Numerical Solution of ODEs

$$\begin{cases} x'(t) = f(t, x(t)), & t > t_0 \\ x(t_0) = x_0 \end{cases}$$

§1 Euler's Method

explicit

$$x'(t_n) = f(t_n, x(t_n))$$

$$x'(t_n) \approx \frac{x(t_n+h) - x(t_n)}{h}$$

$$\Longrightarrow \quad x(t_n+h) \approx x(t_n) + h f(t_n, x(t_n))$$

$$\boxed{x_{n+1} = x_n + h f(t_n, x_n)}$$

local truncation error Assume that $X_n = x(t_n)$

$$\Longrightarrow \quad x_{n+1} = x(t_n) + h f(t_n, x(t_n))$$

$$= x(t_n) + h\, x'(t_n)$$

$$x(t_n+h) = x(t_n) + h\, x'(t_n) + \frac{h^2}{2} x''(\xi), \quad \xi \in (t_n, t_{n+1})$$

$$\Longrightarrow \quad \boxed{x(t_n+h) - \overset{x_{n+1}}{\cancel{x(t_n)}} = \frac{h^2}{2} x''(\xi) = O(h^2)}$$

<u>implicit</u>

$$x'(t_{n+1}) = f(t_{n+1}, x(t_{n+1}))$$

$$x'(t_{n+1}) = \frac{x(t_{n+1}) - x(t_n)}{h} + \frac{h}{2} x''(\xi)$$

$$\Longrightarrow \boxed{x_{n+1} = x_n + h f(t_{n+1}, x_{n+1})}$$

$$\boxed{x(t_{n+1}) - x_{n+1} = O(h^2)}$$

Assume that $x_n = x(t_n)$

$f(t_{n+1}, x_{n+1}) = f(t_n, x(t_n)) + O(h)$

<u>2-step</u>

$$x'(t_n) = f(t_n, x(t_n))$$

$$x'(t_n) = \frac{x(t_{n+1}) - x(t_{n-1})}{2h} + O(h^2)$$

$$\Longrightarrow x_{n+1} = x_{n-1} + 2h f(t_n, x_n)$$

Assume that $x_{n-1} = x(t_{n-1})$ and $x_n = x(t_n)$

$$x(t_{n+1}) - x_{n+1} = x(t_{n+1}) - x(t_{n-1}) - 2h f(t_n, x(t_n))$$

$$= O(h^3)$$

§2 Improved Euler Method

trapezoid

$$x'(t) = f(t,x) \Longrightarrow x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(t, x(t))\,dt$$

$$\approx x(t_n) + \frac{h}{2}\left[f(t_n, x(t_n)) + f(t_{n+1}, x(t_{n+1}))\right]$$

$$\Longrightarrow \boxed{x_{n+1} = x_n + \frac{h}{2}\left[f(t_n, x_n) + f(t_{n+1}, x_{n+1})\right]}$$

$$= \frac{1}{2}\left[x_n + hf(t_n, x_n)\right] + \frac{1}{2}\left[x_n + hf(t_{n+1}, x_{n+1})\right]$$

Assume that $x_n = x(t_n)$

$$x(t_{n+1}) - x_{n+1} = x(t_{n+1}) - x(t_n) - \frac{h}{2}\left[f(t_n, x(t_n)) + f\left(t_n + h, x(t_n) + \frac{h}{2}\left(x'(t_n) + f(t_{n+1}, x_{n+1})\right)\right)\right]$$

$$= h\cancel{x'(t_n)} + \frac{h^2}{2}x''(t_n) - \frac{h}{2}\Big[\cancel{x'(t_n)} + \cancel{f(t_n, x(t_n))} + hf_t(t_n, x(t_n))$$

$$+ \frac{h}{2}\Big(\cancel{f(t_n, x(t_n))} + \underbrace{f(t_{n+1}, x_{n+1})}_{= f(t_n, x(t_n)) + O(h)}\Big) f_x(t_n, x(t_n))\Big] + O(h^3)$$

$$= \frac{h^2}{2}x''(t_n) - \frac{h^2}{2}\left[f_t + ff_x\right] + O(h^3) = O(h^3)$$

