6. Verify Formula (16) for derivatives of splines.

7. Prove that

$$\int_{-\infty}^{\infty} B_i^k(x)dx = \frac{t_{i+k+1} - t_i}{k+1}$$

8. Prove that if $\sum_{i=-\infty}^{\infty} c_i B_i^k(x) = 0$ for all x, then $c_i = 0$ for all i.

9. Define functions of s by the formula

$$U_i^k(s) = (t_{i+1} - s)(t_{i+2} - s) \cdots (t_{i+k} - s)$$

For $k = 0$, let $U_i^0(s) = 1$. Prove that

$$U_i^k(s)V_i^k(x) + U_{i-1}^k(s)\left[1 - V_i^k(x)\right] = (x - s)U_i^{k-1}(s)$$

10. (Continuation) Prove that

$$\sum_{i=-\infty}^{\infty} U_i^k(s)B_i^k(x) = (x - s)\sum_{i=-\infty}^{\infty} U_i^{k-1}(s)B_i^{k-1}(x)$$

11. (Continuation) Prove **Marsden's Identity**:

$$\sum_{i=-\infty}^{\infty} U_i^k(s)B_i^k(x) = (x - s)^k$$

12. (Continuation) Prove that every polynomial of degree $\leq k$ is expressible in the form $\sum_{i=-\infty}^{\infty} c_i B_i^k$.

13. Using the notation in the text, prove that B_i^2 is given by the formulas

$$B_i^2(x) = \begin{cases} V_i^2 V_i^1 & x \in [t_i, t_{i+1}) \\ V_i^k - V_{i+1}^1(V_i^2 - 1 + V_{i+1}^2) & x \in [t_{i+1}, t_{i+2}) \\ (1 - V_{i+1}^2)(1 - V_{i+2}^1) & x \in [t_{i+2}, t_{i+3}) \\ 0 & \text{elsewhere} \end{cases}$$

14. Verify these two equations:

$$\frac{d}{dx}B_i^1 = \frac{B_i^0}{t_{i+1} - t_i} - \frac{B_{i+1}^0}{t_{i+2} - t_{i+1}}$$

$$\frac{d}{dx}B_i^2 = \frac{2B_i^1}{t_{i+2} - t_i} - \frac{2B_{i+1}^1}{t_{i+3} - t_{i+1}}$$

15. Prove that

$$\sup_{-\infty<x<\infty} \left| \sum_{i=-\infty}^{\infty} c_i B_i^k(x) \right| \leq \sup_{-\infty<i<\infty} |c_i|$$

16. Prove that if $\sup_i |t_{i+1} - t_i| \leq m$, then

$$\int_{-\infty}^{\infty} \left| \sum_{i=-\infty}^{\infty} c_i B_i^k(x) \right| dx \leq m \sum_{i=-\infty}^{\infty} |c_i|$$

17. Find an upper bound for

$$\int_{-\infty}^{\infty} \left[\sum_{i=-\infty}^{\infty} c_i B_i^k(x) \right]^2 dx$$

Another similar example is

$$\int_0^x e^{t^2}\,dt = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)k!}(3) \qquad (3)$$

PROBLEMS 6.7

1. Show that $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} a_k (x-c)^k$ have the same radius of convergence. For p as a natural number, what about $\sum_{k=0}^{\infty} a_k x^{k+p}$?
2. **(Ratio test)** If $\lim_{n\to\infty} |A_{n+1}/A_n| < 1$, then $\sum_{k=0}^{\infty} A_k$ converges. Use the test to show that the cosine series in Equation (1) converges for all real x.
3. **(Ratio test**, continuation) If $\lim_{n\to\infty} |A_{n+1}/A_n| > 1$, then $\sum_{k=0}^{\infty} A_k$ diverges. Use this fact together with the preceding problem to find the radius of convergence of

$$\sum_{k=0}^{\infty} (-1)^k (x-1)^k$$

4. Find the radius of convergence of

$$\sum_{k=0}^{\infty} k! x^k$$

5. If $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ and if the radius of convergence is r, then f possesses derivatives of all orders in the interval $|x-c| < r$. Furthermore,

$$f^{(n)}(x) = \sum_{k=n}^{\infty} \frac{a_k k!}{(k-n)!}(x-c)^{k-n} \qquad (|x-c| < r)$$

Use Theorem 2 on radius of convergence to prove this.

