4. a. Consider the series

$$e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \cdots \qquad (|x| \le \pi/2)$$

Retaining three terms in the series, estimate the remaining series using o-notation with the best integer value possible, as $x \to 0$.

b. Repeat the problem using $\mathcal{O}$-notation and the series

$$\ln \tan x = \ln x + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \cdots \qquad (0 < |x| < \pi/2)$$

5. Establish the range of integer values of γ and δ for which the $+\cdots$ in the series

$$\ln(1+x) = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{x^k}{k} + \cdots$$

can be replaced with either $\mathcal{O}(x^\gamma)$ or $o(x^\delta)$ as $x \to 0$.

6. For the pair (x_n, α_n), is it true that $x_n = \mathcal{O}(\alpha_n)$ as $n \to \infty$?

a. $x_n = 5n^2 + 9n^3 + 1, \quad \alpha_n = n^2$

b. $x_n = 5n^2 + 9n^3 + 1, \quad \alpha_n = 1$

c. $x_n = \sqrt{n+3}, \quad \alpha_n = 1$

d. $x_n = 5n^2 + 9n^3 + 1, \quad \alpha_n = n^3$

e. $x_n = \sqrt{n+3}, \quad \alpha_n = 1/n$

7. Choose the correct assertions (in each, $n \to \infty$).

a. $(n+1)/n^2 = o(1/n)$

b. $(n+1)/\sqrt{n} = o(1)$

c. $1/\ln n = \mathcal{O}(1/n)$

d. $1/(n \ln n) = o(1/n)$

e. $e^n/n^5 = \mathcal{O}(1/n)$

8. The expressions e^h, $(1-h^4)^{-1}$, $\cos(h)$, and $1 + \sin(h^3)$ all have the same limit as $h \to 0$. Express each in the following form with the best integer values of α and β.

$$f(h) = c + \mathcal{O}(h^\alpha) = c + o(h^\beta)$$

9. (Continuation) What are the limit and the rate of convergence of the following expression as $h \to 0$?

$$\frac{1}{h^2}\left[(1+h) - e^h\right]$$

Express the limit in the form given in the preceding problem.

10. Show that these assertions are not true.

a. $e^x - 1 = \mathcal{O}(x^2)$ as $x \to 0$

b. $x^{-2} = \mathcal{O}(\cot x)$ as $x \to 0$

c. $\cot x = o(x^{-1})$ as $x \to 0$

11. Let $[a_n] \to 0$ and let $\lambda > 1$. Show that $\sum_{k=0}^{n} a_k \lambda^k = o(\lambda^n)$ as $n \to \infty$.

12. Explain why the least upper bound axiom does not apply to the empty set.

13. Find two functions in explicit form that are defined implicitly by the equation

$$(x^3 - 1)y + e^x y^2 + \cos x - 1 = 0$$

14. In solving differential equations, one often obtains solutions in implicit form. Show that the equation

$$2x^3y^2 + x^2y + e^x = c$$

defines a solution of the differential equation

$$\frac{dy}{dx} = -(6x^2y^2 + 2xy + e^x)/(4x^3y + x^2)$$

15. Kepler's equation in astronomy is $x - y + \varepsilon \sin y = 0$, where ε is a parameter in the range $0 \le \varepsilon \le 1$. Show that for each real x there is a real y that makes the equation true. Show that if $0 \le \varepsilon < 1$, then dy/dx is continuous everywhere. *Hint:* Write $x = y - \varepsilon \sin y$ and consider the behavior of $y - \varepsilon \sin y$ as $y \to +\infty$ and as $y \to -\infty$. Use the Implicit Function Theorem for the second part of this problem.

16. Find the points x at which the equation

$$y - \ln(x + y) = 0$$

defines y implicitly as a function of x. Compute dy/dx.

17. Give an example to show why the least upper bound axiom does not apply to the set of all rational numbers.

18. Does the least upper bound axiom apply to the set of all integers?

19. What are the values of the following?

a. $\sup_{x \in \mathbb{R}} \arctan x$

b. $\sup_{x \ge 0} e^{-x}$

c. $\inf_{x \in \mathbb{R}} e^{-x}$

d. $\sup_{x \in \mathbb{R}} (x^2 + 1)^{-1}$

20. Use the Mean-Value Theorem for Integrals to prove that

$$\int_0^{\pi/2} e^x \cos x \, dx = e^x$$

for some y in $(0, \pi/2)$.