$f(t+h, x+hf)$
$= f + hf_t + hff_x + O(h^2)$
$x''(t) = f_t + ff_x$

improved Euler (Heun)

$$\begin{cases} \bar{x}_{n+1} = x_n + hf(t_n, x_n) & \text{predictor} \\ x_{n+1} = x_n + \frac{h}{2}\left[f(t_n, x_n) + f(t_{n+1}, \bar{x}_{n+1})\right] & \text{corrector} \end{cases}$$

$$= x_n + \frac{h}{2}\left[f(t_n, x_n) + f(t_{n+1}, x_n + hf(t_n, x_n))\right]$$

or

$$\begin{cases} x_p = x_n + hf(t_n, x_n) \\ x_c = x_n + hf(t_{n+1}, x_p) \\ x_{n+1} = \frac{1}{2}(x_p + x_c) \end{cases}$$

§3 Runge-Kutta Method

Mean-Value Thrm $$\frac{x(t_{n+1}) - x(t_n)}{h} = x'(\xi) \qquad \xi \in (t_n, t_{n+1})$$

$$\Longrightarrow \quad x(t_{n+1}) = x(t_n) + h \, \cancel{x'(\xi)} \, f(\xi, x(\xi))$$

$K^* = f(\xi, x(\xi))$ — the average slope over $[t_n, t_{n+1}]$

explicit Euler $\qquad K^* \approx K_1 = f(t_n, x_n)$

improved Euler $\qquad K^* \approx \frac{1}{2}(K_1 + K_2) \qquad$ where $K_2 = f(t_{n+1}, x_n + h K_1)$

2^{nd}-order Runge-Kutta Method

$$t_{n+p} = t_n + ph \in (t_n, t_{n+1}] \qquad \text{with } p \in (0, 1]$$

Let $\quad K_1 = f(t_n, x_n)$ and $K_2 = f(t_{n+p}, x_n + ph K_1)$

$$K^* \approx (1 - \lambda) K_1 + \lambda K_2$$

Assume that $x_n = x(t_n)$

$$K_1 = f(t_n, x(t_n)) = x'(t_n)$$

$$\begin{aligned} K_2 &= f(t_{n+p}, x_n + ph K_1) \\ &= f(t_n, x_n) + ph \left[f_t(t_n, x_n) + f f_x(t_n, x_n) \right] + O(h^2) \\ &= x'(t_n) + ph \, x''(t_n) + O(h^2) \end{aligned}$$

$$x_{n+1} = x(t_n) + h x'(t_n) + \lambda p h^2 x''(t_n) + O(h^3)$$

$$\Rightarrow \boxed{x(t_{n+1}) - x_{n+1} = \left(\frac{1}{2} - \lambda p\right) h^2 x''(t_n) + O(h^3)}$$

$$\boxed{\lambda p = \frac{1}{2}}$$

<u>2nd order R-K</u>

$$\begin{cases} K_1 = f(t_n, x_n) \\ K_2 = f(t_{n+p}, x_n + phK_1) \\ x_{n+1} = x_n + h\left[(1-\lambda)K_1 + \lambda K_2\right] \end{cases} \quad \text{with } \lambda p = \frac{1}{2}$$

<u>Improved Euler (Heun)</u> $\quad p = 1, \quad \lambda = \frac{1}{2}$

<u>Modified Euler</u> (Mid-point) $\quad p = \frac{1}{2}, \quad \lambda = 1$

$$\begin{cases} K_1 = f(t_n, x_n) \\ K_2 = f(t_{n+\frac{1}{2}}, x_n + \frac{h}{2}K_1) \\ x_{n+1} = x_n + h K_2 \end{cases}$$

3rd order R-K $\quad t_{n+q} = t_n + q h \quad$ with $\quad p \le q \le 1$

$$K^* \approx (1-\lambda-\mu) K_1 + \lambda K_2 + \mu K_3$$

$$K_3 = f(t_{n+q}, x_{n+q})$$

$$x_{n+q} = x_n + q h \left[(1-\alpha) K_1 + \alpha K_2\right]$$

where $p, q, \lambda, \mu, \alpha$ are coefficients to be determined so that

$$x(t_{n+1}) - x_{n+1} = O(h^4)$$

e.g.

$$\begin{cases} K_1 = f(t_n, x_n), \quad K_2 = f(t_{n+\frac{1}{2}}, x_n + \frac{h}{2} K_1), \\ K_3 = f(t_{n+1}, x_n + h(-K_1 + 2K_2)) \\ x_{n+1} = x_n + \frac{h}{6}(K_1 + 4K_2 + K_3) \end{cases}$$

