Chapter 6 Approximating Functions

polynomials, spline, trigonometric functions.

§6.1 Polynomial Interpolation

Problem Find $p(x) \in P_n = \{ p(x) \mid p(x) \text{ is a poly. of degree} \leq n \}$ s.t.

$$p(x_i) = y_i \quad \text{for } i = 0, 1, \cdots, n.$$

Given (x_i, y_i) for $i = 0, 1, \cdots, n$.

Thrm $x_i \neq x_j$, $\forall y_i$

$$\Longrightarrow \exists 1 \ p(x) \in P_n \text{ s.t. } p_n(x_i) = y_i \quad \text{for } i = 0, 1, \cdots, n.$$

Proof uniqueness Assume that $p_n, \tilde{p}_n \in P_n$ s.t.

$$p_n(x_i) = \tilde{p}_n(x_i) = y_i \quad \text{for } i = 0, 1, \cdots, n$$

$$\Longrightarrow (p_n - \tilde{p}_n)(x_i) = 0 \quad \text{for } i = 0, 1, \cdots, n \xrightarrow{n+1 \text{ zeros}} p_n(x) \equiv \tilde{p}_n(x)$$

construction $\boxed{\text{Newton's Interpolation}}$

$n=0$ $\quad p_0(x) = y_0 \equiv y[x_0]$

$n=1$ $\quad p_1(x) = y_0 + c_1(x - x_0) \xrightarrow{p_1(x_1) = y_1} c_1 = \dfrac{y_1 - y_0}{x_1 - x_0} = y[x_0, x_1]$

divided differences of $y(x)$

$n=2$ $\quad p_2(x) = p_1(x) + c_2(x - x_0)(x - x_1) \xrightarrow{p_2(x_2) = y_2} c_2 = \dfrac{y_2 - p_1(x_2)}{(x - x_0)(x - x_1)}$

$$c_2 = \frac{y_2 - p_1(x_2)}{(x_2-x_0)(x_2-x_1)} = \frac{y_2 - y_0 - y[x_0, x_1](x_2-x_0)}{(x_2-x_0)(x_2-x_1)}$$

$$= \left(y[x_0, x_2] - y[x_0, x_1]\right) / (x_2 - x_1) = y[x_0, x_1, x_2]$$

<u>$n=k$</u> $\quad p_k(x) = p_{k-1}(x) + c_k\,(x-x_0)\cdots(x-x_{k-1})$

$$y_k = p_k(x_k) \Longrightarrow c_k = y[x_0, x_1, \cdots, x_k]$$

$$p_n(x) = \sum_{k=0}^{n} c_k\, \phi_k(x) = \sum_{k=0}^{n} y[x_0, x_1, \cdots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

x_0	$f[x_0]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
x_1	$f[x_1]$	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	
x_2	$f[x_2]$	$f[x_2, x_3]$		
x_3	$f[x_3]$			

Lagrange Interpolation

$$p_n(x) = \sum_{k=0}^{n} y_k\, l_k(x) \quad \text{where } l_k(x) \in P_n \text{ and}$$

$$l_k(x_j) = \delta_{kj} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

$$\cancel{y_i = p_n(x_i) \Longrightarrow y_i = \sum_{k=0}^{n} y_k\, l_k(x_i)}$$

$$\cancel{\text{choose } \bar{y} = \{0, \cdots 0, \overset{i}{1}, 0, \cdots, 0\} \Longrightarrow l_i(x_i) =} \Rightarrow l_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j}$$

cardinal functions

Ex. 1 and 2

Remark on Newton and Lagrange interpolations.

Error Analysis

$$f \in C^{n+1}[a,b], \quad p_n(x) \in P_n \text{ and } p_n(x_i) = f(x_i)$$

$$\Longrightarrow \quad f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

Proof $\quad f(x_i) - p(x_i) = 0 \quad$ for $i = 0, 1, \cdots, n$

$$\Longrightarrow \quad f(x) - p(x) = A(x) \prod_{i=0}^{n} (x - x_i) \qquad A(x) = ?$$

