PROBLEMS 3.4

1. If F is contractive from $[a, b]$ to $[a, b]$ and $x_{n+1} = F(x_n)$, with $x_0 \in [a, b]$, then $|x_n - s| \leq C\lambda^n$ for an appropriate C. Prove this and give an upper bound for C. Here s is the fixed point of F.

2. Prove that if $F\colon [a, b] \to \mathbb{R}$, if F' is continuous, and if $|F'(x)| < 1$ on $[a, b]$, then F is a contraction. Does F necessarily have a fixed point?

3. Prove that if F is a continuous map of $[a, b]$ into $[a, b]$, then F must have a fixed point. Then determine whether this assertion is true for functions from $\mathbb{R}$ to $\mathbb{R}$.

4. Show that these functions are contractive on the indicated intervals. Determine the best values of λ in Equation (2).

   a. $(1 + x^2)^{-1}$ on an arbitrary interval

   b. $\frac{1}{2}x$ on $1 \leq x \leq 5$

   c. $\tan^{-1} x$ on an arbitrary closed interval excluding 0

   d. $|x|^{\frac{3}{2}}$ on $|x| \leq \frac{1}{3}$

5. Kepler's equation in astronomy reads $x = y - \varepsilon \sin y$, with $0 < \varepsilon < 1$. Show that for each $x \in [0, \pi]$, there is a y satisfying the equation. Interpret this as a fixed-point problem.

6. Consider an iteration function of the form $F(x) = x + f(x)g(x)$, where $f(r) = 0$ and $f'(r) \neq 0$. Find the precise conditions on the function g so that the method of functional iteration will **converge cubically** to r if started near r.

7. If you enter a number into a handheld calculator and then repeatedly press the cosine button, what number will eventually appear? Provide a proof.

8. Prove that the sequence generated by the iteration $x_{n+1} = F(x_n)$ will converge if $|F'(x)| \leq \lambda < 1$ on the interval $[x_0 - \rho, x_0 + \rho]$, where $\rho = |F(x_0) - x_0|/(1 - \lambda)$.

9. What special properties must a function f have if Newton's method applied to f converges cubically to a zero of f?

10. If we attempt to find a fixed point of F by using Newton's method on the equation $F(x) - x = 0$, what iteration formula results?

11. If f' is continuous and positive on $[a, b]$, and if $f(a)f(b) < 0$, then f has *exactly* one zero in (a, b). Prove this, and show that with a suitable parameter λ, the zero can be obtained by applying the method of functional iteration, with $F(x) = x + \lambda f(x)$.

12. Let p be a positive number. What is the value of the following expression?

$$x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}$$

    Note that this can be interpreted as meaning $x = \lim_{n\to\infty} x_n$, where $x_1 = \sqrt{p}$, $x_2 = \sqrt{p + \sqrt{p}}$, and so forth. *Hint:* Observe that $x_{n-1} = \sqrt{p + x_n}$.

13. (Continuation) Let $p > 1$. What is the value of the following continued fraction?

$$x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \cdots}}}$$

    Use the ideas of the preceding problem to solve this one. Prove that the sequence of values converges by using the Contractive Mapping Theorem.

14. (Continuation) Solve for the roots of the quadratic equation $x^2+px+q=0$ by developing an iterative method.

15. Let F be a contractive mapping of an interval $[a, b]$ into itself, and let s be the fixed point of F. If $a \le x \le b$ and $|F(x) - x| < \varepsilon$, does it follow that $|x - s| < \varepsilon$? Prove that $|x - s| < \varepsilon(1 - \lambda)^{-1}$, where λ is the constant in Equation (2).

16. Prove the statement at the end of this section in the text concerning the order of convergence in the method of functional iteration.

17. Most iterative processes are not as simple as the one expressed by $x_{n+1} = F(x_n)$, with $F\colon \mathbb{R} \to \mathbb{R}$. For example, we might have a map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$. Show that the bisection method and the secant method are of this type. In each case, define F explicitly.

18. Prove that if F' is continuous and if $|F'(x)| < 1$ on the interval $[a, b]$, then F is contractive on $[a, b]$. Show that this is not necessarily true for an open interval.

19. If the method of functional iteration is applied to the function $F(x) = 2 + (x - 2)^4$, starting at $x = 2.5$, what order of convergence results? Find the range of starting values for which this functional iteration converges. Note that 2 is a fixed point.

9 **20.** Show that the following functions are contractive on the given domains, yet they have no fixed point on these domains. Why does this not contradict the Contractive Mapping Theorem?

a. $F(x) = 3 - x^2$ on $[-\frac{1}{4}, \frac{1}{4}]$

b. $F(x) = -x/2$ on $[-2, -1] \cup [1, 2]$

21. Prove that if f is continuous on $[a, b]$ and satisfies $a \le f(a)$ and $f(b) \le b$, then f has a fixed point in the interval $[a, b]$. Note that we do not assume $a \le f(x) \le b$ for all x in $[a, b]$.

