Chapter 7 Numerical Differentiation and Integration

§7.1 Numerical Differentiation and Richardson Extrapolation

given $f(x_0), f(x_1), \cdots, f(x_n)$

Can we estimate $f'(c)$ or $\int_a^b f(x)\,dx$?

$f \in P_n \Longrightarrow$ exact $f'(c)$ or exact $\int_a^b f(x)\,dx$

$f \in C[a,b] \Longrightarrow$ possibly approximation to $f'(c)$ or $\int_a^b f\,dx$ is useless.

Numerical Differentiation (Approx. and Estimate)

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{h}{2}f''(\xi)$$

truncation error

Ex. 1 (P466) $f(x)=\cos x$, $x=\frac{\pi}{2}$, $h=0.01$

$f'(x) \approx \frac{1}{h}[f(x+h)-f(x)] = -0.71063051$

$\left|\frac{h}{2}f''(\xi)\right| \le 0.005$

Remark Truncation error and round-off error are equally important.

Ex. 2 (P467) $f(x)=\tan^{-1}x$, $x=\sqrt{2}$, $f'(x)=\frac{1}{x^2+1}$, $f'(\sqrt{2})=\frac{1}{3}$.

for different h see table on P467

the best approx. obtained when $k=12$

central difference

$$f'(x) = \frac{1}{2h}\left[f(x+h) - f(x-h)\right] - \frac{h^2}{12}\left[f'''(\xi_1) + f'''(\xi_2)\right]$$

$$= -\frac{h^2}{6} f'''(\xi)$$

$$f''(x) = \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right] - \frac{h^2}{12} f^{(4)}(\xi)$$

Ex. 2 (P469)

Differentiation via Poly. Interpolation

$$\omega(x) = \prod_{i=0}^{n} (x - x_i)$$

$$f(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x) + \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)\, \omega(x)$$

$$f'(x) = \sum f(x_i)\, l_i'(x) + \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)\, \omega'(x) + \frac{\omega(x)}{(n+1)!} \frac{d}{dx}\left(f^{(n+1)}(\xi_x)\right)$$

$$f'(x_j) = \sum f(x_i)\, l_i'(x_j) + \frac{1}{(n+1)!} f^{(n+1)}(\xi_{x_j}) \cancel{\prod_{\substack{k=0 \\ k \neq j}}^{n} (x_k - x)} \prod_{\substack{k=0 \\ k \neq j}}^{n} (x_j - x_k)$$

Ex. 3 & 4 (P470)

Richardson Extrapolation

$$f(x+h) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x), \quad f(x-h) = \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k h^k f^{(k)}(x)$$

$$\Rightarrow \quad f'(x) = \frac{1}{2h}\left[f(x+h) - f(x-h)\right] - \left[\frac{1}{3!} h^2 f^{(3)}(x) + \frac{1}{5!} h^4 f^{(5)}(x) + \cdots\right]$$

$$\wr \qquad\qquad \wr$$

$$L = \varphi(h) + a_2 h^2 + a_4 h^4 + \cdots$$

$$L = \varphi\left(\frac{h}{2}\right) + a_2\left(\frac{h}{2}\right)^2 + a_4\left(\frac{h}{2}\right)^4 + \cdots$$

$$\Rightarrow \quad L = \underbrace{\left[\frac{4}{3}\varphi\left(\frac{h}{2}\right) - \frac{1}{3}\varphi(h)\right]}_{\psi(h)} - a_4 h^4/4 - \cdots$$

$$\Rightarrow \quad L = \theta(h) + c_6 h^6 + \cdots$$

$$\theta(h) = \frac{16}{15}\psi\left(\frac{h}{2}\right) - \frac{1}{15}\psi(h)$$

$$\begin{array}{lll} D(0,0) & & \\ D(1,0) & D(1,1) & \\ \vdots & \vdots & \ddots \end{array}$$