Set $\quad \varphi(t) = f(t) - p(t) - A(x) \prod_{i=0}^{n} (t - x_i)$

$$\Longrightarrow \quad \varphi(x_i) = 0 \text{ and } \varphi(x) = 0 \qquad n+2 \text{ zeros}$$

$$\Longrightarrow \quad \exists\, \xi_x \text{ s.t. } 0 = \varphi^{(n+1)}(\xi_x)$$

$$= f^{(n+1)}(\xi_x) - \frac{1}{(n+1)!} A(x)$$

$$\Longrightarrow \quad A(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}$$

#

Ex. 3 $\quad |\sin x - p_9(x)|$

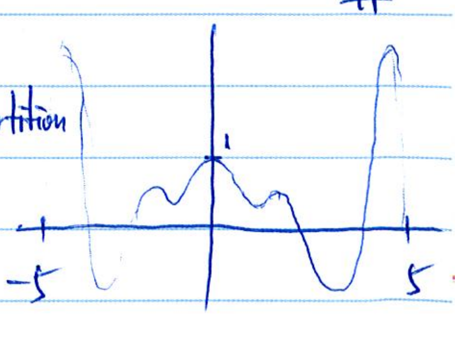
on $[0,1]$ $\quad \le \frac{1}{10!} |f^{(10)}(\xi_x)| \prod_{i=0}^{9} |x - x_i|$

$\le \frac{1}{10!} < 2.8 \times 10^{-7}$

Runge Function, uniform partition

$f(x) = \frac{1}{1+x^2}$

$x \in [-5, 5]$



Chebyshev Polynomials

$$\begin{cases} T_0(x)=1, \quad T_1(x)=x \\ T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x) \end{cases}$$

$$T_2(x) = 2x^2-1$$
$$T_3(x) = 4x^3-3x$$
$$\vdots$$

$$\int_{-1}^{1} T_n(x)\,T_m(x)\,\frac{1}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & n\neq m \\ \pi, & n=m=0 \\ \frac{\pi}{2}, & n=m \end{cases}$$

Thrm 3 **Thrm on Chebyshev Polynomials** $\quad \forall x\in[-1,1]$

$$T_n(x) = \cos\left(n\cos^{-1}x\right) \qquad n\geq 0.$$

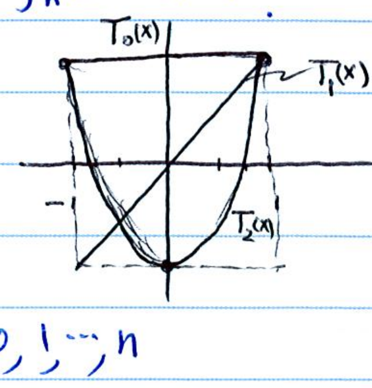
Proof

$$\left.\begin{aligned} \cos(n+1)\theta &= \cos\theta\cos n\theta - \sin\theta\sin n\theta \\ \cos(n-1)\theta &= \cos\theta\cos n\theta + \sin\theta\sin n\theta \end{aligned}\right\} \Longrightarrow \cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta$$

let $\theta = \cos^{-1}x \Longrightarrow x=\cos\theta$

define $\quad f_n(x) = \cos\left(n\cos^{-1}x\right)$

$$\Longrightarrow \begin{cases} f_{n+1}(x) = 2x\,f_n(x) - f_{n-1}(x) \\ f_0(x)=1, \quad f_1(x)=x \end{cases} \Longrightarrow f_n(x) = T_n(x).$$



Properties

(1) $\left|T_n(x)\right| \leq 1 \qquad \forall x\in[-1,1]$

(2) $T_n\left(\cos\frac{j\pi}{n}\right) = (-1)^j \qquad$ for $j=0,1,\cdots,n$

(3) $T_n\left(\cos\frac{2j-1}{2n}\pi\right) = 0 \qquad$ for $j=1,\cdots,n.$

$$\max_{-1\le x\le 1}\left|f(x)-P_n(x)\right| \le \frac{1}{(n+1)!}\max_{-1\le x\le 1}\left|f^{(n+1)}(x)\right| \max_{-1\le x\le 1}\left|\prod_{i=0}^{n}(x-x_i)\right|$$

