## §6.7 Taylor Series

$$f(x) \in C^{n+1}(c-\delta, c+\delta)$$

$$\Longrightarrow \quad f(x) = P_n(x) + E_n(x)$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)\,(x-c)^k, \quad E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Power series

$$\cancel{f(x) =} \sum_{k=0}^{\infty} a_k (x-c)^k$$

$\exists\, r \in [0, +\infty]$ s.t. it converges for $|x-c| < r$

radius of convergence and diverges for $|x-c| > r$.

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad r = +\infty$$

$$\frac{1}{x} = \sum_{k=0}^{\infty} (-1)^k (x-1)^k, \quad r = 1$$

$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ defines a function on $|x-c| < r$

$$\Longrightarrow \quad f'(x) = \sum k a_k (x-c)^{k-1}$$

$$\int_b^x f(t)\,dt = \sum a_k \int_b^x (t-c)^k\,dt$$

## §6.8 Best Approximation: Least-Square Theory

$E$ — a normed linear space

$G \subset E$ — a subspace

Given $f \in E$, find $g \in G$ s.t.

$$\operatorname{dist}(f, G) \equiv \|f - g\| = \inf_{h \in G} \|f - h\|$$ the best approximation

Examples (1) $E = C[a,b]$, $\|f\| = \max_{a \le x \le b} |f(x)|$ — §6.9

(2) $E = L^2[a,b]$, $\|f\| = \left( \int_a^b f^2(x)\,dx \right)^{\frac{1}{2}}$ — §6.8

Thrm 1 (Thrm on Existence of Best Approx.)

$G$ is a finite-dimensional subspace of $E$

$$\Longrightarrow \forall f \in E, \ \exists g \in G, \text{ s.t. } \|f - g\| = \inf_{h \in G} \|f - h\|$$

Proof Let

$$K = \{ g \in G \mid \|g - f\| \le \|f\| \}$$ — closed, bounded $\xRightarrow{G\text{-finite}}$ compact

$L(g) = \|f - g\|$ is cont. $\Longleftarrow$ $|L(g_1) - L(g_2)| = \big| \|f - g_1\| - \|f - g_2\| \big| \le \|g_1 - g_2\|$

$\Longrightarrow L(g)$ attains its infimum. #

# Inner-Product Space

## Inner Product Axioms

(1) $\langle f, g\rangle = \langle g, f\rangle$

(2) $\langle f, \alpha h + \beta g\rangle = \alpha \langle f, h\rangle + \beta \langle f, g\rangle$

(3) $\langle f, f\rangle > 0$ if $f \neq 0$

(4) $\|f\| = \sqrt{\langle f, f\rangle}$

Examples (1) $E = R^n$, $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$

(2) $E = L^2[a,b]$, $\langle f, g\rangle = \int_a^b w(x) f(x) g(x)\, dx$

weight-function $w(x) \geq 0$

Def. $\langle f, g\rangle = 0 \iff f \perp g$.

Properties (1) $\langle \sum_{i=1}^{n} a_i f_i, g\rangle = \sum a_i \langle f_i, g\rangle$

(2) $\|f+g\|^2 = \|f\|^2 + 2\langle f, g\rangle + \|g\|^2$

(3) $f \perp g \implies \|f+g\|^2 = \|f\|^2 + \|g\|^2$

(4) $|\langle f, g\rangle| \leq \|f\| \, \|g\|$

(5) $\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right)$

Thrm ( Charactering Best Approx )

E — inner product space, $G \subset E$ — subspace

$\forall f \in E$

$$\Rightarrow \quad \|f-g\| = \inf_{h\in G} \|f-h\| \iff f-g \perp G.$$

Proof "$\Leftarrow$" $\|f-h\|^2 = \|(f-g)+(g-h)\|^2$

$$= \left(\|f-g\|^2 + \|g-h\|^2\right) \geq \|f-g\|^2$$

"$\Rightarrow$" Let $l(\lambda) = \|f-g+\lambda h\|^2 \quad \forall \lambda \in R, \forall h \in G$

$$= \|f-g\|^2 + 2\lambda \langle f-g, h\rangle + \lambda^2 \|h\|^2$$

$$\Rightarrow \quad l(0) = \min_{\lambda} l(\lambda)$$

$$\Rightarrow \quad 0 = l'(0) = 2\langle f-g, h\rangle \Rightarrow f-g \perp G. \quad \#$$

Normal Equations

Find $g(x) \in G = \operatorname{span}\{x, x^3, x^5\}$ s.t.

