§3.4 Fixed Points and Functional Iteration

Newton's method $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \qquad F(x) = x - \dfrac{f(x)}{f'(x)}$

Steffensen's method $\quad x_{n+1} = x_n - \dfrac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} \qquad F(x) = x - \dfrac{f^2(x)}{f(x+f(x)) - f(x)}$

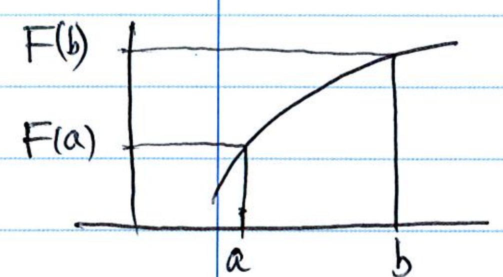
Secant's method $\quad x_{n+1} = x_n - f(x_n)\,\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \qquad \cancel{F(x) = x - f(x)}$

functional iteration $\quad x_{n+1} = F(x_n)$

s is fixed pt of F $\quad F(s) = s$

F is contractive mappings $\quad |F(x) - F(y)| \leq \lambda |x-y|$ with $\lambda \in (0,1)$

$\forall x, y \in Dom(F)$



Thrm Let $C \subset R$ be closed, and $F: C \to C$ is a contractive mapping.

$\Longrightarrow$ (1) $\exists\, s \in C$ s.t. $F(s) = s$

(2) $s = \lim_{n\to\infty} x_n$ where $x_{n+1} = F(x_n)$ with $x_0 \in C$.

<u>Proof</u> convergence of $\{x_n\}_{n=1}^{\infty} \Longleftrightarrow$ convergence of $\sum_{n=1}^{\infty}(x_n - x_{n-1})$

$$\sum_{n=1}^{\infty}(x_n - x_{n-1}) = \lim_{n\to\infty} x_n - x_0$$

$$\Uparrow$$

convergence of $\sum_{n=1}^{\infty}|x_n - x_{n-1}|$

$$|x_n - x_{n-1}| = |F(x_{n-1}) - F(x_{n-2})|$$
$$\le \lambda |x_{n-1} - x_{n-2}| \le \cdots \le \lambda^{n-1}|x_1 - x_0|$$

$$\Rightarrow \sum_{n=1}^{\infty}|x_n - x_{n-1}| \le \left(\sum_{n=1}^{\infty}\lambda^{n-1}\right)|x_1 - x_0| = \frac{1}{1-\lambda}|x_1 - x_0|$$

$\Rightarrow$ $\{x_n\}$ converges.

Let $\lim_{n\to\infty} x_n = s \Longrightarrow s = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}) = F\left(\lim x_{n-1}\right) = F(s)$

<u>uniqueness</u> Let x, y be fixed pts, then $x = F(x)$ and $y = F(y)$

$$|x-y| = |F(x) - F(y)| \le \lambda|x-y| \xrightarrow{\lambda\in(0,1)} x = y \quad . \#$$

<u>Examples 1, 2, 3</u> (p103)

Error Analysis
(order of Convergence)

$s = F(s)$, $x_{n+1} = F(x_n)$

<u>error</u> $e_n = x_n - s$

$$\Rightarrow e_{n+1} = x_{n+1} - s$$
$$= F(x_n) - F(s) = F'(\xi_n) e_n$$ if F' exists & is cont.

if $\max_{|x-s|<\delta} |F'(x)| < 1 \Rightarrow \{e_n\}$ converges

if $F^{(k)}(s) = 0$ for $1 \le k < p$, but $F^{(p)}(s) \neq 0$

$$\Rightarrow e_{n+1} = \frac{1}{p!} e_n^p F^{(p)}(\xi_n)$$

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = \frac{1}{p!} F^{(p)}(s)$$

<u>Example</u> Newton's method $F(x) = x - \frac{f(x)}{f'(x)}$

$$s = F(s) \Longleftrightarrow f(s) = 0$$

$$F'(x) = \frac{f(x) f''(x)}{(f'(x))^2} \Rightarrow F'(s) = 0$$

$$F''(s) = \frac{f''(s)}{f'(s)} \neq 0$$

§3.5 Computing Roots of Polynomials

polynomial $\quad p(z) = a_n z^n + \cdots + a_1 z + a_0, \quad a_i, z \in \mathbb{C}$

complex numbers; complex field

Thrm (Fundamental Thrm of Algebra)
Every nonconstant polynomial has at least one root in $\mathbb{C}$.