22. What is the weakest condition that can be put on the interval $[c, d]$ so that each continuous map of $[a, b]$ into $[c, d]$ shall have a fixed point?

23. Find the order of convergence of these sequences.

a. $x_n = (1/n)^{\frac{1}{2}}$

b. $x_n = \sqrt[n]{n}$

c. $x_n = (1 + 1/n)^{\frac{1}{2}}$

d. $x_{n+1} = \tan^{-1} x_n$

24. To find a zero of the function f, we can look for a fixed point of the function $F(x) = x - f(x)/f'(x)$. To find a fixed point of F, we can solve $F(x) - x = 0$ by Newton's method. When this is done, what is the formula for generating the sequence x_n?

25. Prove that the function F defined by $F(x) = 4x(1 - x)$ maps the interval $[0, 1]$ into itself and is not a contraction. Prove that it has a fixed point. Why does this not contradict the Contractive Mapping Theorem?

26. If the method of functional iteration is used on $f(x) = \frac{1}{2}(1 + x^2)^{-1}$ starting at $x_0 = 7$, will the resulting sequence converge? If so, what is the limit? Establish your answers rigorously.

27. Prove or disprove: If $F\colon \mathbb{R} \to [a, b]$, and if F is contractive on $[a, b]$, then F has a unique fixed point, which can be obtained by the method of functional iteration, starting at any real value.

28. Give examples of functions that *do not* have fixed points but *do* have these characteristics:

a. $f\colon [0, 1] \to [0, 1]$

b. $f: (0, 1) \to (0, 1)$ and is continuous

c. $f: A \to A$ and is continuous, with $A = [0, 1] \bigcup [2, 3]$

d. $f: \mathbb{R} \to \mathbb{R}$ and is continuous

29. Prove that the function $f(x) = 2 + x - \tan^{-1} x$ has the property $|f'(x)| < 1$. Prove that f does not have a fixed point. Explain why this does not contradict the Contractive Mapping Theorem.

30. This problem concerns the function $F(x) = 10 - 2x$. Prove that F has a fixed point. Let x_0 be arbitrary, and define $x_{n+1} = F(x_n)$ for $n \geq 0$. Find a nonrecursive formula for x_n. Prove that the method of functional iteration does not produce a convergent sequence unless x_0 is given a particular value. What is this special value for x_0? Why does this not contradict the Contractive Mapping Theorem?

31. Let F be continuously differentiable in an open interval, and suppose that F has a fixed point s in this open interval. Prove that if $|F'(s)| < 1$, then the sequence defined by functional iteration will converge to s if started sufficiently close to s. *Hint:* Select λ so that $|F'(r)| < \lambda < 1$, and consider an interval centered at r in which $|F'(x)| < \lambda$.

32. Let $\frac{1}{2} \leq q \leq 1$, and define $F(x) = 2x - qx^2$. On what interval can it be guaranteed that the method of iteration using F will converge to a fixed point? (This problem is related to Problem **3.2.5**, p. 90.)

33. Write down two different fixed-point procedures for finding a zero of the function $f(x) = 2x^2 + 6e^{-x} - 4$.

34. On which of these intervals $[\frac{1}{2}, \infty]$, $[\frac{1}{8}, 1]$, $[\frac{1}{4}, 2]$, $[0, 1]$, $[\frac{1}{5}, \frac{3}{2}]$ is the function $f(x) = \sqrt{x}$ contractive?

35. A function F is called an **iterated contraction** if

$$|F(F(x)) - F(x)| \leq \lambda |F(x) - x| \qquad (\lambda < 1)$$

Show that every contraction is an iterated contraction. Show that an iterated contraction need not be a contraction nor continuous.

36. If the method of functional iteration is used on $F(x) = x^2 + x - 2$ and produces a convergent sequence of positive numbers, what is the limit of that sequence and what was the starting point?

37. Consider a function of the form $F(x) = x - f(x)f'(x)$, where $f(r) = 0$ and $f'(r) \neq 0$. Find the precise conditions on the function f so that the method of functional iteration will converge at least **cubically** to r if started near r.

38. Analyze Steffensen's method (Problem **3.2.4**, p. 90) as an example of functional iteration. Determine its order of convergence.

39. A student incorrectly recalls Newton's method and writes $x_{n+1} = f(x_n)/f'(x_n)$. Will this method find a zero of f? What is the order of convergence?

40. Show that the following method has third-order convergence for computing $\sqrt{R}$:

$$x_{n+1} = \frac{x_n(x_n^2 + 3R)}{3x_n^2 + R}$$

41. Consider an iterative method of the form $x_{n+1} = x_n - f(x_n)/g(x_n)$. Assume that it converges to a point ξ that is a simple zero of the function f but not a zero of the function g. Establish the relationship between f and g so that the order of convergence of the method is 3 or greater.