$$\|f-g\| = \inf_{h\in G} \|f-h\|, \quad \text{where } f = \sin x$$
$$\text{and } \|f\| = \left(\int_{-1}^{1} f^2(x)\,dx\right)^{\frac{1}{2}}$$

Solution Let $g = c_1 x + c_3 x^3 + c_5 x^5$

# Orthonormal Systems

- $\{f_1, f_2, \cdots\}$ is orthogonal $\Longleftrightarrow$ $\langle f_i, f_j \rangle = 0 \ \forall\, i \neq j$
- $\{f_1, f_2, \cdots\}$ is orthonormal $\Longleftrightarrow$ $\langle f_i, f_j \rangle = \delta_{ij}$

Thrm Let $G = \text{span}\{g_1, \cdots, g_n\} \subset E$ and $\forall f \in E$.

If $g = \sum_{i=1}^{n} c_i g_i$ is the best approx of $f$

$\Longrightarrow \quad c_i = \langle f, g_i \rangle$

Proof $0 = \langle f - g, g_j \rangle = \langle f, g_j \rangle - c_j$ #

Example $\text{span}\{x, x^3, x^5\} = \text{span}\{g_1, g_2, g_3\}$

$g_1 = x / \sqrt{\tfrac{2}{3}}$, $g_2 = (5x^3 - 3x) / (2\sqrt{2/7})$ Legendre poly.

$g_3 = (63x^5 - 70x^3 + 15x) / \left(8\sqrt{\tfrac{2}{11}}\right)$

$\Longrightarrow \quad c_i = \int_{-1}^{1} f(x)\, g_i(x)\, dx$

## Generalized Pythagorean Law

$\{g_1, \cdots, g_n\}$ is orthogonal $\Longrightarrow \left\| \sum_{i=1}^{n} a_i g_i \right\|^2 = \sum_{i=1}^{n} a_i^2 \|g_i\|^2$.

Proof

## Bessel's Inequality

$\{g_1, \cdots, g_n\}$ is orthogonal $\Longrightarrow \sum_{i=1}^{n} |\langle f, g_i \rangle|^2 \leq \|f\|^2$

Proof Let $g^* = \sum_{i=1}^{n} \langle f, g_i \rangle g_i$ — the best approx.

$$\Longrightarrow \|f\|^2 = \|(f-g^*)+g^*\|^2 = \|f-g^*\|^2 + \|g^*\|^2$$

$$\geq \|g^*\|^2$$

#

## The Gram-Schmidt Process

how to obtain orthonormal bases?

Thrm Let $\{v_1, \cdots, v_n\}$ be a basis for a subspace $U$ of an inner-product space. and let

$$u_i = \frac{v_i - \sum_{j=1}^{i-1} \langle v_i, u_j \rangle u_j}{\left\| v_i - \sum_{j=1}^{i-1} \langle v_i, u_j \rangle u_j \right\|}$$

$\Longrightarrow \{u_1, \cdots, u_n\}$ is an orthonormal bases for $U$.

**Proof** $\underline{n=1}$ $\quad u_1 = \dfrac{v_1}{\|v_1\|}$

$\underline{n=2}$ $\quad u_2 = \left(v_2 + \alpha_1 u_1\right) \Big/ \|v_2 + \alpha_1 u_1\|$

$$0 = \langle u_2, u_1 \rangle = \langle v_2, u_1 \rangle + \alpha_1 \|u_1\|^2 \Longrightarrow u_2 = \left(v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2} u_1\right) \Big/ \|\cdot\|$$

$\underline{n=3}$ $\quad u_3 = \left(v_3 + \alpha_1 u_1 + \alpha_2 u_2\right) \Big/ \|v_3 + \alpha_1 u_1 + \alpha_2 u_2\|$

$$0 = \langle u_3, u_1 \rangle = \langle v_3, u_1 \rangle + \alpha_1 \|u_1\|^2$$

$$0 = \langle u_3, u_2 \rangle = \langle v_3, u_2 \rangle + \alpha_2 \|u_2\|^2$$

#

## ~~Polyno~~ Orthogonal Polynomials $\operatorname{span}\{1, x, \cdots, x^n\} = U$

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx, \qquad w(x) \geq 0 \text{ — weight function}$$

$$p_0(x) = 1, \quad p_1(x) = x - a_1$$

$$p_n(x) = (x - a_n)\, p_{n-1}(x) - b_n\, p_{n-2}(x)$$

with $\quad a_n = \dfrac{\langle x p_{n-1}, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle}, \quad b_n = \dfrac{\langle x p_{n-1}, p_{n-2} \rangle}{\langle p_{n-2}, p_{n-2} \rangle}$