Remainder Thrm $\quad p(z) = (z-c)\,q(z) + r, \quad c, r \in \mathbb{C}$

Factor Thrm $\quad$ If $p(c)=0$, then $p(z) = (z-c)\,q(z)$

$\Longrightarrow$ $p(z)$ is a poly. of degree $n \geq 1$, then $p(z) = (z-r_1)\cdots(z-r_n)$

Localization Thrm $\quad \forall\, r$ s.t. $p(r)=0$

$$\Longrightarrow \quad |r| < \rho = 1 + \frac{\max_{0\le k\le n} |a_k|}{|a_n|}.$$

Proof (1) $\max_k |a_k| = 0 \Longrightarrow p(z) \equiv 0$

(2) $c = \max_k |a_k| > 0 \Longrightarrow \rho > 1$.

Assume that $|r| \geq \rho \Longrightarrow |p(r)| > 0$ this is a contradiction.

$$\Longrightarrow |p(r)| \geq |a_n r^n| - |a_{n-1} r^{n-1} + \cdots + a_0| \geq |a_n r^n| - c\sum_{k=0}^{n-1} |r|^k$$

$$> |a_n r^n| - \frac{c|r|^n}{|r|-1} = |a_n r^n|\left[1 - \frac{c}{|a_n|(|r|-1)}\right] \geq |a_n r^n|\left[1 - \frac{c}{|a_n|(\rho-1)}\right] = 0.$$

Location Thrm 2

If all roots of $s(z) = z^n p\left(\frac{1}{z}\right)$ are in $\{z \mid |z| \leqslant \rho\}$

$\Longrightarrow$ .. .. .. $p(z)$ .. .. $\left\{z \mid |z| > \frac{1}{\rho}\right\}$.

Proof

$$s(z) = a_n + a_{n-1} z + \cdots + a_0 z^n$$

$$p(z_0) = 0 \Longleftrightarrow s\left(\frac{1}{z_0}\right) = 0$$

Example

$$p(z) = z^4 - 4z^3 + 7z^2 - 5z - 2$$

$$\rho_p = 1 + \frac{\max|a_k|}{1} = 8$$

$$\rho_s = 1 + \frac{\max|a_k|}{2} = 1 + \frac{7}{2} = \frac{9}{2}$$

if $p(z) = 0$

$\Longrightarrow \frac{9}{2} < |z| < 8$

Horner's Algorithm

Given z_0, compute $p(z_0)$ and $q(z)$

Let

$$q(z) = \frac{p(z) - p(z_0)}{z - z_0} = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1}$$

$$\begin{aligned}
\Longrightarrow \quad p(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\
&= (z - z_0) q(z) + p(z_0) \\
&= b_{n-1} z^n + (b_{n-2} - z_0 b_{n-1}) z^{n-1} + \cdots + (b_0 - z_0 b_1) z + (p(z_0) - z_0 b_0)
\end{aligned}$$

$$\Longrightarrow \boxed{b_{n-1} = a_n, \quad b_{n-2} = a_{n-1} - z_0 b_{n-1}, \quad \cdots, \quad b_0 = a_1 + z_0 b_1, \quad p(z_0) = a_0 + z_0 b_0}$$

$$\begin{array}{c|ccccc} & a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \\ z_0 & & z_0 b_{n-1} & z_0 b_{n-2} & \cdots & z_0 b_0 \\ \hline & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_{-1} = p(z_0) \end{array}$$

$$p(z_0) = a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_2 z_0^2 + a_1 z_0 + a_0$$

$$= \left(a_n z_0^{n-1} + a_{n-1} z_0^{n-2} + \cdots + a_2 z_0 + a_1\right) z_0 + a_0$$

$$= \left(\left(a_n z_0^{n-2} + a_{n-1} z_0^{n-3} + \cdots + a_3 z_0 + a_2\right) z_0 + a_1\right) z_0 + a_0$$

or $$= \left(a_n z_0 + a_{n-1}\right) z_0^{n-1} + a_{n-2} z_0^{n-2} + \cdots$$

$$= \left(\left(a_n z_0 + a_{n-1}\right) z_0 + a_{n-2}\right) z_0^{n-3} + \cdots$$

nested multiplication.