- $$\max_{-1\le x\le 1}\left|\prod_{i=0}^{n}(x-x_i)\right| \ge 2^{-n}$$
- choosing $x_i = \cos\left(\frac{2i+1}{2n+2}\pi\right)$ for $i=0,1,\cdots,n$ — zeros of $T_{n+1}(x)$

$$\Rightarrow \left|\prod_{i=0}^{n}(x-x_i)\right| = 2^{-n}\left|T_{n+1}(x)\right| \le 2^{-n}$$

$$\Rightarrow \min_{x_i}\max_{-1\le x\le 1}\left|\prod_{i=0}^{n}(x-x_i)\right| = 2^{-n}$$

$$\Rightarrow \|f-P_n\|_{\infty,[-1,1]} \le \frac{1}{2^n(n+1)!}\|f^{(n+1)}\|_{\infty,[-1,1]}.$$

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

monic poly. $\quad a_n = 1$

$$T_n(x) = 2^{n-1} x^n + \cdots$$

Thrm $\quad p$ is a monic poly of deg. $n \Longrightarrow \|p\|_\infty \equiv \max_{-1\le x\le 1} |p(x)| \geq 2^{1-n}$.

Proof $\quad$ Assume that $\quad |p(x)| < 2^{1-n} \quad \forall x \in [-1,1]$

Let $\tilde{p}(x) = 2^{1-n} T_n(x)$ and $x_i = \cos\frac{i\pi}{n}$ for $i = 0, 1, \cdots, n$

$$\Longrightarrow \tilde{p}(x_i) = 2^{1-n} T_n(x_i) = 2^{1-n} (-1)^i$$

$$\Longrightarrow (-1)^i p(x_i) \le |p(x_i)| < 2^{1-n} = (-1)^i \tilde{p}(x_i)$$

$$\Longrightarrow (-1)^i \left(\tilde{p}(x_i) - p(x_i) \right) > 0 \quad \forall i = 0, 1, \cdots, n$$

$\Longrightarrow \tilde{p}(x) - p(x)$ oscillates in sign $n+1$ times in $[-1, 1]$

$\Longrightarrow \tilde{p}(x) - p(x)$ have at least n roots in $(-1, 1)$

but $\tilde{p}(x) - p(x) \in P_{n-1}$. $\quad$ #

§6.2 Divided Differences

k^{th}-order divided difference $f[x_0, x_1, \cdots, x_k]$

is the coefficient of interpolation using basis functions

$$\left\{ 1,\ x-x_0,\ (x-x_0)(x-x_1),\ \cdots,\ \prod_{i=0}^{k-1}(x-x_i) \right\}.$$

$$f[x_0] = f(x_0)$$

$$f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$\cdots$

$$f[x_0, x_1, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0}$$ recursive formula

let $p_{k-1}(x)$ interpolate $f(x)$ at $\{x_0, \cdots, x_{k-1}\}$

$q_{k-1}(x)$ interpolate $f(x)$ at $\{x_1, \cdots, x_k\}$

$$\Longrightarrow \quad p_k(x) = p_{k-1}(x) + f[x_0, x_1, \cdots, x_k] \prod_{j=0}^{k-1}(x-x_j)$$

$$\overset{?}{=} \cancel{q_{k-1}(x) +} \ \frac{x-x_0}{x_k-x_0}\, q_{k-1}(x) + \frac{x-x_k}{x_0-x_k}\, p_{k-1}(x)$$

$$= q_{k-1}(x) + \cancel{\frac{x-x_k}{x_k-x_0}} \ \frac{x-x_k}{x_0-x_k}\left(p_{k-1}(x) - q_{k-1}(x)\right)$$

comparing coefficients of x^k $\Rightarrow$ $$f[x_0, x_1, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, x_1, \cdots, x_{k-1}]}{x_k - x_0}$$

Ex. 1 & Ex. 2

Properties (1) $\{z_0, z_1, \cdots, z_n\}$ is a permutation of $\{x_0, x_1, \cdots, x_n\}$

$$\Longrightarrow f[z_0, \cdots, z_n] = f[x_0, \cdots, x_n]$$

(2) $p \in P_n$ interpolates f at $\{x_0, x_1, \cdots, x_n\}$

$$\Longrightarrow f(t) - p(t) = f[x_0, x_1, \cdots, x_n, t] \prod_{i=0}^{n} (t - x_i) \qquad (*)$$

