given by

$$p(x) = \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

The coefficient of x^m in $p(x)$ is $f^{(m)}(x_0)/m!$, and this establishes Equation (11) when $n = m$ and $x_m = x_0$. In the other case, the argument used to prove Theorem 1 in Section 6.2 (p. 330) is valid. ∎

PROBLEMS 6.3

1. Use the extended Newton divided difference method to obtain a quartic polynomial that takes these values:

x	0	1	2
$p(x)$	2	−4	44
$p'(x)$	−9	4	

2. (Continuation) Find a quintic polynomial that takes the values given in the preceding problem and, in addition, satisfies $p(3) = 2$. *Hint:* Add a suitable term to the polynomial found in the preceding problem.

3. Obtain a formula for the polynomial p of least degree that takes these values:

$$p(x_i) = y_i \qquad p'(x_i) = 0 \qquad (0 \leq i \leq n)$$

4. What condition will have to be placed on the nodes x_0 and x_1 if the interpolation problem

$$p(x_i) = c_{i0} \qquad p''(x_i) = c_{i2} \qquad (i = 0, 1)$$

is to be solvable by a cubic polynomial (for arbitrary c_{ij})?

5. Show that the polynomial given in Equation (6), with the divided differences in Equations (4) and (5), fulfills the conditions in Equation (3).

6. Verify the properties claimed for the functions A_i and B_i.

7. Prove that the Taylor polynomial

$$p(x) = \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(x_0)(x - x_0)^j$$

interpolates f at $x_0, x_0, \ldots, x_0$ (k repetitions).

8. Prove that a polynomial interpolates 0 at $x_0, x_1, \ldots, x_n$ (repetitions permitted) if and only if it contains the factor $\prod_{j=0}^{n}(x - x_j)$.

9. Prove that if f interpolates g at $x_0, x_1, \ldots, x_n$ and if h interpolates 0 at these points, then $f \pm ch$ interpolates g at these points.

10. Fix $x_0, x_1, \ldots, x_n$ and show that the set of functions that interpolate 0 at these points is an *algebra*; that is, it is closed under addition, multiplication, and multiplication by scalars.

11. Prove that if f interpolates 0 at the nodes $x_0, x_1, \ldots, x_n$, then it interpolates 0 at the nodes $x_0, x_1, \ldots, x_{n-1}$.

12. Let $x_0 < x_1 < \cdots < x_n$, and let f be continuously differentiable. Show that

$$\frac{\partial}{\partial x_i} f[x_0, x_1, \ldots, x_n] = f[x_0, x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n]$$

$$\int_a^b \left[f^{(m+1)}(x)\right]^2 dx = \int_a^b \left[S^{(m+1)}(x) + g^{(m+1)}(x)\right]^2 dx$$
$$= \int_a^b \left[S^{(m+1)}(x)\right]^2 dx + 2\int_a^b S^{(m+1)}(x) g^{(m+1)}(x) dx + \int_a^b \left[g^{(m+1)}(x)\right]^2 dx$$
$$= \int_a^b \left[S^{(m+1)}(x)\right]^2 dx + \int_a^b \left[g^{(m+1)}(x)\right]^2 dx$$
$$\geq \int_a^b \left[S^{(m+1)}(x)\right]^2 dx$$

■

PROBLEMS 6.4

1. Refer to the algorithm for determining the values of z_i, and prove that for all $i = 1, 2, \ldots, n-1$, we have $u_i z_i + h_i z_{i+1} - v_i = 0$.
2. (Continuation) Denote the left side of the equation in the preceding problem by E_i. Then $(h_i/u_i)E_i + E_{i+1} = 0$. Show that the latter equation can be reduced to Equation (10) by using the formulas in the algorithms. This will establish that the algorithm produces a solution to Equation (10).
3. Show that if we replace t_{i+1} by $t_i + h_i$ in Equation (6), the result is

$$S_i(x) = \frac{z_{i+1}}{6h_i}(x - t_i)^3 - \frac{z_i}{6h_i}(x - t_i - h_i)^3 + C(x - t_i) - D(x - t_i - h_i)$$

4. (Continuation) By expanding the term $(x - t_i - h_i)^3$ in the preceding equation and by using the correct values of C and D, show that Equation (11) in the text is correct.
5. Determine whether this is a quadratic spline function:

$$f(x) = \begin{cases} x & x \in (-\infty, 1] \\ -\frac{1}{2}(2 - x)^2 + \frac{3}{2} & x \in [1, 2] \\ \frac{3}{2} & x \in [2, \infty) \end{cases}$$