Ex.3 $p(z) = z^4 - 4z^3 + 7z^2 - 5z - 2$

$$\begin{array}{c|ccccc} & 1 & -4 & 7 & -5 & 2 \\ 3 & & 3 & -3 & 12 & 21 \\ \hline & 1 & -1 & 4 & 7 & 13 \end{array} \Longrightarrow p(z) = (z-3)(z^3 - z^2 + 4z + 7) + 13$$

deflation

$$\begin{array}{c|ccccc} & 1 & -4 & 7 & -5 & 2 \\ 2 & & 2 & -4 & 6 & 2 \\ \hline & 1 & -2 & 3 & 1 & 0 \end{array} \Longrightarrow p(z) = (z-2)(z^3 - 2z^2 + 3z + 1)$$

Taylor expansion

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$
$$= c_n (z - z_0)^n + c_{n-1}(z - z_0)^{n-1} + \cdots + c_0$$

Compute $c_k = \dfrac{p^{(k)}(z_0)}{k!}$

$\boxed{c_0 = p(z_0)}$ — Horner's Algorithm for p at z_0

Newton's Method

$z_{k+1} = z_k - \dfrac{p(z_k)}{p'(z_k)}$

$$g(z) = \frac{p(z) - p(z_0)}{z - z_0} = c_n (z - z_0)^{n-1} + \cdots + c_1$$

$\boxed{c_1 = g(z_0) = p'(z_0)}$ Horner's Algorithm for g at z_0

....

Ex. 5 $p(z) = z^4 - 4z^3 + 7z^2 - 5z + 2$

compute its Taylor expansion at $z_0 = 3$

$$\begin{array}{c|ccccc}
 & 1 & -4 & 7 & -5 & -2 \\
3 & & 3 & -3 & 12 & 21 \\
\hline
 & 1 & -1 & 4 & 7 & \boxed{19} = p(3) \\
3 & & 3 & 6 & 30 & \\
\hline
 & 1 & 2 & 10 & \boxed{37} = g(3) = p'(3) & \\
3 & & 3 & 15 & & \\
\hline
 & 1 & 5 & \boxed{25} & & \\
3 & & 3 & & & \\
\hline
 & \boxed{1} & \boxed{8} & & &
\end{array}$$

$$p(z) = (z-3)^4 + 8(z-3)^3 + 25(z-3)^2 + 37(z-3) + 19$$

Thrm on Horner's Method

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

$$\begin{cases} (\alpha_n, \beta_n) = (a_n, 0) \\ (\alpha_j, \beta_j) = (a_j + x\alpha_{j+1}, \ \alpha_{j+1} + x\beta_{j+1}) & \text{for } j = n-1, n-2, \cdots, 0 \end{cases}$$

Proof

	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_1	a_0
x	$\beta_n = 0$	$x\alpha_n$	$x\alpha_{n-1}$		$x\alpha_2$	$x\alpha_1$
	$\alpha_n = a_n$	$\alpha_{n-1} = a_{n-1} + x\alpha_n$	$\alpha_{n-2} = a_{n-2} + x\alpha_{n-1}$	$\cdots$	$\alpha_1 = a_1 + x\alpha_2$	$\alpha_0 = a_0 + x\alpha_1 = p(x)$
x		$x\beta_{n-1}$	$x\beta_{n-2}$	$\cdots$	$x\alpha_1$	
	$\beta_{n-1} = a_n$	β_{n-2}	β_{n-3}		$\beta_0 = p'(x)$	

Thrm on Successive Newton Iterations

$$x_{k+1} = x_k - \frac{p(x_k)}{p'(x_k)}$$

deg. n

$\Longrightarrow \exists$ ~~a root of p~~ r s.t. $p(r) = 0$ and

$$(*) \qquad |x_k - r| \le n\,|x_k - x_{k+1}| = n\left|\frac{p(x_k)}{p'(x_k)}\right|$$

Proof Assume that $p(r_j) = 0$ for $j = 1, 2, \cdots, n$.

$$\Longrightarrow p(z) = c\prod_{j=1}^{n}(z - r_j) \quad \text{and} \quad p'(z) = p(z)\sum_{k=1}^{n}\frac{1}{z - r_k}$$