Proof Let $g \in P_{n+1}$ interpolate f at $\{x_0, \cdots, x_n, t\}$

$$\Longrightarrow g(x) = p(x) + f[x_0, x_1, \cdots, x_n, t] \prod_{i=0}^{n} (t - x_i)$$

$(*)$ follows from $f(t) = g(t)$. #

(3) $f \in C^n[a,b]$, $x_i \in [a,b]$ with $x_i \neq x_j$ if $i \neq j$

$$\Longrightarrow \exists\, \xi \in (a,b) \text{ s.t. } f[x_0, x_1, \cdots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

Proof Let $p \in P_{n-1}$ interpolate f at $\{x_0, x_1, \cdots, x_{n-1}\}$

$$\Longrightarrow \exists\, \xi \in (a,b) \text{ s.t. } f(x_n) - p(x_n) = \frac{1}{n!} f^{(n)}(\xi) \prod_{i=0}^{n-1} (x_n - x_i)$$

$$(2) \ \Big\|$$

$$f[x_0, \cdots, x_n] \prod_{i=0}^{n-1} (x_n - x_i) \qquad \#$$

§6.3 Hermite Interpolation

- Find $p(x) \in P_3$ s.t.

$$p(x_i) = f(x_i) \quad \text{and} \quad p'(x_i) = f'(x_i) \qquad i = 1, 2$$

Lagrange Formula $\quad p(x) = \sum_{i=0}^{1} f(x_i) A_i(x) + \sum_{i=0}^{1} f'(x_i) B_i(x)$

$$\Rightarrow \begin{cases} A_i(x_j) = \delta_{ij} \\ A_i'(x_j) = 0 \end{cases} \quad \text{and} \quad \begin{cases} B_i(x_j) = 0 \\ B_i'(x_j) = \delta_{ij} \end{cases}$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad l_0'(x) = \frac{1}{x_0 - x_1}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0}, \quad l_1'(x) = \frac{1}{x_1 - x_0}$$

$$A_0(x) = \left[a + b(x - x_0) \right] l_0^2(x)$$

$$1 = A_0(x_0) = a\, l_0^2(x_0) = a$$

$$0 = A_0'(x_0) = b\, l_0^2(x_0) + a\, 2\, l_0(x_0)\, l_0'(x_0) = b + 2\, l_0'(x_0)$$

$$\Rightarrow b = -2\, l_0'(x_0)$$

$$A_0(x) = \left[1 - 2\, l_0'(x_0)\, (x - x_0) \right] l_0^2(x), \quad A_1(x) = \left[1 - 2\, l_1'(x_1)\, (x - x_1) \right] l_1^2(x)$$

$$B_0(x) = a\, l_1(x)\, l_0^2(x)$$

$$1 = B_0'(x) = a\, l_1'(x_0)\, l_0^2(x_0) + a\, l_1(x_0)\, 2\, l_0(x_0)\, l_0'(x_0) = a \left[l_1'(x_0) \right] \Rightarrow a = \frac{\cancel{\;\;}}{x_1 - x_0}$$

$$\Rightarrow B_0(x) = (x - x_0)\, l_0^2(x), \qquad B_1(x) = (x - x_1)\, l_1^2(x).$$

- Find $p(x) \in P_m$ s.t.

$$p^{(j)}(x_i) = c_{ij} \quad (0 \le j \le k_i - 1,\ 0 \le i \le n)$$

where $k_0 + k_1 + \cdots + k_n = m+1$.

Thrm It has a unique solution.

Proof $m+1$ equations and $m+1$ unknowns $\Longrightarrow$ square system $\Longleftrightarrow$ uniqueness.

Homogeneous problem $\quad p^{(j)}(x_i) = 0$.

$$\Longrightarrow \boxed{\cancel{\underset{P_m}{\underset{\cap}{p(x)}} = \underset{P_{m+1}}{\underset{\cap}{\prod_{i=0}^{n}(x-x_i)^{k_i}}}}} \quad p(x) \in P_m \text{ has } k_0 + \cdots + k_n = m+1 \text{ zeros}$$

$$\Longrightarrow p(x) \equiv 0$$

#

Example $\quad p^{(j)}(x_0) = c_{0j} \quad (0 \le j \le k)$

$$\Longrightarrow p(x) = c_{00} + c_{01}(x-x_0) + \cdots + \frac{c_{0k}}{k!}(x-x_0)^k.$$

