set of p, so named in honor of G. Julia, a French mathematician who published an important memoir on this subject in 1918. If all roots of p are simple, then the basins of attraction are open sets in the complex plane, and the Julia set is the boundary of each basin of attraction.

The basins of attraction for the polynomial $p(z) = z^5 + 1$ are shown in Figure 3.8. The five roots of p are:

$$\omega_k = \cos\left(\tfrac{2}{5}\pi k\right) + i \sin\left(\tfrac{2}{5}\pi k\right) \qquad k = 0, 1, 2, 3, 4$$

We took a square region in the complex plane and generated a large number of lattice points in that square. For each lattice point, we ran a rough test to determine to which basin of attraction it belonged. This test consisted of monitoring the first 20 Newton iterates and checking whether any of them was within distance 0.25 of a root. If so, then subsequent iteration would produce quadratic convergence to that root. This fact was established by use of the standard convergence theory of Newton's method in the complex plane. In this way, a list of lattice points belonging to each basin of attraction was generated. The basin of attraction for root ω_0 was assigned one color, and the basins for roots ω_1, ω_2, ω_3, and ω_4 were assigned four other colors. These five basins of attraction (actually just the lattice points in them) were displayed on the color screen of a workstation, and then a color plot was produced from it. The five color sets fit together in an incredible way that displays a **fractal** appearance. That is, on magnifying a portion of the plane where two sets meet, we see the same general patterns repeated. This persists on repeated magnification. Furthermore, each boundary point of any one of these five sets is also a boundary point of the other three sets!

Recently, a number of articles and books have been published on fractals and chaos. Additional information can be found, for example, in Barnsley [1988], Curry, Garnett, and Sullivan [1983], Dewdney [1988], Glieck [1987], Mandelbrot [1982], Peitgen and Richter [1986], Peitgen, Saupe, and Haeseler [1984], Pickover [1988], and Sander [1987].

PROBLEMS 3.5

9 1. Use Horner's algorithm to find $p(4)$, where

$$p(z) = 3z^5 - 7z^4 - 5z^3 + z^2 - 8z + 2$$

2. (Continuation) For the polynomial of preceding problem, find its expansion in a Taylor series about the point $z_0 = 4$.
3. (Continuation) For the polynomial of Problem **3.5.1** (above), start Newton's method at the point $z_0 = 4$. What is z_1?
4. 9 (Continuation) For the polynomial of Problem **3.5.1** (above), apply Bairstow's method with the initial point $(u, v) = (3, 1)$. Compute the corrections δu and δv.
5. (Continuation) For the polynomial of Problem **3.5.1** (above), find a disk centered at the origin that contains all the roots.
6. (Continuation) For the polynomial of Problem **3.5.1** (above), find a disk centered at the origin that contains none of the roots.
7. Does Theorem 3 sometimes give the radius of the smallest circle centered at the origin that contains all the roots of a given polynomial?

8. Prove that every polynomial having real coefficients can be factored into a product of linear and quadratic factors having real coefficients.

9. Verify the recurrence relations and starting values given for c_k and d_k in the discussion of Bairstow's method.

10. For the polynomial $p(z) = 9z^4 - 7z^3 + z^2 - 2z + 5$, compute $p(6)$, $p'(6)$, and the next point in the Newton iteration starting at $z = 6$.

11. Using the definition of multiplicity adopted in the text, prove that if z is a root of multiplicity m of a polynomial p, then $p(z) = p'(z) = \cdots = p^{(m-1)}(z) = 0$ and $p^{(m)}(z) \neq 0$.

12. (Continuation) Prove the converse of the assertion in the preceding problem.

13. Does Bairstow's method yield a quadratically convergent sequence (u_k, v_k)?

14. Write an algorithm that deflates $p(z)$ when a root z_0 is known, but computes the coefficients of the reduced polynomial in ascending order—that is, constant term first.

15. Derive Equation (4) in the discussion of Bairstow's method.

16. Concerning the polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$, prove the following result. For a given x, we set $(\alpha_n, \beta_n, \gamma_n) = (a_n, 0, 0)$ and define inductively

$$(\alpha_j, \beta_j, \gamma_j) = (a_j + x\alpha_{j+1}, \alpha_{j+1} + x\beta_{j+1}, \beta_{j+1} + x\gamma_{j+1})$$

for $j = n-1, n-2, \ldots, 0$. Then $p(x) = \alpha_0$, $p'(x) = \beta_0$, and $p''(x) = 2\gamma_0$.

17. In the analysis of Laguerre's method, prove that

$$C^2 + (n-1)D^2 = \sum_{j=1}^{n} u_j^2$$

18. In the description of Laguerre's method, the quantities A and B are functions of z and depend on the given polynomial p. Let r be a root of p. Show that the corresponding functions A and B for $p(z)/(z-r)$ are, respectively,

$$A + (z-r)^{-1} \quad \text{and} \quad B - (z-r)^2$$

19. Prove that if p is a polynomial of degree n having real coefficients, then

$$(n-1)\left[p'(x)\right]^2 \geq np(x)p''(x)$$

Assume that the roots are real.

COMPUTER PROBLEMS 3.5

1. Write a program that takes as input the coefficients of a polynomial p and a specific point z_0 and produces as output the values $p(z_0)$, $p'(z_0)$, and $p''(z_0)$. Write the pseudocode with only one loop. Test on the polynomial in Problem **3.5.1** (p. 128), taking $z_0 = 4$.

2. Write a complex Newton method for a polynomial having complex coefficients, with a given starting point in the complex plane and a given number of iterations. Test your program on the polynomial of Problem **3.5.1** (p. 128), starting with $z_0 = 3 - 2i$.

3. Experiment with Laguerre's method coded in complex arithmetic. Find all four roots of the polynomial used in the examples of this section.

4. Using Newton's method and the polynomial $p(z) = z^3 - 1$, find three nearby starting points (within 0.01 of each other) so that the resulting sequences converge to different roots. Using a plotter, show the paths of these sequences of points within a square containing the roots by connecting successive points with line segments.

PROBLEMS 3.6

1. Solve the system of equations

$$x - 2y + y^2 + y^3 - 4 = -x - y + 2y^2 - 1 = 0$$

by the homotopy method used in Example 2, starting with the point $(0, 0)$. (All the calculations can be performed without recourse to numerical methods.)

2. Consider the homotopy $h(t, x) = tf(x) + (1 - t)g(x)$, in which

$$f(x) = x^2 - 5x + 6 \qquad g(x) = x^2 - 1$$

Show that there is no path connecting a root of g to a root of f.

3. Let $y = y(s)$ be a differentiable function from $\mathbb{R}$ to $\mathbb{R}^n$ satisfying the differential equation (9). Assume that $h(y(0)) = 0$. Prove that $h(y(s)) = 0$.

4. If the homotopy method of Example 2 is to be used on the system

$$\sin x + \cos y + e^{xy} = \tan^{-1}(x + y) - xy = 0$$

starting at $(0, 0)$, what system of differential equations will govern the path? A computer program to seek the solution will be instructive.

5. Prove that homotopy is an equivalence relation among the continuous maps from one topological space to another.

6. Are the functions $f(x) = \sin x$ and $g(x) = \cos x$ homotopic?

7. Consider these maps of $[0, 1]$ into $[0, 1] \bigcup [2, 3]$:

$$f(x) = 0 \qquad g(x) = 2$$

Are they homotopic?