Algorithm (1) choose h and compute

$$D(n,0) = \varphi(h/2^n) \qquad n = 0, 1, \cdots, M$$

(2) compute

$$D(n,k) = \frac{4^k}{4^k - 1} D(n, k-1) - \frac{1}{4^k - 1} D(n-1, k-1) \qquad \text{for } k = 1, 2, \cdots, M$$
$$n = k, k+1, \cdots, M$$

$$D(n,0) = L + O(h^2) \qquad\qquad D(n,k-1) = L + O(h^{2k}).$$

$$D(n,1) = L + O(h^4)$$

$$\vdots$$

§7.2 Numerical Integration Based on Interpolation

$$\int_a^b f(x)\,dx \approx \int_a^b g(x)\,dx \quad \text{where } f(x) \approx g(x)$$

Integration via Poly. Interpolation

Let $\quad p(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$

$$\Longrightarrow \int_a^b f(x)\,dx \approx \sum_{i=0}^{n} f(x_i) \int_a^b l_i(x)\,dx$$

$$= \sum_{i=0}^{n} A_i f(x_i) \quad \text{where } A_i = \int_a^b l_i(x)\,dx$$

Newton-Cotes Formula if x_i are equally spaced.

$x_i = \frac{b-a}{n} i$

Trapezoid Rule $(n=1)$ $\quad x_0 = a, \; x_1 = b$

$$l_0(x) = \frac{b-x}{b-a}, \; l_1(x) = \frac{x-a}{b-a} \Longrightarrow A_0 = A_1 = \frac{b-a}{2}$$

$$\boxed{\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a)+f(b)\right]} = Q_1(f)$$

Properties (1) $\int_a^b f(x)\,dx = Q_1(f) \quad$ for $f \in P_1$

(2) $\int_a^b f(x)\,dx - Q_1(f) = -\frac{1}{12}(b-a)^3 f''(\xi)$

$$= \frac{1}{2}\int_a^b f''(\eta)(x-a)(x-b)\,dx = \frac{1}{2} f''(\xi) \int_a^b (x-a)(x-b)\,dx$$

Composite Trapezoid Rule

partition $\quad a = x_0 < x_1 < \cdots < x_n = b$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$\approx \frac{1}{2} \sum_{i=1}^{n} (x_i - x_{i-1}) \left[f(x_{i-1}) + f(x_i) \right]$$

$$\boxed{\begin{aligned} x_i &= a + ih \\ h &= \frac{b-a}{n} \end{aligned}}$$

$$\int_a^b f(x)\,dx \approx \frac{h}{2} \left[f(a) + 2 \sum_{i=1}^{n-1} f(a+ih) + f(b) \right]$$

$$\| $$

$$Q_{1,h}(f)$$

$$\int_a^b f(x)\,dx - Q_{1,h}(f) = -\frac{1}{12}(b-a)\,h^2 f''(\xi)$$

Ex. 1 $\quad [a,b] = [0,1]$, $\quad n = 2 \qquad$ N-C formula

$$\int_0^1 f(x)\,dx \approx \frac{1}{6} f(0) + \frac{2}{3} f\left(\frac{1}{2}\right) + \frac{1}{6} f(1)$$

$$\int_0^1 l_0(x)\,dx = \int_0^1 \frac{(x-\frac{1}{2})(x-1)}{(0-\frac{1}{2})(0-1)} = \int_0^1 (2x-1)(x-1)\,dx = \frac{1}{6}$$

Method of Undetermined Coefficients

Assume that $\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$ is exact for all $f \in P_n$

$$\Longrightarrow \quad A_i = \int_a^b l_i(x)\,dx$$

Proof $f = l_j(x)$

$$\Longrightarrow \quad \int_a^b l_j(x)\,dx = \sum_{i=0}^{n} A_i\, l_j(x_i) = A_j \qquad \#$$

Ex. 1 $\int_0^1 f(x)\,dx \approx A_0 f(0) + A_1 f(\tfrac{1}{2}) + A_2 f(1)$

$$f = 1, x, x^2 \quad \Longrightarrow \quad \begin{cases} A_0 + A_1 + A_2 = 1 \\ \frac{1}{2}A_1 + A_2 = \frac{1}{2} \\ \frac{1}{4}A_1 + A_2 = \frac{1}{3} \end{cases} \quad \Longrightarrow \quad A_0 = A_2 = \frac{1}{6},\ A_1 = \frac{2}{3}$$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = Q_2(f)$$

Properties (1) $\int_a^b f\,dx = Q_2(f) \quad \forall f \in P_2$

(2) $\int_a^b f\,dx - Q_2(f) = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$

$$\int_a^b f\,dx - Q_2(f) = \int_a^b \frac{f'''(\eta)}{3!}(x-a)\left(x-\frac{a+b}{2}\right)(x-b)\,dx$$

— the mean-value theorem does not apply.