$\boxed{\text{~~Newton Divided Differ~~}}$ Lagrange Form

$$\begin{cases} p(x_i) = c_{i0} \\ p'(x_i) = c_{i1} \end{cases} \quad 0 \le i \le n$$

$$p(x) = \sum_{i=0}^{n} c_{i0} A_i(x) + \sum_{i=0}^{n} c_{i1} B_i(x) \qquad \begin{cases} A_i(x_j) = \delta_{ij} \\ A_i'(x_j) = 0 \end{cases}, \quad \begin{cases} B_i(x_j) = 0 \\ B_i'(x_j) = \delta_{ij} \end{cases}$$

$$A_i(x) = \left[1 - 2(x-x_i)\, l_i'(x_i)\right] l_i^2(x)$$

$$B_i(x) = (x - x_i)\, l_i^2(x) \qquad l_i(x) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{x - x_j}{x_i - x_j}$$

<u>Thrm</u> (Error Estimate) $f \in C^{2n+2}[a,b]$, $p \in P_{2n+1}$ s.t.

$$p(x_i) = f(x_i) \text{ and } p'(x_i) = f'(x_i) \quad \text{for } i = 0, 1, \cdots, n$$

$$\Longrightarrow \quad f(x) - p(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{i=0}^{n} (x - x_i)^2$$

<u>Proof</u>

$$f(x) - p(x) = W(x) \prod_{i=0}^{n} (x - x_i)^2$$

$$\varphi(t) = f(t) - p(t) - W(x) \prod_{i=0}^{n} (t - x_i)^2 \quad \text{has } 2n+3 \text{ zeros}$$

$$\Longrightarrow \exists\, \xi \in (a,b) \text{ s.t.}$$

$$0 = \varphi^{(2n+2)}(\xi) = f^{(2n+2)}(\xi) - W(x)(2n+2)!$$

#

Newton Divided Difference Method

- Find $p \in P_2$ s.t.

$$p(x_0) = c_{00}, \quad p'(x_0) = c_{01}, \quad p(x_1) = c_{10}$$

$$\lim_{x \to x_0} f[x_0, x] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \Longrightarrow f[x_0, x_0] = f'(x_0)$$

$$p(x) = p[x_0] + p[x_0, x_0](x - x_0) + p[x_0, x_0, x_1](x - x_0)(x - x_1)$$

x_0	$f(x_0)$	$f'(x_0)$	$f[x_0, x_0, x_1]$	$f[x_0, x_0, x_1, x_1]$
x_0	$f(x_0)$	$f[x_0, x_1]$	$f[x_0, x_1, x_1]$	
x_1	$f(x_1)$	$f'(x_1)$		
x_1	$f(x_1)$			

Ex 3 Find $p \in P_4$ s.t.

$p(1) = 2,\ p'(1) = 3,\ p(2) = 6,\ p'(2) = 7,\ p''(2) = 8$

1	2	3	?	?	?
1	2	?	?	?	
2	6	7	4		
2	6				
2	6				

Divided Differences with Repetitions Assume that f is sufficiently differentiable

$f[x_0, x_1, \cdots, x_n]$ is defined as the coefficients of x^n in $p(x) \in P_n$ where <u>p interpolates f at $x_0, x_1, \cdots, x_n$</u>, where $x_0, x_1, \cdots, x_n$ may repeat.

$$p(x) = \sum_{j=0}^{n} f[x_0, x_1, \cdots, x_j] \prod_{i=0}^{j-1} (x - x_i)$$

~~Example~~ <u>Recursive Formula</u>

$$f[x_0, x_1, \cdots, x_n] = \begin{cases} \dfrac{f[x_1, \cdots, x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0} & \text{if } x_n \neq x_0 \\[2ex] \dfrac{f^{(n)}(x_0)}{n!} & \text{if } x_n = x_0 \end{cases}$$

