2. The polynomial p of degree $\leq n$ that interpolates a given function f at $n+1$ prescribed nodes is uniquely defined. Hence, there is a mapping $f \mapsto p$. Denote this mapping by L and show that

$$Lf = \sum_{i=0}^{n} f(x_i)\ell_i$$

Show that L is **linear**; that is, $L(af + bg) = aLf + bLg$.

3. (Continuation) Refer to the preceding problem and define another mapping, G, by the formula

$$Gf = \sum_{i=0}^{n} f(x_i)\ell_i^2$$

Prove that Gf is a polynomial of degree at most $2n$, that Gf interpolates f at the nodes, and that Gf is nonnegative whenever f is nonnegative.

4. (Continuation) Prove that the mapping, L, in Problem **6.1.2** (above) has the property that $Lq = q$ for every polynomial q of degree at most n.

5. (Continuation) Prove that $\sum_{i=0}^{n} \ell_i(x) = 1$ for all x.

6. (Continuation) Prove that if p is a polynomial of degree $\leq n$ that interpolates the function f at $x_0, x_1, \ldots, x_n$ (distinct points), then

$$f(x) - p(x) = \sum_{i=0}^{n} [f(x) - f(x_i)]\ell_i(x)$$

7. Prove that the algorithm for computing the coefficients c_i in the Newton form of the interpolating polynomial involves n^2 *long* operations (multiplications and divisions).

8. Discuss the problem of determining a polynomial of degree at most 2 for which $p(0)$, $p(1)$, and $p'(\xi)$ are prescribed, ξ being any preassigned point.

9. Prove that if g interpolates the function f at $x_0, x_1, \ldots, x_{n-1}$ and if h interpolates f at $x_1, x_2, \ldots, x_n$, then the function

$$g(x) + \frac{x_0 - x}{x_n - x_0}[g(x) - h(x)]$$

interpolates f at $x_0, x_1, \ldots, x_n$. Notice that h and g need not be polynomials.

10. Prove that the coefficient of x^n in the polynomial p of Equation (9) is

$$\sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)^{-1}$$

11. Prove that for any polynomial q of degree $\leq n-1$,

$$\sum_{i=0}^{n} q(x_i) \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)^{-1} = 0$$

12. Determine whether the algorithm

```
x ← a_n b_n
for i = 1 to n do
    x ← (x + a_{n-i}) b_i
end do
```

computes

$$x = \sum_{i=0}^{n} a_i \prod_{j=0}^{i} b_j$$

13. Prove that if we take *any* set of 23 nodes in the interval $[-1, 1]$ and interpolate the function $f(x) = \cosh x$ with a polynomial p of degree 22, then the relative error $|p(x) - f(x)|/|f(x)|$ is no greater than 5×10^{-16} on $[-1, 1]$.

14. Let p be a polynomial of degree $\leq n-1$ that interpolates the function $f(x) = \sinh x$ at any set of n nodes in the interval $[-1, 1]$, subject only to the condition that one of the nodes is 0. Prove that the error obeys this inequality on $[-1, 1]$:

$$|p(x) - f(x)| \leq \frac{2^n}{n!}|f(x)|$$

15. What is the final value of v in the algorithm shown?

```
v ← c_{i-1}
for j = i to n do
    v ← vx + c_j
end do
```

What is the number of additions and subtractions involved in this algorithm?

16. Write an efficient algorithm for evaluating

$$u = \sum_{i=1}^{n} \prod_{j=1}^{i} d_j$$

17. Suppose that p is a polynomial of degree greater than n that interpolates f at $n+1$ nodes. What can you discover about $f(x) - p(x)$?

18. Prove or disprove: If n is a divisor of m, then each zero of T_n is a zero of T_m.

19. Find a polynomial that assumes these values:

x	1	2	0	3
y	3	2	−4	5

20. Prove that for $x \geq 1$, $T_n(x) = \cosh(n \cosh^{-1} x)$. *Hint:* Imitate the proof of Theorem 3 on Chebyshev polynomials.