21. Give an example to show why the continuity of u cannot be dropped from the hypotheses in Theorem 1.

22. Prove that if $0 < \theta < 1$, then $(1 + a\theta^n)/(1 + a\theta^{n-1})$ converges to 1 linearly.

23. Are the following equivalent?

a. $|f(x)| = \mathcal{O}(|x|^{-n-\varepsilon})$ for some $\varepsilon > 0$ as $|x| \to \infty$

b. $|f(x)| = o(|x|^{-n})$

24. Prove that the set of upper bounds for a set S in $\mathbb{R}$ is either $\mathbb{R}$, the empty set, or an interval of the form $[a, \infty)$.

25. Prove by induction that the Horner algorithm is correct.

26. When the sequence $x_n = (1 + 1/n)^n$ is computed, it appears to be monotone increasing. Prove that this is so. *Hints:* First, if $\ln f(x)$ is increasing, then so is $f(x)$. Second, if $f'(x) > 0$, then f is increasing. Finally, $\ln x$ is defined to be $\int_1^x t^{-1} dt$.

27. (Continuation) Show that the elements of the sequence in the preceding problem remain less than 3.

28. Prove that $x_n = x + o(1)$ if and only if $\lim_{n\to\infty} x_n = x$.

An example of a difference equation that has *nonconstant* coefficients arises in the theory of Bessel functions. The **Bessel functions** J_n are defined by the formula

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\,d\theta$$

It is obvious from the definition that $|J_n(x)| \le 1$. Not so obvious, but true, is the recurrence formula

$$J_n(x) = 2(n-1)x^{-1}J_{n-1}(x) - J_{n-2}(x)$$

If (for a certain x) we know $J_0(x)$ and $J_1(x)$, then the recurrence relation can be used to compute $J_2(x), J_3(x), \ldots, J_n(x)$. This procedure becomes unstable and useless when $2n > |x|$ because roundoff errors that inevitably occur will be multiplied by the factor $2nx^{-1}$. This factor eventually becomes very large. (See Computer Problem **1.3.2**, p. 36.)

For further information on computing functions by means of recurrence relations, see Abramowitz and Stegun [1964, p. xiii], Cash [1979], Gautschi [1961, 1967, 1975], and Wimp [1984].

PROBLEMS 1.3

1. For the sequences following Equation (2), express the first as a linear combination of the second and third.
2. Let p be a polynomial of degree m. Is the solution space of $p(E)x = 0$ necessarily of dimension m?
3. Let p be a polynomial of degree m, with $p(0) \neq 0$. Prove that if a sequence x contains m consecutive zeros and $p(E)x = 0$, then $x = 0$.
4. Is the operator E **injective (one to one)**? Does it have a right or left inverse? Is it **surjective (onto)**? Define an operator F by $(Fx)_n = x_{n-1}$, with $(Fx)_1 = 0$, and answer the same questions for F. Explore the relationship between E and F. Suppose that V were redefined as the space of all functions on the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, and suppose that F were defined simply by $(Fx)_n = x_{n-1}$. How are the answers to the previous questions affected?
5. What are the eigenvalues and eigenvectors of the operator E?
6. Consider the infinite series $\sum_{n=1}^{\infty} x_n v^{(n)}$. What can you say about the question of convergence? Prove that $x = \sum_{n=1}^{\infty} x_n v^{(n)}$ in the pointwise sense.
7. If $\{v^{(1)}, v^{(2)}, \ldots\}$ is adopted as a basis for V, show that $\sum_{i=0}^{m} c_i E^i$ can be represented by an infinite matrix.
8. (Continuation) Prove that any two operators of the form described in the preceding problem commute with each other.
9. Prove that if L_1 and L_2 are linear combinations of powers of E, and if $L_1 x = 0$, then $L_1 L_2 x = 0$.
10. Develop a complete theory for the difference equation $E^r x = 0$.
11. Give bases consisting of real sequences for each solution space.
    a. $(4E^0 - 3E^2 + E^3)x = 0$
    b. $(3E^0 - 2E + E^2)x = 0$

c. $(2E^6 - 9E^5 + 12E^4 - 4E^3)x = 0$

d. $(\pi E^2 - \sqrt{2}E + \log 2 \cdot E^0)x = 0$

12. Prove that if p is a polynomial with real coefficients, and if $z \equiv [z_1, z_2, \dots]$ is a (complex solution of $p(E)z = 0$, then the conjugate of z, the real part of z, and the imaginary par of z are also solutions.

