{1+\frac{1}{\alpha}}\,|C|^{-1}$$

$$\Longrightarrow\ 1-\alpha+\frac{1}{\alpha}=0 \ \Longrightarrow\ \alpha=\frac{1+\sqrt{5}}{2} \approx 1.62$$ <u>superlinear</u>

$$A^{1+\frac{1}{\alpha}}\,|C|^{-1} \sim 1 \ \Longrightarrow\ A=|C|^{\frac{\alpha}{1+\alpha}}=|C|^{0.62} \approx \left|\frac{f''(r)}{2 f'(r)}\right|^{0.62}$$

<u>Comparison with Newton's method</u>

<u>Newton</u> 2 evaluation $f(x_n)$, $f'(x_n)$ $\Longrightarrow$ 1 step Newton $\approx$ 2 steps Secant

Secant 1 evaluation $f(x_n)$

$$|e_{n+2}| \sim A\,|e_{n+1}|^{\alpha} \sim A^{1+\alpha}\,|e_n|^{\alpha^2} = A^{1+\alpha}\,|e_n|^{\frac{3+\sqrt{5}}{2}\approx 2.62}$$