6. We can determine a polynomial q of degree $n+1$ that is orthogonal to Π_n by writing

$$q(x) = x^{n+1} + c_1 x^n + \cdots + c_{n+1}$$

and imposing the conditions $\int_a^b q(x)x^k w(x)\,dx = 0$ for $0 \le k \le n$. The resulting system of $n+1$ equations in the $n+1$ unknowns $c_1, c_2, \ldots, c_{n+1}$ can then be solved. Carry out this process to obtain q_5 as needed in Problem **7.3.3** (p. 498). Do you think that this is a good way to obtain q?

7. **a.** Find a formula of the form

$$\int_0^1 x f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

with $n = 1$, that is exact for all polynomials of degree 3.

b. Repeat with $n = 2$, making the formula exact on Π_5.

8. **a.** Determine appropriate values of A_i and x_i so that the quadrature formula

$$\int_{-1}^1 x^2 f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

will be correct when f is any polynomial of degree 3. Use $n = 1$.

b. Repeat when f is any polynomial of degree 5, using $n = 2$.

9. Find a quadrature formula

$$\int_{-1}^1 f(x)\,dx \approx c \sum_{i=0}^{2} f(x_i)$$

that is exact for all quadratic polynomials.

10. **a.** If the integration formula

$$\int_{-1}^1 f(x)\,dx \approx f(\alpha) + f(-\alpha)$$

is to be exact for all quadratic polynomials, what value of α should be used? Answer the same question for all cubic polynomials.

Repeat Part **a** for polynomials of the forms:

b. $f(x) = a + bx + cx^3 + dx^4$

c. $f(x) = a + \sum_{i=1}^{n} b_i x^{2i-1} + cx^{2n}$

11. For what value of α is this formula exact on Π_3?

$$\int_0^2 f(x)\,dx \approx f(\alpha) + f(2-\alpha)$$

12. Prove that if the interval is symmetric with respect to the origin and if w is an even function, then the Gaussian nodes will be symmetric and $A_i = A_{n-i}$ for $0 \le i \le n$.

13. Prove that every quadrature formula of the type

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

is exact on some infinite-dimensional subspace of $C[a, b]$.

Proof We begin with the first column, which contains trapezoidal estimates of the integral, I. The trapezoid rule with k subintervals can be written in the form

$$h \sum_{i=0}^{k}{}'' f(a+ih) = \frac{1}{2} h \sum_{i=0}^{k-1} f(a+ih) + \frac{1}{2} h \sum_{i=1}^{k} f(a+ih)$$

The right side of this equation represents the average of two Riemann sums for I. Because $h = (b-a)/k$, the maximum width of subintervals converges to 0 as $k \to \infty$. Hence, by the theory of the Riemann integral, both Riemann sums converge to I. Of course, their average also converges to I. This proves that $\lim_{n\to\infty} R(n, 0) = I$. As for the second column, we note that

$$R(n, 1) = \frac{4}{3} R(n, 0) - \frac{1}{3} R(n-1, 0)$$

from which we get

$$\lim_{n\to\infty} R(n, 1) = \frac{4}{3} I - \frac{1}{3} I = I$$

All subsequent columns can be analyzed in the same way. ■

PROBLEMS 7.4

1. Derive Equation (7) starting with Equation (6).
2. Derive Equation (8) from Equation (7), and in particular, justify the conversion from

$$h^{2m+1} \sum_{i=0}^{2^n-1} f^{(2m)}(\xi_i) \qquad \text{to} \qquad (b-a)h^{2m} f^{(2m)}(\xi)$$

3. Establish the following equation in which $h = 1/2^n$:

$$I = \frac{4}{3} T\left(f, \frac{h}{2}\right) - \frac{1}{3} T(f, h) - \sum_{n=1}^{\infty} \frac{4^n - 1}{3(4^n)} c_{2n+2} h^{2n+2}$$

where

$$I = \int_0^1 f(x)\,dx \qquad \text{and} \qquad T(f, h) = h \sum_{i=0}^{2^n}{}'' f(ih)$$

4. Show that the second column in the Romberg array is the result of using Simpson's rule on f. (See Equation (6) in Section 7.2, p. 483.)
5. Prove by induction that

$$I - R(n, m-1) = ah^{2m} + bh^{2m+2} + ch^{2m+4} + \cdots$$

6. Apply the Romberg algorithm to find $R(2, 2)$ for these integrals:

   a. $\displaystyle\int_1^3 \frac{dx}{x}$

   b. $\displaystyle\int_0^{\pi/2} \left(\frac{x}{\pi}\right)^2 dx$ (in terms of π)

7. Suppose that $S(f, h)$ is a quadrature rule for the integral I in Equation (1) and that the error series is $c_4 h^4 + c_6 h^6 + \cdots$. Combine $S(f, h)$ with $S(f, h/3)$ to find a more accurate approximation to I.