21. Write the Lagrange and Newton interpolating polynomials for these data:

x	2	0	3
$f(x)$	11	7	28

22. Find the Lagrange and Newton forms of the interpolating polynomial for these data:

x	−2	0	1
$f(x)$	0	1	−1

Write both polynomials in the form $a + bx + cx^2$ to verify that they are identical as functions.

23. Consider the data

x	$-\sqrt{\frac{3}{5}}$	0	$\sqrt{\frac{3}{5}}$
$f(x)$	$f\left(-\sqrt{\frac{3}{5}}\right)$	$f(0)$	$f\left(\sqrt{\frac{3}{5}}\right)$

What are the Newton interpolation polynomial and the Lagrange interpolation polynomial for these data?

24. The formula for the leading coefficient in T_n is 2^{n-1}. What is the formula for the coefficient of x^{n-2}? What about x^{n-1}?

25. Find the interpolating polynomial for the table

x	1	3	2	6
y	-2	-22	-1	-37

26. The equation $x - 9^{-x} = 0$ has a solution in $[0, 1]$. Find the interpolation polynomial on $x_0 = 0, x_1 = 0.5, x_2 = 1$ for the function on the left side of the equation. By setting the interpolation polynomial equal to 0 and solving the equation, find an approximate solution to the equation.

27. If we interpolate the function $f(x) = e^{x-1}$ with a polynomial p of degree 12 using 13 nodes in $[-1, 1]$, what is a good upper bound for $|f(x) - p(x)|$ on $[-1, 1]$?

28. Let p_k be the polynomial of degree $\leq k$ such that $p_k(x_i) = y_i$ for $0 \leq i \leq k$. Prove that $p_k = p_{k-1}$ if and only if $p_{k-1}(x_k) = y_k$.

29. Devise a method for solving an equation $f(x) = 0$ that gives the correct root in $n + 1$ steps if f^{-1} is a polynomial of degree n in a neighborhood of the root sought. Here f^{-1} is an inverse function: $f^{-1}(f(x)) = x$.

30. Prove the following: If g is a function (not necessarily a polynomial) that interpolates a function f at nodes $x_0, x_1, \ldots, x_{n-1}$, and if h is a function such that $h(x_i) = \delta_{in}$ $(0 \leq i \leq n)$, then for some c the function $g + ch$ interpolates f at $x_0, x_1, \ldots, x_n$.

31. Refer to the Lagrange interpolation process and define

$$w_i = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)^{-1}$$

Show that if x is not a node, then the interpolating polynomial can be evaluated by the formula

$$p(x) = \left[\sum_{i=0}^{n} y_i w_i (x - x_i)^{-1}\right] \bigg/ \left[\sum_{i=0}^{n} w_i (x - x_i)^{-1}\right]$$

This is called the **barycentric form** of the Lagrange interpolation process.

32. (Continuation) Show that the evaluation of p in the preceding problem is stable in the sense that if the w_i are incorrectly computed, we still have the interpolation property: $\lim_{x \to x_k} p(x) = y_k$ $(0 \leq k \leq n)$.

33. Let E be an $(n + 1)$-dimensional vector space of functions defined on a domain D. Let $x_0, x_1, \ldots, x_n$ be distinct points in D. Show that the interpolation problem

$$f(x_i) = y_i \qquad (0 \leq i \leq n) \qquad f \in E$$

has a unique solution for any choice of ordinates y_i if and only if no element of E other than 0 vanishes at all the points $x_0, x_1, \ldots, x_n$.

3. Let $f \in C^n[a, b]$. Prove that if $x_0 \in (a, b)$ and if $x_1, x_2, \ldots, x_n$ all converge to x_0, then $f[x_0, x_1, \ldots, x_n]$ will converge to $f^{(n)}(x_0)/n!$.

4. Prove that if f is a polynomial of degree k, then for $n > k$,

$$f[x_0, x_1, \ldots, x_n] = 0$$

5. Prove that if p is a polynomial of degree at most n, then

$$p(x) = \sum_{i=0}^{n} p[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

6. Show that the divided differences are linear maps on functions. That is, prove the equation

$$(\alpha f + \beta g)[x_0, x_1, \ldots, x_n] = \alpha f[x_0, x_1, \ldots, x_n] + \beta g[x_0, x_1, \ldots, x_n]$$

7. The divided difference $f[x_0, x_1]$ is analogous to a first derivative, as indicated in Theorem 4. Does it have a property analogous to $(fg)' = f'g + fg'$?