4th order R-K

$$\begin{cases} K_1 = f(t_n, x_n), \quad K_2 = f(t_{n+\frac{1}{2}}, x_n + \frac{h}{2} K_1) \\ K_3 = f(t_{n+\frac{1}{2}}, x_n + \frac{h}{2} K_2), \quad K_4 = f(t_{n+1}, x_n + h K_3) \\ x_{n+1} = x_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \end{cases}$$

Convergence and Stability

For fixed $t_n = t_0 + nh$,

does x_n converge to $x(t_n)$ as $h \to 0$ (and $n \to \infty$)?

<u>explicit Euler</u> $\quad x_{n+1} = x_n + h f(t_n, x_n)$

let $\quad \bar{x}_{n+1} = x(t_n) + h f(t_n, x(t_n))$

$\Rightarrow x_{n+1} - \bar{x}_{n+1} = x_n - x(t_n) + h\left(f(t_n, x_n) - f(t_n, x(t_n))\right)$

local truncation error

$$\Rightarrow \quad x(t_{n+1}) - \bar{x}_{n+1} = \frac{h^2}{2} x''(\xi), \quad \xi \in (t_n, t_{n+1})$$

$$\Rightarrow \quad |x(t_{n+1}) - \bar{x}_{n+1}| \le C h^2 \quad \text{if } x \in C^2[a,b]$$

Let $\quad e_n = |x(t_n) - x_n|$

$$\Rightarrow e_{n+1} = |x(t_{n+1}) - \bar{x}_{n+1} + \bar{x}_{n+1} - x_{n+1}|$$

$$\le |x(t_{n+1}) - \bar{x}_{n+1}| + |\bar{x}_{n+1} - x_{n+1}|$$

$|f(t_n, x_n) - f(t_n, x(t_n))| \le L |x_n - x(t_n)|$

$$|\bar{x}_{n+1} - x_{n+1}| = |x_n - x(t_n) + h\left(f(t_n, x_n) - f(t_n, x(t_n))\right)|$$

$$\le e_n + h\,|f(t_n, x_n) - f(t_n, x(t_n))|$$

$$\le e_n + hL e_n = (1 + hL) e_n$$

$$\le ch^2 + (1+hL) e_n \le ch^2 [1 + (1+hL)] + (1+hL)^2 e_{n-1}$$

$$\le ch^2 \left[1 + (1+hL) + \cdots + (1+hL)^n\right] + (1+hL)^{n+1} e_0$$

$$\Rightarrow e_{n+1} \leq \frac{Ch}{L}\left[(1+hL)^{n+1}-1\right] + (1+hL)^{n+1} e_0$$

$$1+hL \leq e^{hL}, \quad t_n - t_0 = nh \leq T \sim \text{fixed}$$

$$\Rightarrow (1+hL)^n \leq e^{nhL} \leq e^{TL}$$

$$\Rightarrow e_{n} \leq \frac{C}{L}\left(e^{TL}-1\right)h + e^{TL} e_0$$

$$\overset{e_0=0}{=} \frac{C}{L}\left(e^{TL}-1\right)h \to 0 \quad \text{as } h \to 0$$

<u>Stability</u> model problem $x'(t) = \lambda x(t)$ with $\lambda < 0$

<u>explicit Euler</u> $x_{n+1} = (1+h\lambda)x_n$

Assume that there is an perturbation ε_n at x_n and that there is no ~~new~~ other error in calculation of $x_{n+1} = (1+h\lambda)x_n$

$$\Rightarrow \varepsilon_{n+1} = (1+h\lambda)\varepsilon_n$$

$|\varepsilon_{n+1}| \leq |\varepsilon_n| \Longrightarrow$ the scheme is <u>stable</u>

$$\Downarrow$$

$$|1+h\lambda| \leq 1 \Rightarrow 0 \leq h \leq -\frac{2}{\lambda}$$

<u>implicit Euler</u> $x_{n+1} = x_n + h\lambda x_{n+1}$

$$\Rightarrow x_{n+1} = \frac{1}{1-h\lambda} x_n$$

$$\Rightarrow \varepsilon_{n+1} = \frac{1}{1-\lambda h}\varepsilon_n \Rightarrow |\varepsilon_{n+1}| \leq |\varepsilon_n|$$ unconditionally stable.