Assume that $(*)$ is incorrect

$$\Longrightarrow \forall j, \quad |x_k - r_j| > n\left|\frac{p(x_k)}{p'(x_k)}\right|$$

$$\Rightarrow \frac{1}{|x_k - r_j|} < \frac{1}{n}\left|\frac{p'(x_k)}{p(x_k)}\right| = \frac{1}{n}\left|\sum_{l=1}^{n}\frac{1}{x_k - r_l}\right| \le \underbrace{\frac{1}{n}\sum_{l=1}^{n}\frac{1}{|x_k - r_l|}}_{\text{average of } n \text{ numbers}}$$

$\forall j = 1, \cdots, n$

impossible. #

<u>Bairstow's Method</u> $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

where $a_k \in R$.

$$p(w) = 0 \Rightarrow p(\bar{w}) = 0 \Rightarrow p(z) = (z-w)(z-\bar{w})\, q(z)$$

$$(z-w)(z-\bar{w}) = z^2 - (w+\bar{w})z + w\bar{w} \qquad |w| = |w_1 + i w_2| = \sqrt{w_1^2 + w_2^2}.$$

real quadratic factor

<u>Thrm on Quotient and Remainder</u> $p(z) = a_n z^n + \cdots + a_1 z + a_0$

$$p(z) = q(z)(z^2 - uz - v) + r(z)$$

where $q(z) = b_n z^{n-2} + \cdots + b_3 z + b_2$ and $r(z) = b_1(z-u) + b_0$

$$\Rightarrow \begin{cases} b_k = a_k + u b_{k+1} + v b_{k+2} & \text{for } k = n, n-1, \cdots, 0 \\ b_{n+1} = b_{n+2} = 0 \end{cases}$$

Problem Finding a real quadratic factor of $p(z)$ with real coefficients.

$z^2 - uz - v$ is a real quadratic factor of $p(z)$

$$\Longrightarrow r(z) = b_1(z-u) + b_0 \equiv 0 \Longrightarrow \begin{cases} b_0(u,v) = 0 \\ b_1(u,v) = 0 \end{cases}$$

Newton's Method Given initial guess (u, v), compute correction $(\delta u, \delta v)$

$$\begin{cases} 0 = b_0(u+\delta u, v+\delta v) \approx b_0(u,v) + \frac{\partial b_0}{\partial u}\delta u + \frac{\partial b_0}{\partial v}\delta v \\ 0 = b_1(u+\delta u, v+\delta v) \approx b_1(u,v) + \frac{\partial b_1}{\partial u}\delta u + \frac{\partial b_1}{\partial v}\delta v \end{cases}$$

$$\Longrightarrow \begin{pmatrix} \frac{\partial b_0}{\partial u} & \frac{\partial b_0}{\partial v} \\ \frac{\partial b_1}{\partial u} & \frac{\partial b_1}{\partial v} \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = -\begin{pmatrix} b_0(u,v) \\ b_1(u,v) \end{pmatrix}$$

Let $c_k = \frac{\partial b_k}{\partial u}$, $d_k = \frac{\partial b_{k-1}}{\partial v}$

$$\Longrightarrow \begin{cases} c_k = b_{k+1} + u\,c_{k+1} + v\,c_{k+2} \\ d_k = b_{k+1} + u\,d_{k+1} + v\,d_{k+2} \end{cases} \qquad \begin{cases} c_{n+1} = c_n = 0 \\ d_{n+1} = d_n = 0 \end{cases}$$

$$\Longrightarrow c_k = d_k \Longrightarrow \frac{\partial b_0}{\partial u} = c_0, \quad \frac{\partial b_0}{\partial v} = d_1 = c_1$$

$$\frac{\partial b_1}{\partial u} = c_1, \quad \frac{\partial b_1}{\partial v} = d_2 = c_2$$

$$\Longrightarrow \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = \frac{1}{J}\begin{pmatrix} c_1 b_1 - c_2 b_0 \\ c_1 b_0 - c_0 b_1 \end{pmatrix} \quad \text{with } J = c_0 c_2 - c_1^2$$

Jacobian determinant

<u>Thrm</u> Let (u_0, v_0) be a pt s.t. the roots of $z^2 - u_0 z - v_0$ are simple roots of $p(z)$.

$$\Longrightarrow J(u_0, v_0) \neq 0$$ (proof on p120)