6. (Continuation) Is the function in the preceding problem a cubic spline function?
7. Determine all the values of a, b, c, d, e for which the following function is a cubic spline:

$$f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3 & x \in (-\infty, 1] \\ c(x-2)^2 & x \in [1, 3] \\ d(x-2)^2 + e(x-3)^3 & x \in [3, \infty) \end{cases}$$

Next, determine the values of the parameters so that the cubic spline interpolates this table:

x	0	1	4
y	26	7	25

8. Verify that Equations (7), (9), and (10) in the text are correct.
9. Using the development of the cubic splines as a model, derive the appropriate equations and algorithm to provide a quadratic spline interpolant to data (t_i, y_i) for $0 \leq i \leq n$, where $t_0 < t_1 < \cdots < t_n$. If Q is the spline interpolant, then the numbers $z_i = Q'(t_i)$ are well defined. Find the equations governing $z_0, z_1, \ldots, z_n$. You should discover that one of the z points can be arbitrary, say $z_0 = 0$.
10. Show that in the algorithm for solving the system (10), $u_i > h_i + h_{i-1}$ for all $i = 1, 2, \ldots, n-1$.

11. Determine the values of a, b, and c so that this is a cubic spline having knots 0, 1, and 2:

$$f(x) = \begin{cases} 3 + x - 9x^2 & x \in [0, 1] \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & x \in [1, 2] \end{cases}$$

Next, determine d so that $\int_0^2 [f''(x)]^2 dx$ is a minimum. Finally, find the value of d that makes $f''(2) = 0$ and explain why this value is different from the one previously determined.

12. Determine whether this is a cubic spline:

$$f(x) = \begin{cases} x^3 + x & x \le 0 \\ x^3 - x & x \ge 0 \end{cases}$$

Show that

$$\lim_{x \uparrow 0} f''(x) = \lim_{x \downarrow 0} f''(x)$$

13. Determine whether the natural cubic spline that interpolates the table

x	0	1	2	3
y	1	1	0	10

is or is not the function

$$f(x) = \begin{cases} 1 + x - x^3 & x \in [0, 1] \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 & x \in [1, 2] \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3 & x \in [2, 3] \end{cases}$$

14. Determine whether this function is a natural cubic spline:

$$f(x) = \begin{cases} 2(x+1) + (x+1)^3 & x \in [-1, 0] \\ 3 + 5x + 3x^2 & x \in [0, 1] \\ 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3 & x \in [1, 2] \end{cases}$$

15. A theorem in calculus asserts that if a function is differentiable at a point, then it is necessarily continuous at that point. The reason for this is elementary: If the limit defining $f'(x)$ exists,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

then the limit of the numerator alone must be 0, and this implies the continuity of f at x. Suppose now that for a certain function f and a certain point x_0, we have

$$\lim_{x \downarrow x_0} f'(x) = \lim_{x \uparrow x_0} f'(x)$$

Can we conclude that f' is continuous at x_0?

16. (Continuation) In checking to determine whether a piecewise cubic function f is a cubic spline, does it suffice to verify the equation

$$\lim_{x \downarrow t} f''(x) = \lim_{x \uparrow t} f''(t)$$

for each knot t?

17. Find the natural cubic spline function whose knots are -1, 0, and 1 and that takes the values $S(-1) = 13$, $S(0) = 7$, and $S(1) = 9$.

18. Which properties of a natural cubic spline does the following function possess, and which properties does it not possess?

$$f(x) = \begin{cases} (x+1) + (x+1)^3 & x \in [-1, 0] \\ 4 + (x-1) + (x-1)^3 & x \in (0, 1] \end{cases}$$

19. Find a natural cubic spline function whose knots are -1, 0, and 1 and that takes these values:

x	-1	0	1
y	5	7	9

20. Determine whether the coefficients a, b, c, and d exist so that the function

$$S(x) = \begin{cases} 1 - 2x & x \in (-\infty, -3] \\ a + bx + cx^2 + dx^3 & x \in [-3, 4] \\ 157 - 32x & x \in [4, +\infty) \end{cases}$$

is a natural cubic spline for the interval $[-3, 4]$.