Let $H(x) \in P_3$ and $\begin{cases} H(a) = f(a),\ H(b) = f(b) \\ H(c) = f(c),\ H'(c) = f'(c) \end{cases}$

$c = \frac{a+b}{2}$

$$\Longrightarrow \quad f(x) - H(x) = \frac{f^{(4)}(\xi)}{3!}(x-a)(x-c)^2(x-b)$$

$$\int_a^b H(x)\,dx = Q_2(H) = Q_2(f)$$

$$\Longrightarrow \quad \int_a^b f\,dx - Q_2(f) = \int_a^b f\,dx - Q_2(H)$$

$$= \int_a^b (f-H)\,dx = \frac{1}{3!} f^{(4)}(\eta) \int_a^b (x-a)(x-c)^2(x-b)\,dx$$

$$= -\frac{b-a}{180}\left(\frac{b-a}{2}\right)^4 f^{(4)}(\eta).$$

Composite Simpson's Rule

$$\int_a^b f\,dx \approx Q_{2,h}(f) = \frac{h}{3}\left[f(x_0) + 2\sum_{i=1}^{\frac{n}{2}} f(x_{2i-2}) + 4\sum_{i=1}^{n} f(x_{2i-1}) + f(x_n)\right]$$

$$\int_a^b f\,dx - Q_{2,h}(f) = -\frac{1}{180}(b-a)h^4 f^{(4)}(\eta)$$

General Integration Formula

$$\int_a^b f(x)\, w(x)\, dx \approx \sum_{i=0}^{n} A_i f(x_i) \qquad w(x) \text{ — weight function}$$

Ex. 2

Change of Intervals

given $\displaystyle\int_c^d f\, dt \approx \sum_{i=0}^{n} A_i f(t_i)$ is exact for all $f \in P_n$

want $\displaystyle\int_a^b f(x)\, dx \approx ?$

$$\lambda(t) = \frac{b-a}{d-c}\, t + \frac{ad-bc}{b-c} = a\frac{t-d}{c-d} + b\frac{t-c}{d-c} \ : \ [c,d] \longrightarrow [a,b]$$

$$\Longrightarrow \int_a^b f(x)\, dx \overset{x=\lambda(t)}{=\!=\!=} \frac{b-a}{d-c} \int_c^d f(\lambda(t))\, dt$$

$$\approx \frac{b-a}{d-c} \sum_{i=0}^{n} A_i f\left(\frac{b-a}{d-c} t_i + \frac{ad-bc}{d-c}\right)$$

§7.3 Gaussian Quadrature

$$(*) \qquad \int_a^b w(x) f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i), \qquad w(x) \text{ — weighted function}$$

(1) for fixed x_i, $(*)$ is exact for $f \in P_n$.

$$\Longleftrightarrow \quad A_i = \int_a^b w(x) \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}\,dx$$

(2) Find x_i s.t. $(*)$ is exact for all $f \in P_{2n+1}$

Thrm on Gaussian Quadrature $\quad w(x)$ — positive weight function

Let $q(x) \in P_{n+1}$ be w-orthogonal to P_n, i.e.,

$$\int_a^b w(x)\, q(x)\, p(x)\,dx = 0 \quad \forall\, p \in P_n$$

$\Longrightarrow$ If $\{x_i\}_{i=0}^{n}$ are zeros of $q(x)$, then $(*)$ is exact for all $f \in P_{2n+1}$.