<u>Laguerre Iteration</u>

§3.6 Homotopy and Continuation Methods

finding the roots

$$f(x) = 0 \qquad \text{where } f: X \longrightarrow Y$$

basic concepts of continuation method

$$h(t, x) = t\, f(x) + (1-t)\, g(x) \qquad t \in [0,1]$$

$t=0$ $\quad 0 = h(0,x) = g(x)$ $\quad$ having a known solution

$t=1$ $\quad 0 = h(1,x) = f(x)$ $\quad$ the original problem

partition $\quad 0 = t_0 < t_1 < \cdots < t_m = 1$

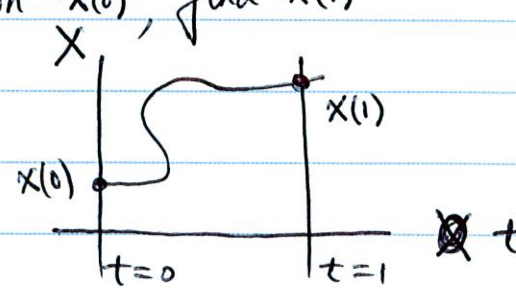
$t = t_i$ $\quad$ solve $m+1$ equations: $h(t_i, x) = 0$ for $i = 0, 1, \cdots, m$

by iterative method, e.g., Newton, with initial guess x_{i-1} from the previous step.

$$0 = h(t, x(t)) \qquad \{x(t) \mid 0 \le t \le 1\} \text{ curve in } X$$

$$\Rightarrow \quad 0 = \frac{\partial h}{\partial t}(t, x(t)) + \frac{\partial h}{\partial x}(t, x(t))\, x'(t) \qquad \text{known } x(0), \text{ find } x(1)$$

$$\Rightarrow \quad \begin{cases} x'(t) = -\left[\dfrac{\partial h}{\partial x}(t, x(t))\right]^{-1} h_t(t, x(t)) \\ x(0) \end{cases}$$



homotopy f is homotopic to g, if

$$h : [0,1]\times X \longrightarrow Y$$

is a continuous map s.t. $h(0,x)=g(x)$ and $h(1,x)=f(x)$.

example

$$h(t,x) = t f(x) + (1-t)\left[f(x)-f(x_0)\right] = f(x)-(1-t)f(x_0)$$

Ex. 1 $X = Y = R^2$

$$0 = f(x) = \begin{pmatrix} \xi_1^2 - 3\xi_2^2 + 3 \\ \xi_1 \xi_2 + 6 \end{pmatrix} \qquad (\xi_1,\xi_2)\in R^2$$

$$x_0 = (1,1), \quad h_x = f'(x) = \begin{pmatrix} 2\xi_1 & -6\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix}$$

$$h_t = f(x_0) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} x'(t) = -\left[f'(x)\right]^{-1} h_t = -\begin{pmatrix} 2\xi_1 & -6\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 7 \end{pmatrix} \\ x_0 = (1,1) \end{cases}$$

Thrm (1) $f : R^n \to R^n$ is cont. diff. (2) $\left\| [f'(x)]^{-1} \right\| \le M$ on R^n

$\Longrightarrow_{(1)}$ $\forall\ x_0 \in R^n$, $\exists 1$ curve $\{x(t) \mid 0\le t\le 1\}$ in R^n such that

$$f(x(t)) + (t-1) f(x_0) = 0 \quad \text{with } t\in[0,1]$$

(2) $x(t)$ is a cont. diff solution of $\begin{cases} x'(t) = -\left[f'(x(t))\right]^{-1} f(x_0) \\ x(0) = x_0 \end{cases}$

Relation to Newton's Method

$$h(t,x) = f(x) - e^{-t} f(x_0)$$

- Find $x(t)$ $t \in [0,\infty)$ s.t.

$$0 = h(t, x(t)) = f\big(x(t)\big) - e^{-t} f(x_0)$$

$$\Rightarrow \quad 0 = f'(x(t))\, x'(t) + e^{-t} f(x_0)$$

$$\Rightarrow \quad \begin{cases} x'(t) = -\left[f'(x(t))\right]^{-1} f(x(t)) \\ x(0) = x_0 \end{cases}$$

Euler's method with step size 1

$$x_{n+1} = x_n - \left[f'(x_n)\right]^{-1} f(x_n)$$