§6.4 Spline Interpolation

partition of $[t_0, t_n]$ $\quad \Delta: t_0 < t_1 < \cdots < t_n \qquad t_i$ – knot

spline function $\quad \mathcal{S}_\Delta^k = \left\{ s \in C^{k-1}[t_0, t_n] \,\middle|\, s \in P_k([t_{i-1}, t_i)) \ \forall i = 1, \cdots, n \right\}$

dimensions

k=0 $\quad \mathcal{S}_\Delta^0$ is the collection of piecewise constants $\Rightarrow \dim \mathcal{S}_\Delta^0 = n$

k=1 $\quad \mathcal{S}_\Delta^1$ is the collection of continuous piecewise linear functions

$$\underbrace{2 \times n}_{\#\text{ of subintervals}} \quad \text{degrees of freedom}$$

$$s^-(x_i) = s^+(x_i) \quad (i = 1, \cdots, n-1) \quad \text{interior continuity constraints}$$

$$\Rightarrow \dim \mathcal{S}_\Delta^1 = 2n - (n-1) = n+1$$

degrees of freedoms : values at knots $\quad f(t_i) \quad i = 0, 1, \cdots, n$.

k=3 $\quad$ On each subinterval, $s \in P_3 \Rightarrow 4n$

Continuities at interior knots t_i $(i = 1, \cdots, n-1) \Rightarrow 3(n-1)$

$$\dim \mathcal{S}_\Delta^3 = 4n - 3(n-1) = n+3 = \underbrace{(n+1)}_{\text{values at knots}} + 2$$

Let $s(x) \in \mathcal{S}_\Delta^3$ and $s(x_i) = y_i$ for $i = 0, 1, \cdots, n$.

2 degrees of freedom to be determined.

Set $S''(t_i) = z_i$ for $i = 0, 1, \cdots, n$

On $[t_i, t_{i+1}]$, $S''(t_i) = z_i$, $S''(t_{i+1}) = z_{i+1}$, $S''(t) \in P_1[t_i, t_{i+1}]$

$$\Longrightarrow \quad S_i''(x) = \frac{z_i}{h_i}(t_{i+1} - x) + \frac{z_{i+1}}{h_i}(x - t_i), \quad h_i = t_{i+1} - t_i$$

$$\Longrightarrow \quad S_i(x) = \frac{z_i}{6h_i}(t_{i+1} - x)^3 + \frac{z_{i+1}}{6h_i}(x - t_i)^3 + C(x - t_i) + D(t_{i+1} - x)$$

$S(t_i) = y_i$ & $S(t_{i+1}) = y_{i+1}$

$$C = \frac{y_{i+1}}{h_i} - \frac{z_{i+1} h_i}{6}, \quad D = \frac{y_i}{h_i} - \frac{z_i h_i}{6}$$

$S_{i-1}'(t_i) = S_i'(t_i)$ for $i = 1, \cdots, n-1$

$$h_{i-1} z_{i-1} + 2(h_i + h_{i-1}) z_i + h_i z_{i+1} = \frac{6}{h_i}(y_{i+1} - y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1})$$

Natural Cubic Spline $\quad S''(t_0) = 0$ and $S''(t_n) = 0$

$$\begin{bmatrix} u_1 & h_1 & & & 0 \\ h_1 & u_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & h_{n-2} \\ 0 & & & h_{n-2} & u_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \\ v_{n-1} \end{bmatrix}, \quad \begin{aligned} u_i &= 2(h_i + h_{i-1}) \\ v_i &= b_i - b_{i-1} \\ b_i &= \frac{6}{h_i}(y_{i+1} - y_i) \end{aligned}$$

Thrm on Optimality of Natural Cubic Spline

$f \in C^2[a,b]$, $\Delta: a = t_0 < t_1 < \cdots < t_n = b$,

$S(x) \in S_\Delta^3$ interpolates f at t_i

$$\Rightarrow \quad \int_a^b (S'')^2 dx \le \int_a^b (f'')^2 dx .$$

curvature of $y = f(x)$ is $|f''| \left[1 + (f')^2\right]^{-\frac{3}{2}}$

Proof Let $g(x) \equiv f(x) - S(x)$, then $f = g + S$

$$\int_a^b f''^2 = \int_a^b S''^2 + \int_a^b g''^2 + 2\int_a^b g'' S''$$

Need to prove

$$0 \le \int_a^b g'' S'' = \sum_i \int_{t_{i-1}}^{t_i} g'' S''$$

$$= \sum_i \left[S'' g'(t_i) - S'' g'(t_{i-1}) - \int_{t_{i-1}}^{t_i} S''' g' dx \right]$$