$\underline{n=1}$ $\quad 0 = \langle p_0, p_1 \rangle \Longrightarrow a_1 = \dfrac{\langle x p_0, p_0 \rangle}{\langle p_0, p_0 \rangle}$

$\underline{n=2}$ $\quad p_2(x) = (x - a_2)\, p_1(x) - b_2\, p_0(x)$

$\vdots$

<u>Example 2</u> Legendre polynomials $\quad w(x) = 1, \quad [a,b] = [-1,1]$

$$p_0(x) = 1, \quad p_1(x) = x$$

$$p_2(x) = x^2 - \frac{1}{3}, \quad \cdots$$

<u>Example 3</u> Chebyshev polynomials, is orthogonal $w(x) = \frac{1}{\sqrt{1-x^2}}$, $[a,b] = [-1,1]$

<u>Evaluation of $p_n(x)$</u> see Algorithm on P401.

<u>Thrm on Extremal Property</u>

$p_n(x) \in P_n$ is the monic poly and $\|p_n\| = \inf_{g \in P_n} \|g\|$

<u>Proof</u> $\forall g \in P_n$ being monic

$$\Rightarrow g = p_n - \sum_{i=0}^{n-1} c_i p_i$$

If $\|g\|$ is minimum $\Rightarrow g \perp P_{n-1} \Leftrightarrow p_n - \sum_{i=0}^{n-1} c_i p_i \perp P_{n-1}$

$$\Rightarrow 0 = \langle p_n - \sum_{i=0}^{n-1} c_i p_i, p_j \rangle \quad \text{for } j = 0, 1, \cdots, n-1$$

$$= 0 - c_j \langle p_j, p_j \rangle$$

$$\Rightarrow c_j = 0.$$

#

$\{u_1, u_2, \cdots\}$ — orthonormal system in $E$.

$\forall f \in E$, projection operator $P_n : f \in E \longrightarrow \underbrace{\mathrm{span}\{u_1, \cdots, u_n\}}_{= U_n}$ by

$$P_n f = \sum_{i=1}^{n} \langle f, u_i \rangle u_i$$

Properties

(1) $P_n$ is a projection: $P_n^2 = P_n$

(2) $f - P_n f \perp U_n$

(3) $P_n f$ is the best approx of $f$ in $U_n$

(4) $P_n$ is self-adjoint: $\langle P_n f, g \rangle = \langle f, P_n g \rangle$

The Gram Matrix

Let $\{u_1, \cdots, u_n\}$ be a basis for a subspace of $U$

and $u$ be the best approx of $f$

$\Rightarrow \; u - f \perp U \iff 0 = \langle u - f, u_i \rangle = \langle u, u_i \rangle - \langle f, u_i \rangle$

$u = \sum_{j=1}^{n} c_j u_j \implies \sum_{j=1}^{n} c_j \langle u_j, u_i \rangle = \langle f, u_i \rangle$ for $i = 1, \cdots, n$

Gram Matrix Let $G = (G_{ij})_{n \times n}$ with $G_{ij} = \langle u_j, u_i \rangle$ $\implies G\vec{c} = \vec{F}$

- $\{u_1, \cdots, u_n\}$ is linearly indep. $\iff G$ is non-singular.

## §6.10 Interpolation in Higher Dimensions (in $R^2$)

give an overview with no homework

## §6.12 Trigonometric Interpolation

**Fourier Series** span $\{1, \cos x, \cos 2x, \cdots, \sin x, \sin 2x, \cdots\}$

$f$ is $2\pi$-periodic and has a cont. $1^{st}$-order der.

$$\Longrightarrow \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos kx + b_k \sin kx\right) \longrightarrow f(x) \text{ uniformly}$$

where $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt$

**Complex F-Series**

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{ikx} \text{ with } \hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\,dt$$

**Euler's formula** $e^{i\theta} = \cos\theta + i\sin\theta$

**$f(x)$ is real** $\hat{f}(k) = \frac{1}{2}\left(a_k - i b_k\right) \Longrightarrow R_e\left(\sum \hat{f}(k) e^{ikx}\right) = \frac{a_0}{2} + \sum\left(a_k \cos kx + b_k \sin kx\right)$

$I_m\left(\sum \hat{f}(k) e^{ikx}\right) = 0$

$\cancel{\frac{1}{2}(a_k - i b_k)(\cos kx - i \sin kx)}$

## Thrm on F-series

Given real sequences $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$, define

$$b_0 = 0, \quad a_{-k} = a_k, \quad b_{-k} = -b_k, \quad c_k = \frac{1}{2}(a_k - i\, b_k)$$

$$\Longrightarrow \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right) = \sum_{k=-n}^{n} c_k e^{ikx}.$$

