Let $\alpha = (m+1)\varphi + \pi$. Then, using Re($x$) to signify the real part of x, we have

$$
\begin{aligned}
(m+1)I &= \sum_{k=0}^{n}\left[\cos(k+1)\alpha + \cos k\alpha\right] \\
&= \text{Re}\left\{\sum_{k=0}^{n}[e^{i\alpha(k+1)} + e^{i\alpha k}]\right\} \\
&= \text{Re}\left\{\frac{e^{i\alpha(n+2)} - e^{i\alpha} + e^{i\alpha n} - 1}{e^{i\alpha} - 1}\right\} \\
&= \frac{\text{Re}\left[(e^{-i\alpha} - 1)(e^{i\alpha(n+2)} - e^{i\alpha} + e^{i\alpha n} - 1)\right]}{|e^{i\alpha} - 1|^2}
\end{aligned}
$$

Here the numerator is

$$
\begin{aligned}
\text{Re}\left(e^{i\alpha n} - e^{i\alpha(n+2)}\right) &= \cos n\alpha - \cos(n+2)\alpha \\
&= \cos\left[(n+1)\alpha - \alpha\right] - \cos\left[(n+1)\alpha + \alpha\right] \\
&= \cos(k\pi - \alpha) - \cos(k\pi + \alpha) = 0
\end{aligned}
$$

where $k = m + n + 2$. To complete the proof, let p be a monic polynomial of degree n. It can be expressed as

$$p = 2^{-n}U_n + a_{n-1}U_{n-1} + \cdots + a_0U_0$$

Hence, by the orthogonality relation,

$$
\begin{aligned}
\int_{-1}^{1} |p|\,dx &\geq \int_{-1}^{1} p\,\text{sign}[U_n]\,dx = 2^{-n}\int_{-1}^{1} U_n\,\text{sign}[U_n]\,dx \\
&= 2^{-n}\int_{-1}^{1} |U_n|\,dx
\end{aligned}
$$

■

The subject of numerical integration can be studied further in Davis and Rabinowitz [1984], Krylov [1962], and Ghizetti and Ossiccini [1970].

PROBLEMS 7.2

1. Derive the Newton-Cotes formula for $\int_0^1 f(x)\,dx$ based on the nodes 0, $\frac{1}{3}$, $\frac{2}{3}$, and 1.
2. Prove (without using its error term) that Simpson's rule, Equation (6), correctly integrates all cubic polynomials.
3. Obtain Formula (6) from Formula (5) by a suitable change of variable.
4. Verify that the following formula is exact for polynomials of degree ≤ 4:

$$\int_0^1 f(x)\,dx \approx \frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right]$$

5. (Continuation) From the formula in the preceding problem, obtain a formula for $\int_a^b f(x)\,dx$ that is exact for all polynomials of degree 4.

6. (Continuation) Calculate $\ln 2$ approximately by applying the formula in the preceding problem to

$$\int_0^1 \frac{dt}{t+1}$$

Compare your answer to the correct value and compute the error.

7. Calculate $\int_0^1 e^{x^2}\,dx$ to eight-decimal-place accuracy by use of the series in the text.

8. Find the formula

$$\int_0^1 f(x)\,dx \approx A_0 f(0) + A_1 f(1)$$

that is exact for all functions of the form $f(x) = ae^x + b\cos(\pi x/2)$.

9. Find a formula of the form

$$\int_0^{2\pi} f(x)\,dx = A_1 f(0) + A_2 f(\pi)$$

that is exact for any function having the form

$$f(x) = a + b\cos x$$

Prove that the resulting formula is exact for any function of the form

$$f(x) = \sum_{k=0}^{n} [a_k \cos(2k+1)x + b_k \sin kx]$$

10. Use the Lagrange interpolation polynomial to derive the formula of the form

$$\int_0^1 f(x)\,dx \approx Af\left(\frac{1}{3}\right) + Bf\left(\frac{2}{3}\right)$$

Transform this formula to one for integration over $[a, b]$.

11. Using the polynomial of lowest order that interpolates $f(x)$ at x_1 and x_2, derive a numerical integration formula for

$$\int_{x_0}^{x_3} f(x)\,dx$$

Do not assume uniform spacing. Here $x_0 < x_1 < x_2 < x_3$.

12. Derive a formula for approximating

$$\int_1^3 f(x)\,dx$$

in terms of $f(0)$, $f(2)$, and $f(4)$. It should be exact for all f in Π_2.

13. Determine values for A, B, and C that make the formula

$$\int_0^2 xf(x)\,dx \approx Af(0) + Bf(1) + Cf(2)$$

exact for all polynomials of degree as high as possible. What is the maximum degree?

14. Derive the Newton-Cotes formula for

$$\int_0^1 f(x)\,dx$$

based on the Lagrange interpolation polynomial at the nodes -2, -1, and 0. Apply this result to evaluate the integral when $f(x) = \sin \pi x$.

15. Calculate

$$\int_0^{10^{-2}} \left(\frac{\sin x}{x}\right) dx$$

to seven-decimal-place accuracy using a series.

16. We intend to use $\int_0^1 p(x)\,dx$ as an estimate of $\int_0^1 f(x)\,dx$, where p is a polynomial of degree n that interpolates f at nodes $x_0, x_1, \ldots, x_n$ in $[0, 1]$. Assume that $|f^{(n+1)}(x)| < M$ on $[0, 1]$. What upper bound can be given for the error $|\int_0^1 f(x)\,dx - \int_0^1 p(x)\,dx|$ if nothing is known about the location of the nodes? Can you find the best upper bound?

17. Determine the composite numerical integration rule based on the simple **right-side rectangle rule**:

$$\int_0^1 f(x)\,dx \approx f(1)$$

Assume unequal spacing $a = x_0 < x_1 < \cdots < x_n = b$.

18. Derive the composite rule for $\int_a^b f(x)\,dx$ based on the midpoint rule

$$\int_{-1}^1 f(x)\,dx \approx 2f(0)$$

Give formulas for unequal spacing and equal spacing of nodes.

19. (Continuation) The **midpoint rule** over the interval $[x_{i+1}, x_{i-1}]$ is given by

$$\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx = (x_{i+1} - x_{i-1}) f(x_i)$$

Determine the composite midpoint rule over the interval $[a, b]$ with uniform spacing of $h = (b - a)/n$ such that $x_i = a + ih$ for $i = 0, 1, 2, \ldots, n$ (n is even).

20. Determine the integration rule for $\int_a^b f(x)\,dx$ based on the Gaussian quadrature rule

$$\int_{-1}^1 f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

21. There are two Newton-Cotes formulas for $n = 2$ and $[a, b] = [0, 1]$; namely,

$$\int_0^1 f(x)\,dx \approx af(0) + bf\left(\frac{1}{2}\right) + cf(1)$$

$$\int_0^1 f(x)\,dx \approx \alpha f\left(\frac{1}{4}\right) + \beta f\left(\frac{1}{2}\right) + \gamma f\left(\frac{3}{4}\right)$$

Which is better?

22. Is there a formula of the form

$$\int_0^1 f(x)\,dx \approx \alpha\left[f(x_0) + f(x_1)\right]$$

that correctly integrates all quadratic polynomials?

23. Prove that if the formula

$$\int_{-1}^1 f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i) \qquad (n \text{ is even})$$