4. Write and test a computer program to compute the cubic spline function S having prescribed knots $t_0 < t_1 < \cdots < t_n$ and satisfying these conditions:

$$\begin{cases} S(t_i) = y_i \quad (0 \le i \le n) \\ S''(t_0) = \alpha \\ S''(t_n) = \beta \end{cases}$$

5. Program the algorithm for tension splines and test it with a variety of values of the tension parameter τ.

6. Start with the silhouette of a car body as shown in Figure 6.6. Prepare a table of abscissas and ordinates for 10 to 20 points. Using values of the tension $\tau = 0.25$, 4, and 10, generate and plot values of the interpolating tension spline to see which one produces the most pleasing appearance.

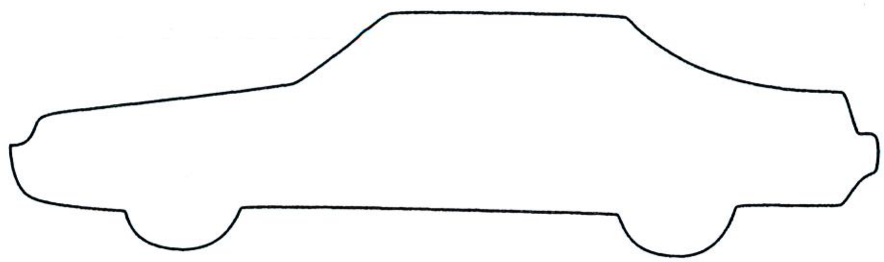

FIGURE 6.6
Car silhouette

7. Draw a script letter, such as the one shown in Figure 6.7. Then reproduce it with the aid of cubic splines and a plotter. Proceed as follows: Select a modest number of points on the curve, say $n = 11$. Label these $t = 1, 2, \ldots, n$. For each point, obtain the corresponding x- and y-coordinates. Then fit $x = S_x(t)$ and $y = S_y(t)$, using cubic spline interpolating functions S_x and S_y. This will produce a parametric representation of the original curve. Compute a large number of values of $S_x(t)$ and $S_y(t)$ to give to the plotter. To learn more about how spline curves are used in designing typefaces, the reader should consult Knuth [1979].

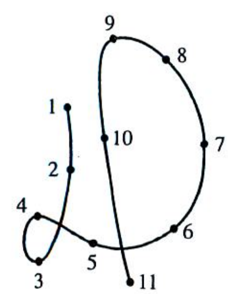

FIGURE 6.7
Script letter from 11 knots

8. Interpret the results of the following numerical experiment and draw some conclusions.

a. Define p to be the polynomial of degree 20 that interpolates the function $f(x) = (1 + 6x^2)^{-1}$ at 21 equally spaced nodes in the interval $[-1, 1]$. Include the endpoints as nodes. Print a table of $f(x)$, $p(x)$, and $f(x) - p(x)$ at 41 equally spaced points on the interval.

b. Repeat the experiment using the **Chebyshev nodes** given by

$$x_i = \cos[(i-1)\pi/20] \qquad (1 \le i \le 21)$$

c. With 21 equally spaced knots, repeat the experiment using a cubic interpolating spline.

6.5 *B*-Splines: Basic Theory

This section is devoted to a system of spline functions from which all other spline functions can be obtained by forming linear combinations. These splines provide bases for certain spline spaces and are therefore called **B-splines**. Once the knots are known, the B-splines are easily generated by recurrence relations and the algorithm is relatively simple. The B-splines are distinguished by their elegant theory and their model behavior in numerical calculations. Moreover, B-splines can be generalized.

We begin with a system of knots, t_i, on the real line. For practical purposes, only a finite set of knots is ever needed, but for the theoretical development it is much easier to suppose that the knots form an infinite set extending to $+\infty$ on the right and to $-\infty$ on the left:

$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots$$

This knot sequence is assumed to be fixed throughout this section, and all of our splines will be based on it.

B-Splines of Degree 0

The B-splines of degree 0 are denoted by B_i^0 and have the appearance shown in Figure 6.8. The index i ranges over all the integers. The heavy dots on the graph indicate that we define $B_i^0(t_i) = 1$ and $B_i^0(t_{i+1}) = 0$. The formal definition is

$$B_i^0(x) = \begin{cases} 1 & \text{if } t_i \le x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

These B-splines form an infinite sequence, $\{B_i^0 : i \in \mathbb{Z}\}$. (Here $\mathbb{Z}$ denotes the set of all integers, positive, negative, or 0.) We observe some of their salient properties:

1. The **support** of B_i^0, defined as the set of x where $B_i^0(x) \ne 0$, is the interval $[t_i, t_{i+1})$.

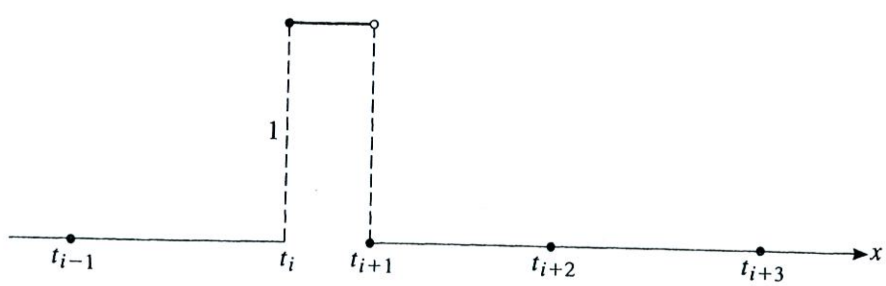

FIGURE 6.8
The B-spline B_i^0