Proof $\forall f \in P_{2n+1} \xRightarrow{f/q} f = \overset{\in P_{n+1}}{q(x)}\,\overset{\in P_n}{p(x)} + \overset{\in P_n}{r(x)}$

and $f(x_i) = r(x_i) \quad i = 0, 1, \cdots, n$

$$\int_a^b f w\,dx = \int_a^b w\,q\,p\,dx + \int_a^b w\,r\,dx$$
$$= \int_a^b w\,r\,dx = \sum_{i=0}^{n} A_i\,r(x_i) = \sum_{i=0}^{n} A_i\,f(x_i) \qquad \#$$

<u>Remark</u> q has $n+1$ simple roots in $[a,b]$.

<u>Theorem on Number of Sign Changes</u> ~~w = positive weight~~

Let $w \in C[a,b]$ be positive and let $q(x) \in C[a,b]$ be nonzero and w-orthogonal to P_n.

$\Longrightarrow$ $q(x)$ changes signs at least $n+1$ times on (a,b).

<u>Proof</u> (1) $0 = \int_a^b q\,w\,dx \Longrightarrow q$ changes sign at least once

(2) Assume that q changes sign only r times and $r \le n$

$\Rightarrow \exists\ a = t_0 < t_1 < \cdots < t_r < t_{r+1} = b$ s.t.

q has one sign on ~~eat~~ each interval

$(t_0, t_1), (t_1, t_2), \cdots, (t_r, t_{r+1})$

$\Longrightarrow p(x) = \prod_{i=1}^{r} (x - t_i)$ has the same sign property as $q(x)$

$\Longrightarrow \int_a^b w\,q\,p\,dx \neq 0 \Longrightarrow$ contradiction since $p \in P_n$ #

Example $\omega(x)=1$, $[a,b]=[-1,1]$ $\boxed{\text{Legender Poly}}$

n=1 $\int_{-1}^{1} f(x)\,dx \approx f(-\alpha)+f(\alpha)$ with $\alpha=\frac{1}{\sqrt{3}}$, $P_2(x)=$

n=4 $\int_{-1}^{1} f(x)\,dx \approx \sum_{i=0}^{3} A_i f(x_i)$

Legender Polynomials

n=2 $P_2(x)=\frac{1}{2}(3x^2-1)=0 \implies x_{1,2}=\pm\frac{1}{\sqrt{3}}$

n=5 $P_5(x)=\frac{1}{8}(35x^4-30x^2+3)$

Convergence and Error Analysis

Lemma on Gaussian Quadrature Formula $\int_a^b \omega f\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$

$$A_i>0 \quad \text{and} \quad \sum_{i=0}^{n} A_i = \int_a^b \omega\,dx$$

Proof Let $q \in P_{n+1}$ be ω-orthogonal to P_n and $q(x_i)=0$ for $i=0,\dots,n$.

Let $p(x)=\frac{q(x)}{x-x_j}$ for a fixed j

$$0<\int_a^b \omega p^2(x)\,dx = \sum_{i=0}^{n} A_i p^2(x_i) = A_j p^2(x_j) \implies A_j>0$$

$$\int_a^b \omega\,dx = \sum_{i=0}^{n} A_i$$

#

Thrm on Gaussian Quadrature Convergence

$$f \in C[a,b] \implies \lim_{n\to\infty} \sum_{i=0}^{n} A_i f(x_i) = \int_a^b w f \, dx .$$

~~Proof~~

Thrm on Gaussian Formula with Error Term

$$\int_a^b f w \, dx = \sum_{i=0}^{n-1} A_i f(x_i) + E_n$$

$$f \in C^{2n}[a,b] \implies E_n = \frac{f^{(2n)}(\xi)}{(2n)!} \int_a^b q^2 w \, dx$$

where $\xi \in (a,b)$ and $q = \prod_{i=0}^{n-1} (x - x_i)$

Proof Let $p \in P_{2n-1}$ and

$p(x_i) = f(x_i)$ and $p'(x_i) = f'(x_i)$ for $i = 0, 1, \cdots, n-1$.