$S''g'(b) - S''(a)g'(a) = 0$

$$\Rightarrow = -\sum_i \int_{t_{i-1}}^{t_i} S''' g' = -\sum S_i''' \int_{t_{i-1}}^{t_i} g' dx = 0 \qquad \#$$

Remark Instead of $S''(a) = S''(b) = 0$,

assuming $S'(a) = f'(a)$ and $S'(b) = f'(b)$

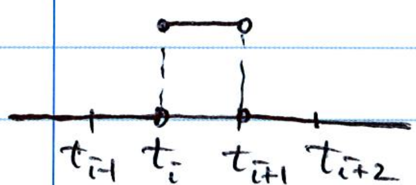
$\Rightarrow$ $S''(b)g'(b) - S''(a)g'(a) = 0 \Rightarrow$ the condition is true.

§6.5 B-Splines: Basic Theory

$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots$$

B-Splines of Degree 0

$$B_i^0(x) = \begin{cases} 1 & x \in [t_i, t_{i+1}) \\ 0 & x \notin [t_i, t_{i+1}) \end{cases}$$



Properties

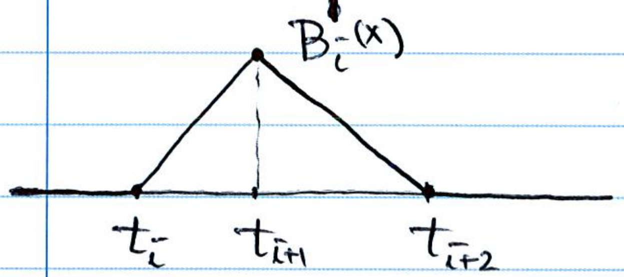
1. $\operatorname{supt} B_i^0 = \left\{ x \,\middle|\, B_i^0(x) \neq 0 \right\} = [t_i, t_{i+1})$
2. $B_i^0(x) \geq 0 \quad \forall x, \ \forall i$
3. $B_i^0(x)$ is cont. on $[t_{i+1}, +\infty)$
4. $\displaystyle\sum_{i=-\infty}^{\infty} B_i^0(x) = 1 \quad \forall x$
5. $S_\Delta^0 = \left\{ p \in P_0 \,\middle|\, p = c_i \text{ on } [t_i, t_{i+1}) \right\} = \operatorname{span}\left\{ B_i^0(x) \right\}$

Recursive Formula for $k \geq 1$

$$B_i^k(x) = \frac{x - t_i}{t_{i+k} - t_i} B_i^{k-1}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} B_{i+1}^{k-1}(x)$$

B-Splines of Degree 1

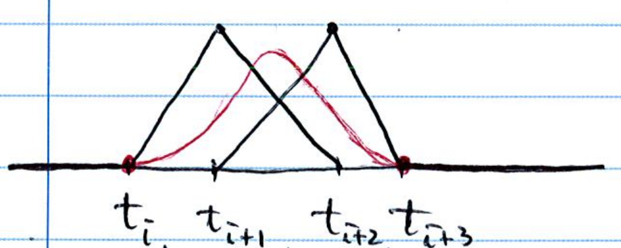
$$B_i^1(x) = \frac{x-t_i}{t_{i+1}-t_i} B_i^0(x) + \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} B_{i+1}^0(x) = \begin{cases} \dfrac{x-t_i}{t_{i+1}-t_i}, & [t_i, t_{i+1}) \\ \dfrac{t_{i+2}-x}{t_{i+2}-t_{i+1}}, & [t_{i+1}, t_{i+2}) \\ 0 & \text{otherwise} \end{cases}$$



$$\sum_{i=-\infty}^{\infty} B_i^1(x) \equiv 1$$

B-Splines of Degree 2

$$B_i^2(x) = \frac{x-t_i}{t_{i+2}-t_i} B_i^1(x) + \frac{t_{i+3}-x}{t_{i+3}-t_{i+1}} B_{i+1}^1(x) = \begin{cases} \dfrac{(x-t_i)^2}{(h_i+h_{i+1})h_i}, & [t_i, t_{i+1}) \\ \dfrac{(x-t_i)(t_{i+2}-x)}{(h_i+h_{i+1})\ h_{i+1}} + \dfrac{(t_{i+3}-x)(x-t_{i+1})}{(h_{i+1}+h_{i+2})\ h_{i+1}}, & [t_{i+1}, t_{i+2}) \\ \dfrac{(t_{i+3}-x)^2}{(h_{i+1}+h_{i+2})h_{i+2}}, & [t_{i+2}, t_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$