**Proof**
$$\sum_{k=-n}^{n} c_k e^{ikx} = \underbrace{\frac{1}{2} \sum_{k=-n}^{n} \left[ a_k \cos kx + b_k \sin kx \right]}_{\displaystyle = \frac{a_0}{2} + \sum_{k=1}^{n} \left( a_k \cos kx + b_k \sin kx \right)} + i\, \frac{1}{2} \underbrace{\sum_{k=-n}^{n} \left[ a_k \sin kx - b_k \cos kx \right]}_{= 0}$$

## Inner Product, Pseudo-Inner Product, and Pseudonorm

- inner product in the complex Hilbert space $L_2[-\pi, \pi]$

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, dx$$

- $\left\{ E_k(x) = e^{ikx} \;\middle|\; k = 0, \pm 1, \cdots \right\}$ — orthonormal system

$$\langle E_k, E_n \rangle = \begin{cases} 1, & n = k \\ 0 & n \neq k \end{cases}$$

- pseudo-inner product $\quad \langle f, g \rangle_N = \dfrac{1}{N} \displaystyle\sum_{j=0}^{N-1} f\left(\frac{2\pi j}{N}\right) \overline{g\left(\frac{2\pi j}{N}\right)}$

$$0 = \langle f, f \rangle \;\not\Longrightarrow\; f = 0.$$

(1) $\langle f, f\rangle_N \geq 0$

(2) $\langle f, g\rangle_N = \boxed{\cancel{\langle g, g\rangle_N}}\ \overline{\langle g, f\rangle_N}$

(3) $\langle \alpha f + \beta g, h\rangle_N = \alpha \langle f, h\rangle_N + \beta \langle g, h\rangle_N$

- pseudo-norm $\quad \|f\|_N = \sqrt{\langle f, f\rangle_N}$.

- $0 = \|f\|_N \iff f\left(\frac{2\pi j}{N}\right) = 0$ for $j = 0, 1, \cdots, N-1$.

Thrm on Pseudo-Inner Product $\quad$ For any $N \geq 1$

$$\langle E_k, E_m\rangle_N = \begin{cases} 1, & k-m \text{ is divisible by } N \\ 0, & \text{otherwise} \end{cases}$$

Proof $\quad \langle E_k, E_m\rangle_N = \frac{1}{N}\sum_{j=0}^{N-1} E_k\left(\frac{2\pi j}{N}\right)\overline{E_m\left(\frac{2\pi j}{N}\right)} = \frac{1}{N}\sum\left\{e^{i\frac{2\pi(k-m)}{N}}\right\}^j$

$\frac{k-m}{N}$ is integer $\implies e^{i\frac{2\pi(k-m)}{N}} = 1 \implies \langle E_k, E_m\rangle_N = 1$

$\frac{k-m}{N}$ is not an integer $\implies \underbrace{e^{i\frac{2(k-m)\pi}{N}}}_{=\lambda} \neq 1 \implies \langle E_k, E_m\rangle_N = \frac{1}{N}\sum_{j=0}^{N-1}\lambda^j = \frac{1}{N}\frac{\lambda^N - 1}{\lambda - 1} = 0$

#

## Exponential Polynomials (of degree at most n)

$$P(x) = \sum_{k=0}^{n} c_k e^{ikx} = \sum_{k=0}^{n} c_k E_k(x) = \sum_{k=0}^{n} c_k \left(e^{ix}\right)^k$$

## Thrm on Orthonormal Functions

$\{E_0, E_1, \cdots, E_{N-1}\}$ is orthonormal w.r.t $\langle \cdot, \cdot \rangle_N$.

## Corollary on Exponential Poly.

$P(x) = \sum_{k=0}^{N-1} c_k e^{ikx}$ interpolates $f$ at $x_j = \frac{2\pi j}{N}$, $j = 0, 1, \cdots, N-1$

$$\Longrightarrow \quad c_k = \langle f, E_k \rangle_N$$

Proof at $x_\nu = \frac{2\pi \nu}{N}$,

$$\sum_{k=0}^{N-1} \langle f, E_k \rangle_N e^{ik\cancel{x}x_\nu} = \sum_{k=0}^{N-1} \langle f, E_k \rangle_N E_k(x_\nu) = \sum_{j=0}^{N-1} f(x_j) \langle E_\nu, E_j \rangle_N = f(x_\nu).$$

## Corollary

$$\min_{c_k} \left\| f - \sum_{k=0}^{N-1} c_k E_k(x) \right\|_N \Longrightarrow c_k = \langle f, E_k \rangle_N .$$