$$\implies f(x) - p(x) = \frac{f^{(2n)}(\xi)}{(2n)!} q^2(x)$$

$$\implies \int_a^b f w \, dx - \sum_{i=0}^{n-1} A_i f(x_i) = \int_a^b (f - p) w \, dx = \frac{1}{(2n)!} \int_a^b f^{(2n)}(\xi) q^2 w \, dx$$

$$= \frac{f^{(2n)}(\xi)}{(2n)!} \int_a^b q^2 w \, dx$$

mean-value thrm

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§7.4 Romberg Integration

$$I = \int_a^b f(x)\,dx$$

Recurcive Trapezoid Rule $\quad h = \frac{b-a}{n}$

$$T(n) = \frac{b-a}{n}\left[\frac{1}{2}f(a) + \sum_{i=1}^{n-1} f\big(a+ih\big) + \frac{1}{2}f(b)\right]$$

$[a,b]=[0,1]$ $\quad h_0 = 1,\quad h_i = \frac{h_{i-1}}{2}$

$$T(1) = \frac{1}{2}f(0) + \frac{1}{2}f(1)$$

$$T(2) = \frac{1}{4}f(0) + \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{4}f(1) \Rightarrow T(2^1) = \frac{1}{2}T(2^0) + h_1 f(h_1)$$

$$= \frac{1}{2}T(1) + \frac{1}{2}f\left(\frac{1}{2}\right)$$

$$T(4) = \frac{1}{8}f(0) + \frac{1}{4}\left[f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right] + \frac{1}{8}f(1)$$

$$= \frac{1}{2}T(2) + \frac{1}{4}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right] \Rightarrow T(2^2) = \frac{1}{2}T(2^1) + h_2\left[f(h_2) + f(3h_2)\right]$$

$$T(8) = \frac{1}{2}T(4) + \frac{1}{8}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]$$

$$\Rightarrow T(2^3) = \frac{1}{2}T(2^2) + h_3\left[f(h_3) + f(3h_3) + f(5h_3) + f(7h_3)\right]$$

$$\Rightarrow T(2^n) = \frac{1}{2}T(2^{n-1}) + h_n \sum_{i=1}^{2^{n-1}} f\big((2i-1)h_n\big)$$

Let $h_0 = b-a$, $h_n = h_{n-1}/2$

$$T(2^n) = \frac{1}{2} T(2^{n-1}) + h_n \sum_{i=1}^{2^{n-1}} f\big(a + (2i-1) h_n\big) \quad \text{for } n \geq 1$$

<u>Romberg Algorithm</u> $\quad R(n,0) = T(2^n)$

$$\begin{cases} R(0,0) = \frac{1}{2}(b-a)\left[f(a) + f(b)\right] \\ R(n,0) = \frac{1}{2} R(n-1,0) + h_n \displaystyle\sum_{i=1}^{2^{n-1}} f\big(a + (2i-1) h_n\big) \end{cases}$$

$$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}\left[R(n,m-1) - R(n-1,m-1)\right]$$

$$= \frac{4^m}{4^m-1} R(n,m-1) - \frac{1}{4^m-1} R(n-1,m-1)$$

$R(0,0)$			
$R(1,0)$	$R(1,1)$		
$R(2,0)$	$R(2,1)$	$R(2,2)$	
$R(3,0)$	$R(3,1)$	$R(3,2)$	$R(3,3)$

$$\int_a^b f\,dx = I = \underset{\substack{\parallel \\ T(2^n)}}{R(n,0)} + c_2 h^2 + c_4 h^4 + c_6 h^6 + \cdots \quad \text{with } h = \frac{b-a}{2^n}$$

$$R(1,1) = \frac{4}{4-1} R(1,0) - \frac{1}{4-1} R(0,0)$$
$$= \frac{4}{3} T(2) - \frac{1}{3} T(1) =$$

$$R(n,1) = \frac{4}{3} R(n,0) - \frac{1}{3} R(n-1,0)$$
$$= \frac{4}{3} T(2^n) - \frac{1}{3} T(2^{n-1}) = I + d_4 h^4 + \cdots$$

Assume that $I = \int_a^b f\,dx$

$$T(n) = I + c_2 h^2 + c_4 h^4 + \cdots \qquad \text{with } h = \frac{b-a}{n}$$

$$T(2n) = I + c_2 \left(\frac{h}{2}\right)^2 + c_4 \left(\frac{h}{2}\right)^4 + \cdots$$

$$\Longrightarrow \frac{2^2 T(2n) - T(n)}{2^2 - 1} = I + d_4 h^4 + d_6 h^6 + \cdots$$