$h_i = t_{i+1}-t_i$ $\quad h_{i+1} = t_{i+2}-t_{i+1}$ $\quad h_{i+2} = t_{i+3}-t_{i+2}$

Properties of B-Splines

(1) $\text{suppt } B_i^k(x) = (t_i, t_{i+k+1})$ $\quad$ for $k \geq 0$

(2) $B_i^k(x) > 0 \quad \forall x \in (t_i, t_{i+k+1})$ $\quad$ for $k \geq 0$

(3) $$\sum_{i=-\infty}^{\infty} c_i B_i^k(x) = \sum_{i=-\infty}^{\infty} \left[c_i V_i^k(x) + c_{i-1} \left(1 - V_i^k(x)\right)\right] B_i^{k-1}(x)$$

where $V_i^k(x) = \dfrac{x - t_i}{t_{i+k} - t_i}$

<u>Evaluation</u> $$f(x) = \sum_{i=-\infty}^{\infty} c_i^k(x) B_i^k(x)$$

$$= \sum c_i^{k-1}(x) B_i^{k-1}(x) \quad \text{with } c_i^{k-1}(x) = c_i^k V_i^k + c_{i-1}^k \left(1 - V_i^k\right)$$

$$\ldots \qquad = \frac{(x - t_i) c_i^k + (t_{i+j} - x) c_{i-1}^j}{t_{i+j} - t_i}$$

$$= \sum c_i^0(x) B_i^0(x)$$

Given c_i^k ~~for~~ ~~Compute~~ $S(x_0) = \sum c_i^k B_i^k(x_0)$

(1) determine index m s.t. $t_m \leq x_0 < t_{m+1}$

(2) Compute

$$\begin{array}{llll} c_m^k & c_m^{k-1} & \cdots\ c_m^1 & c_m^0 = S(x_0) \\ c_{m-1}^k & c_{m-1}^{k-1} & \cdots\ c_{m-1}^1 & \\ \vdots & \vdots & & \\ c_{m-k}^k & & & \end{array}$$

(4) $$\sum_{i=-\infty}^{\infty} B_i^k(x) = 1$$

Proof Let $C_i^k = 1$ for all i

$$\Rightarrow C_i^{k-1} = C_i^k V_i^k + C_{i-1}^k \left[1 - V_i^k\right]$$
$$= V_i^k + \left(1 - V_i^k\right) = 1 \qquad \forall i$$
$$\Rightarrow C_i^j = 1 \qquad \forall i, \ \forall j = k, k-1, k-2, \cdots, 0$$
$$\Rightarrow \sum_i B_i^k(x) = \sum_i B_i^{k-1}(x) = \cdots = \sum B_i^0(x) = 1 .$$

(5) $$\frac{d}{dx} B_i^k(x) = \left(\frac{k}{t_{i+k} - t_i}\right) B_i^{k-1}(x) - \left(\frac{k}{t_{i+k+1} - t_{i+1}}\right) B_{i+1}^{k-1}(x)$$

for $k \geq 2$.

When $k=1$, the formula is valid except $\{t_i\}_{i=-\infty}^{\infty}$

(6) $\forall k \geq 1, \quad B_i^k(x) \in C^{k-1}(R)$

(7) $$\int_{-\infty}^{x} B_i^k(x)\,dx = \left(\frac{t_{i+k+1} - t_i}{k+1}\right) \sum_{j=i}^{\infty} B_i^{k+1}(x)$$

(8) $\{B_j^k, B_{j+1}^k, \cdots, B_{j+k}^k\}$ is l. indep. on (t_{k+j}, t_{k+j+1})

(9) $\{B_{-k}^k, B_{-k+1}^k, \cdots, B_{n-1}^k\}$ is l. indep. on (t_0, t_n)