8. In the Romberg algorithm, $R(n, 0)$ is an estimate of $\int_a^b f(x)\,dx$ using the trapezoid rule with 2^n subintervals. How many evaluations of $f(x)$ are needed to compute $R(i, j)$ for $0 \le i \le N$ and $0 \le j \le N$?

9. If the trapezoid rule satisfied the equation

$$\int_a^b f(x)\,dx = T(f, h) + c_1 h + c_2 h^2 + c_3 h^3 + \cdots$$

instead of Equation (9), then how would we have to modify Formula (5)?

10. In the Romberg algorithm, the elements in the second column satisfy

$$R(i, 1) = I + C_4 h_i^4 + C_6 h_i^6 + \cdots$$

where $I = \int_a^b f(x)\,dx$ and $h_i = (b - a)/2^i$. Derive the formula for computing elements in the third column and the first term in its error series.

11. **(Milne's rule)** Express $R(2, 2)$ in terms of elements in the first column of the Romberg array. Show that $R(3, 3)$ is *not* a Newton-Cotes formula but that $R(2, 2)$ is.

12. Show that Equation (3) follows immediately from the fact that

$$\sum_{0\le i\le 2n} f(a + ih) - \sum_{\substack{0\le i\le 2n\\ i \text{ even}}} f(a + ih) = \sum_{\substack{0\le i\le 2n\\ i \text{ odd}}} f(a + ih)$$

COMPUTER PROBLEM 7.4

1. Write a subprogram to carry out the Romberg algorithm for a function f defined on an arbitrary interval $[a, b]$. The user will specify the number of rows to be computed in the array and will want to see the entire array when it has been computed. Write a main program and test your Romberg subprogram on these three examples:

   a. $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$

   b. $\displaystyle\int_{-1}^1 \frac{\cos x - e^x}{\sin x}\,dx$

   c. $\displaystyle\int_1^\infty (xe^x)^{-1}\,dx$

   The routines for these integrals should be written to avoid serious loss of significance due to subtraction. Also, it is customary to define a function f at any questionable point x_0 by the equation $f(x_0) = \lim_{x\to x_0} f(x)$. If the limit exists, this method guarantees continuity of f at x_0. For the third example, make a suitable change of variable, such as $x = 1/t$. Compute seven rows in the Romberg array. Print the array in each case with a format that enables the convergence to be observed.

7.5 Adaptive Quadrature

Adaptive quadrature methods are intended to compute definite integrals by automatically taking into account the behavior of the integrand. Ideally, the user supplies only the integrand f, the interval $[a, b]$, and the accuracy ε desired for computing

Show that, for example,

$$P_4(t) = 12 - 48t^2 + 16t^4$$

$$P_5(t) = -120t + 160t^3 - 32t^5$$

2. Verify that the function $x(t) = t^2/4$ solves the initial-value problem

$$\begin{cases} x' = \sqrt{x} \\ x(0) = 0 \end{cases}$$

Apply the Taylor-series method of order 1, and explain why the numerical solution differs from the solution $t^2/4$.

3. Compute $x(0.1)$ by solving the differential equation

$$\begin{cases} x' = -tx^2 \\ x(0) = 2 \end{cases}$$

with one step of the Taylor-series method of order 2. (Use a calculator.)

4. Using the ordinary differential equation

$$\begin{cases} x' = x^2 + xe^t \\ x(0) = 1 \end{cases}$$

and one step of the Taylor-series method of order 3, calculate $x(0.01)$.

5. Consider the ordinary differential equation

$$\begin{cases} 5tx' + x^2 = 2 \\ x(4) = 1 \end{cases}$$

Calculate $x(4.1)$ using one step of the Taylor-series method of order 2.

6. An **integral equation** is an equation involving an unknown function within an integration. For example, here is a typical integral equation (of a type known by the name **Volterra**):

$$x(t) = \int_0^t \cos(s + x(s))\, ds + e^t$$

By differentiating this integral equation, obtain an equivalent initial-value problem for the unknown function.

7. If the Taylor-series method is used to solve an initial-value problem involving the differential equation

$$x' = \cos(tx)$$

what are the formulas for x'', x''', and $x^{(4)}$?

8. Let $x' = f(t, x)$. Determine x'', x''', and $x^{(4)}$ from this equation.

COMPUTER PROBLEMS 8.2

1. Write and test a computer program to solve the following differential equation with initial condition