$2^2 = 4^1$

$$\parallel$$

$$S(n)$$

$$S(2n) = I + d_4 \left(\frac{h}{2}\right)^4 + d_6 \left(\frac{h}{2}\right)^6 + \cdots$$

$$\Longrightarrow \frac{2^4 S(2n) - S(n)}{2^4 - 1} = I + e_6 h^6 + \cdots$$

$2^4 = 4^2$

$$\parallel$$

Newton-Cotes Formula for $n=4$ $\qquad C(n)$

$$C(2n) = I + e_6 \left(\frac{h}{2}\right)^6$$

$$\Longrightarrow \frac{2^6 C(2n) - C(n)}{2^6 - 1} = I + f_8 h^8 + \cdots$$

$2^6 = 4^3$

$$\parallel$$

Romberg Formula $\qquad R(n)$

which is not part of Newton-Cotes.

§7.5 Adaptive Quadrature

Given f, $[a, b]$, tolerance ε,

output I_n s.t. $|I - I_n| < \varepsilon$ with $I = \int_a^b f\,dx$

Simpson's Rule

$$\int_a^b f(x)\,dx = s(a,b) - \frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$$

$$s(a,b) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Composite

$$\Longrightarrow \int_a^b f\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f\,dx = \sum_{i=1}^{n} S_i + \sum_{i=1}^{n} e_i$$

$$\boxed{|e_i| \le \varepsilon \frac{x_i - x_{i-1}}{b-a}} \Longrightarrow \left|\sum e_i\right| \le \sum |e_i| \le \frac{\varepsilon}{b-a}\sum(x_i - x_{i-1}) = \varepsilon$$

$$\Longrightarrow \left|\int_a^b f\,dx - \sum S_i\right| \le \varepsilon$$

$$e_i = -\frac{1}{90}\left(\frac{x_i - x_{i-1}}{2}\right)^5 \boxed{f^{(4)}(\xi)}$$ a priori error estimation

a posteriori error estimation $\quad h_i = x_i - x_{i-1}$

$$S_i(h_i) = \frac{h_i}{6}\left[f(x_{i-1}) + 4f\Big(\underbrace{\frac{x_{i-1}+x_i}{2}}_{= x_{i-\frac{1}{2}}}\Big) + f(x_i)\right] = I_i + c\,h_i^4 + \cdots$$

$$S_i\left(\frac{h_i}{2}\right) = \frac{h_i}{12}\left[f(x_{i-1}) + 4f(x_{i-\frac{1}{4}}) + 2f(x_{i-\frac{1}{2}}) + 4f(x_{i-\frac{3}{4}}) + f(x_i)\right] = I_i + c\left(\frac{h_i}{2}\right)^4$$

$$\Longrightarrow \frac{1}{15}\left(S_i\left(\frac{h_i}{2}\right) - S_i(h_i)\right) = \cancel{I_i - S_i + O(h_i^6)}\ (I_i - S_i(h_i)) + O(h_i^6)$$

$$e_i \approx I - S_i(h_i) \approx \frac{1}{15}\left(S_i\left(\frac{h_i}{2}\right) - S_i(h_i)\right)$$

If $\left|\frac{1}{15}\left(S_i\left(\frac{h_i}{2}\right) - S_i(h_i)\right)\right| < \frac{\varepsilon(x_i - x_{i-1})}{b-a}$

then $\hat{I}_i = S_i\left(\frac{h_i}{2}\right) + \frac{1}{15}\left(S_i\left(\frac{h_i}{2}\right) - S_i(h)\right)$

is accepted as an approx. to $I_i = \int_{x_{i-1}}^{x_i} f\,dx$

Otherwise, $[x_{i-1}, x_i]$ is divided into 2 subintervals

$$\left[x_{i-1}, \frac{x_{i-1}+x_i}{2}\right] \text{ and } \left[\frac{x_{i-1}+x_i}{2}, x_i\right].$$