6. Find a power series for

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

Note: This function is known as the **error function** and is denoted by erf(x). It plays a role in statistics.

7. Find a power series for the function

$$f(x) = \int_0^x \frac{e^t - 1}{t}\,dt$$

Use the series you have obtained to compute $f(1)$ with three significant figures. (In this example, that means two decimal places.) Finally, prove that the sum of all the terms you did not include is so small that it will not affect the answer. *Cultural note:* The function f is an important one in applied mathematics. For further information, consult Abramowitz and Stegun [1964, chap. 5].

8. Show that the function $f(x) = \sum_{k=0}^{\infty} x^{2k}/(k!2^k)$ solves the differential equation $y' = xy$.
9. Let $a_0 = 1$ and $a_n = a_{n-1}/[2(n+1)]$ for $n \geq 1$. What is the radius of convergence of $\sum_{k=0}^{\infty} a_k x^k$?
10. (Continuation) Let $f(x)$ be the function whose series is in the preceding problem. What is the power series for $f'(x)$? A recurrence relation for the coefficients suffices.

11. Using the ratio test (see Problems **6.7.2–3**, p. 390), prove that if $\lim_{n\to\infty} |a_n/a_{n+1}|$ exists or is $+\infty$, then it is the radius of convergence of $\sum_{k=0}^{\infty} a_k(x-c)^k$.

12. Let $a_0 = 1$ and $a_{n+1} = [2 + (-1)^n]a_n$ for $n \geq 1$. What is the radius of convergence of $\sum_{k=0}^{\infty} a_k(x-c)^k$?

13. If r and r' are the radii of convergence of $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$, respectively, what is the radius of convergence of $\sum_{k=0}^{\infty}(a_k + b_k)x^k$?

14. A function related to the **dilogarithm** is defined by

$$f(x) = -\int_0^x \frac{\ln(1-t)}{t}\,dt \qquad (-\infty < x \leq 1)$$

Find the Maclaurin series for f and determine its radius of convergence. How would you compute $f(-2)$? What about $f(0.001)$?

15. Obtain a series for $\ln x$ by integrating the series in Equation (2).

16. Show how the series for $\sin x$ can be obtained in two ways by differentiating or by integrating the series in Equation (1).

17. Obtain a power series for $\tan^{-1} x$ as follows: Start with $(1+x)^{-1} = \sum_{k=0}^{\infty}(-x)^k$, replace x by x^2, and then integrate the terms in the resulting equation. Compare this method to the alternative procedure of computing the successive derivatives of $\tan^{-1} x$ and obtaining the Taylor series.

18. Criticize the following analysis and correct it:

$$\int_0^x \frac{1-\cos t}{t}\,dt = \int_0^x \left[\frac{1}{t} - \frac{\cos t}{t}\right] dt = \int_0^x \left[\frac{1}{t} - \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k-1}}{(2k)!}\right] dt$$

$$= \ln x - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k(2k)!}$$

Hint: There are several errors.

19. Find the Maclaurin series for the function

$$f(x) = \int_0^x \frac{1-\cos t}{t}\,dt$$

Cultural note: This function is sometimes called the **cosine integral** and is denoted by $\mathrm{Cin}(x)$. The nomenclature has not been standardized.

20. Find the Maclaurin series for the **Fresnel integral**:

$$\varphi(x) = \int_0^x \sin t^2\,dt$$

21. **(Reciprocal function)** Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$, with $a_0 = 1$. Then, in some neighborhood of 0, $1/f(x)$ is well defined. Assume a series $\sum_{k=0}^{\infty} b_k x^k$ for the reciprocal function, and determine recursively the b_k's from the fact that the product of the two series is 1.

22. (Continuation) Use the result of the preceding problem to find the Maclaurin series for $x/(e^x - 1)$.