13. Solve.

a. $x_{n+1} - nx_n = 0$

b. $x_{n+1} - x_n = n$

c. $x_{n+1} - x_n = 2$

14. Define an operator Δ by putting

$$\Delta x = [x_2 - x_1, x_3 - x_2, x_4 - x_3, \dots]$$

Show that $E = I + \Delta$. Show that if p is a polynomial, then

$$p(E) = p(I) + p'(I)\Delta + \frac{1}{2}p''(I)\Delta^2 + \frac{1}{3!}p'''(I)\Delta^3 + \cdots + \frac{1}{m!}p^{(m)}(I)\Delta^m$$

15. (Continuation) Prove that if $x = [\lambda, \lambda^2, \lambda^3, \dots]$ and p is a polynomial, then $p(\Delta)x = p(\lambda - 1)x$. Describe how to solve a difference equation written in the form $p(\Delta)x = 0$.

16. (Continuation) Show that

$$\Delta^n = (-1)^n\left[E^0 - nE + \tfrac{1}{2}n(n-1)E^2 - \tfrac{1}{3!}n(n-1)(n-2)E^3 + \cdots + (-1)^n E^n\right]$$

17. Give a complete proof of Theorem 2.

18. Let p be a polynomial such that $p(0) = 0$. Describe the null space of $p(E)$.

19. For $\lambda \in \mathbb{C}$, define $x(\lambda) = [\lambda, \lambda^2, \lambda^2, \dots]$. Prove that if $\lambda_1, \lambda_2, \dots, \lambda_m$ are distinct nonzero complex numbers, then $\{x(\lambda_1), x(\lambda_2), \dots, x(\lambda_m)\}$ is a linearly independent set in V.

20. Prove that if λ is a nonzero root of p having multiplicity k, then the equation $p(E)x = 0$ has solutions $u^{(1)}, u^{(2)}, \dots, u^{(k)}$ in which $u_n^{(j)} = n^{j-1}\lambda^n$.

21. Prove that if $\mu \in (0, \infty)$ and $|\lambda| < 1$, then $\lim_{n\to\infty} n^\mu \lambda^n = 0$.

22. Prove in detail that a convergent sequence is bounded.

23. Prove Theorem 3 without the hypothesis that $p(0) \neq 0$.

24. Define a sequence inductively by the equation $x_{n+1} = x_n + x_n^{-1}$, where $x_0 > 0$. Determine the behavior of x_n as $n \to \infty$.

25. Determine whether the difference equation $x_n = x_{n-1} + x_{n-2}$ is stable.

26. Prove that if x is a solution of the difference equation $p(E)x = 0$, then so is Ex.

27. Consider the recurrence relation $x_n = 2(x_{n-1} + x_{n-2})$. Show that the general solution is $z_n = \alpha\left(1 + \sqrt{3}\right)^n + \beta\left(1 - \sqrt{3}\right)^n$. Show that the solution with starting values $x_1 = 1$ and $x_2 = 1 - \sqrt{3}$ corresponds to $\alpha = 0$ and $\beta = \left(1 - \sqrt{3}\right)^{-1}$.

COMPUTER PROBLEMS 1.3

1. Consider the difference equation $x_{n+2} - 2x_{n+1} - 2x_n = 0$. One of its solutions is $x_n = \left(1 - \sqrt{3}\right)^{n-1}$. This solution oscillates in sign and converges to 0. Compute and print out the first 100 terms of this sequence by use of the equation $x_{n+2} = 2(x_{n+1} + x_n)$ starting with $x_1 = 1$ and $x_2 = 1 - \sqrt{3}$. Explain the curious phenomenon that occurs.