21. Is the following function a natural cubic spline?

$$S(x) = \begin{cases} x^3 - 1 & x \in [-1, \frac{1}{2}] \\ 3x^3 - 1 & x \in [\frac{1}{2}, 1] \end{cases}$$

22. (Continuation) Repeat the preceding problem for the function

$$S(x) = \begin{cases} x^3 - 1 & x \in [-1, 0] \\ 3x^3 - 1 & x \in [0, 1] \end{cases}$$

23. Can a and b be defined so that the function

$$S(x) = \begin{cases} (x-2)^3 + a(x-1)^2 & x \in (-\infty, 2] \\ (x-2)^3 - (x-3)^2 & x \in [2, 3] \\ (x-3)^3 + b(x-2)^2 & x \in [3, +\infty) \end{cases}$$

is a natural cubic spline? Why or why not?

24. If S is a first-degree spline function that interpolates f at a sequence of knots $0 = t_0 < t_1 < \cdots < t_n = 1$, what is $\int_0^1 S(x)dx$?

25. What value of (a, b, c, d) makes this a cubic spline?

$$f(x) = \begin{cases} x^3 & x \in [-1, 0] \\ a + bx + cx^2 + dx^3 & x \in [0, 1] \end{cases}$$

26. Determine the value of (a, b, c) that makes the function

$$f(x) = \begin{cases} x^3 & x \in [0, 1] \\ \frac{1}{2}(x-1)^3 + a(x-1)^2 + b(x-1) + c & x \in [1, 3] \end{cases}$$

a cubic spline. Is it a natural cubic spline?

27. Let $t_0 < t_1 < \cdots < t_n$ and $-\infty < x < \infty$. What is the output value of k from the following algorithm?

```
for i = 1 to n do
    if x < t_i then
        k ← i
```

```
        exit loop
    end if
end do
```

28. How many conditions are needed to define a quadratic spline interpolation function $S(x)$ over $a = t_0 < t_1 < \cdots < t_{20} = b$? Does the continuity of $S'(x)$ provide all the needed conditions?

29. Prove Theorem 1 when the end conditions on S are changed to $S'(a) = f'(a)$ and $S'(b) = f'(b)$.

30. Develop a suitable procedure for finding the cubic spline interpolant when the end conditions specify values of $S'(t_0)$ and $S'(t_n)$.

31. Give the simplified version of natural cubic spline interpolation that pertains to the case of equally spaced knots.

32. Show that a spline function of degree 1 having knots $t_0 < t_1 < \cdots < t_n$ can be expressed in the form

$$S(x) = ax + b + \sum_{i=1}^{n-1} c_i |x - t_i|$$

33. Show that the function in Equation (12) solves the two-point boundary-value problem preceding Equation (12).

34. Show that Equation (13) follows from the global smoothness conditions, as asserted in the text.

35. Prove that the coefficient matrix in Equation (13) is diagonally dominant.

36. Using the notation and hypotheses of Theorem 4, prove that the inequality

$$\|f^{(m+1)}\| \geq \|f^{(m+1)} - S^{(m+1)}\|$$

is valid.

COMPUTER PROBLEMS 6.4

1. Draw a curve, such as an oval or spiral, on a sheet of graph paper. Select points in a more or less regular distribution along the curve and label them $t_0 = 1.0$, $t_1 = 2.0$, and so on. Read the x- and y-coordinates at each selected point to obtain a table of $x(t)$ and $y(t)$. Fit these functions by spline functions S and S^*. Then the formulas $x = S(t)$ and $y = S^*(t)$ give an approximate parametric representation of the curve. Plot the resulting curves for several test cases using an automatic plotter.

2. In the development of Problem **6.4.9** (p. 361), find a way to solve for the array $(z_0, z_1, \ldots, z_n)$ so that $\sum_{i=0}^{n} z_i^2$ is a minimum. That is, use this condition to remove the arbitrariness referred to in the problem. Incorporate this feature in your algorithm and test it on a computer.

3. Prove this formula:

$$\int_{t_i}^{t_{i+1}} S_i(x)dx = \frac{h_i}{2}(y_i + y_{i+1}) - \frac{h_i^3}{24}(z_i + z_{i+1})$$

Then write and test a program to compute

$$\int_{t_0}^{t_n} S(x)dx$$