8. Using the functions ℓ_i defined in Section 6.1 (p. 312) and based on nodes $x_0, x_1, \ldots, x_n$, show that for any f,

$$\sum_{i=0}^{n} f(x_i)\ell_i(x) = \sum_{i=0}^{n} f[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

9. (Continuation) Prove this formula:

$$f[x_0, x_1, \ldots, x_n] = \sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)^{-1}$$

10. Compare the efficiency of the divided difference algorithm to the procedure described in Section 6.1 (p. 311) for computing the coefficients in a Newton interpolating polynomial.

11. Use Cramer's rule in matrix theory to prove that

$$f[x_0, x_1, \ldots, x_n] = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & f(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & f(x_1) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & f(x_n) \end{vmatrix} \div \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n} \end{vmatrix}$$

12. For the particular function $f(x) = x^m$, $m \in \mathbb{N}$, show that

$$f[x_0, x_1 \ldots, x_n] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n > m \end{cases}$$

13. Prove the **Leibniz formula**

$$(fg)[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k] g[x_k, x_{k+1}, \ldots, x_n]$$

Hint: See Problem **6.2.7** (above).

14. Write the equation in Problem **6.2.9** (above) in this form:

$$f[x_0, x_1, \ldots, x_n] = \sum_{i=0}^{n} \alpha_i f(x_i) \quad \text{where} \quad \alpha_i = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)^{-1}$$

Prove that if the x_i's are ordered thus:

$$x_0 < x_1 < x_2 < \cdots < x_n$$

then the α_i's alternate in sign.

15. (Continuation) Prove that

$$\sum_{i=0}^{n} \alpha_i x_i^n = 1 \quad \text{and} \quad \sum_{i=0}^{n} \alpha_i = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

16. Let $f(x) = 1/x$ and prove that

$$f[x_0, x_1, \ldots, x_n] = (-1)^n \prod_{i=0}^{n} x_i^{-1}$$

17. Find the Newton interpolating polynomial for these data:

x	1	3/2	0	2
$f(x)$	3	13/4	3	5/3

18. Prove that if f is a polynomial, then the divided difference $f[x_0, x_1, \ldots, x_n]$ is a polynomial in the variables $x_0, x_1, \ldots, x_n$.

19. Show that if u is any function that interpolates f at $x_0, x_1, \ldots, x_{n-1}$, and if v is a function that interpolates f at $x_1, x_2, \ldots, x_n$, then the function

$$[(x_n - x)u(x) + (x - x_0)v(x)]/(x_n - x_0)$$

interpolates f at $x_0, x_1, \ldots, x_n$.

20. (Continuation) Consider the array

$$\begin{array}{cc|ccc} x_0 & y_0 & a_0 & b_0 & c_0 \\ x_1 & y_1 & a_1 & b_1 & \\ x_2 & y_2 & a_2 & & \\ x_3 & y_3 & & & \end{array}$$

in which, for some fixed x, the a_i, b_i, and c_i are computed by the formulas

$$a_i = [(x_{i+1} - x)y_i + (x - x_i)y_{i+1}]/(x_{i+1} - x_i)$$

$$b_i = [(x_{i+2} - x)a_i + (x - x_i)a_{i+1}]/(x_{i+2} - x_i)$$

$$c_i = [(x_{i+3} - x)b_i + (x - x_i)b_{i+1}]/(x_{i+3} - x_i)$$

Show that c_0 is the value of the cubic interpolating polynomial at x.

21. (Continuation) Generalize the algorithm suggested in the preceding problem to compute $p_n(x)$ for any n. This is known as **Neville's algorithm**.

22. Determine the Newton interpolating polynomial for this table:

x	0	1	2	7
y	51	3	1	201

23. The polynomial $p(x) = 2 - (x + 1) + x(x + 1) - 2x(x + 1)(x - 1)$ interpolates the first four points in the table: