PROBLEMS 6.8

1. Find the best approximation to $\sin x$ by a function $u(x) = \lambda x$ on the interval $[0, \pi/2]$ using the supremum norm. *Hint:* Draw a sketch. The function $\sin t - \lambda t$ should have a maximum at a point ξ in $(0, \pi/2)$, and it should have a minimum of the same magnitude at $\pi/2$. These conditions should determine ξ and λ.
2. (Continuation) Solve the preceding problem using the usual quadratic norm.
3. Suppose that we wish to approximate an even function by a polynomial of degree $\le n$ using the norm $\|f\| = \left\{\int_{-1}^{1} |f(x)|^2 dx\right\}^{1/2}$. Prove that the best approximation is also even. Generalize.
4. Let $p_0, p_1, p_2, \ldots$ be a sequence of polynomials such that (for each n) p_n has exact degree n. Show that the sequence is linearly independent.
5. Prove the **Parseval identity**:
$$\langle f, g\rangle = \sum_{i=1}^{n} \langle f, u_i\rangle \langle g, u_i\rangle$$
which is valid if f and g are in the span of the orthonormal set $[u_1, u_2, \ldots, u_n]$.
6. Show that the notorious **Hilbert matrix**, with elements
$$a_{ij} = (1 + i + j)^{-1} \qquad (0 \le i, j \le n)$$
is a Gram matrix for the functions $1, x, x^2, \ldots, x^{n-1}$.
7. In a vector space with basis $\{v_1, v_2, \ldots, v_n\}$, any other basis is obtained by a linear transformation
$$u_j = \sum_{i=1}^{n} a_{ij} v_i \qquad (1 \le j \le n)$$
in which the coefficient matrix is nonsingular. Show that the matrix that arises in this way from the Gram-Schmidt process is upper triangular.
8. In the three-term recurrence relation for the orthogonal polynomials, assume that the inner product is $\langle f, g\rangle = \int_{-a}^{a} f(x)g(x)w(x)dx$, where w is an even function. Prove that $a_n = 0$ for all n. Prove that p_n is even if n is even and that p_n is odd if n is odd.
9. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of vectors in an inner-product space. What choice of coefficients produces a minimum value in $\|f - \sum_{i=1}^{n} c_i v_i\|$? Don't overlook the possibility that some of the v's may be 0.
10. In the algorithm for computing a linear combination of orthogonal polynomials, show that at most $2n - 1$ multiplications are required. In the case of the Chebyshev polynomials, n multiplications suffice.
11. In the three-term recurrence formula for orthogonal polynomials, prove that b_n is positive by establishing that $b_n = \|p_{n-1}\|^2/\|p_{n-2}\|^2$.
12. Prove that for the Legendre polynomials, the coefficients in the three-term recurrence are $a_n = 0$ and $b_n = (n-1)^2/[(2n-1)(2n-3)]$.
13. How would the three-term recurrence formula for orthogonal polynomials have to be changed if we desired to produce an *orthonormal* system?
14. Using the inner product $\langle u, v\rangle = \int_{-1}^{1} u(x)v(x)dx$, let the Gram-Schmidt process be applied to the sequence of functions $x \mapsto (x^2 - 1)x^k$, $k = 0, 1, 2, \ldots$. Prove that if the resulting orthonormal sequence is renormalized to form a sequence of *monic* polynomials, then the latter satisfy a three-term recurrence relation of the form
$$q_{n+1}(x) = xq_n(x) - b_n q_{n-1}(x)$$
Give a formula for b_n. Determine the first three q-polynomials.

15. Prove that an orthogonal set of nonzero elements is necessarily linearly independent.

16. Let A be a linear transformation on an inner-product space. Assume that A is **self-adjoint**, which means that $\langle Af, g\rangle = \langle f, Ag\rangle$ for all f and g. Prove that the solutions of the equation $Af = \lambda f$, corresponding to different values of λ, are mutually orthogonal.

17. Let $[u_1, u_2, \dots]$ be an orthonormal sequence in an inner-product space. Prove that for any f in the space, the Fourier coefficients $\langle f, u_n\rangle$ are **square summable**:

$$\sum_{n=1}^{\infty} \langle f, u_n\rangle^2 < \infty.$$

18. Prove Theorem 7. *Suggestion:* Ordering the properties **1, 4, 3, 2, 5** is quite efficient.

19. Let $\widetilde{T}_n$ be the monic multiple of T_n. Find the three-term recurrence relation satisfied by $\widetilde{T}_0, \widetilde{T}_1, \dots$.

20. Find a formula for dist(f, G), where G is the subspace spanned by an orthonormal set $[g_1, g_2, \dots, g_n]$.

21. Derive these Legendre polynomials:

$$p_3(x) = x^3 - \tfrac{3}{5}x$$

$$p_4(x) = x^4 - \tfrac{6}{7}x^2 + \tfrac{3}{35}$$

$$p_5(x) = x^5 - \tfrac{10}{9}x^3 + \tfrac{5}{21}x$$

22. Using Theorem 5 directly, find p_0, p_1, p_2, p_3 for $[a, b] = [0, 1]$ and $w(x) = 1$.

23. (Continuation) Determine p_3 in the form $p_3 = x^3 + Bx^2 + Cx + D$ by making p_3 orthogonal to $\prod_2$. Verify your results by the use of the preceding problem.

24. Devise an algorithm for computing $\sum_{i=0}^{n} c_i p_i$ that computes, in order, each partial sum, $\sum_{i=0}^{k} c_i p_i$, for $k = 0, 1, 2, \dots, n$. Assume that the coefficients a_k and b_k in Theorem 5 are known.

6.9 Best Approximation: Chebyshev Theory

In this section, we work with the space $C(X)$ of all continuous real-valued functions defined on a given topological space X. We assume that X is a compact Hausdorff space. The reader who wishes to avoid considerations of general topology may take X to be a closed and bounded set in the real space $\mathbb{R}^k$, for example, an interval $[a, b]$ in $\mathbb{R}$.

The space $C(X)$ becomes a normed space (indeed, a *Banach* space) if we define the norm to be

$$\|f\| = \max_{x \in X} |f(x)|$$

This norm is used throughout this section.

An important problem of best approximation in the space $C(X)$ is as follows: An element f is given in $C(X)$, and a finite-dimensional subspace G is given in $C(X)$. We want to approximate f as well as possible by an element of G. Hence (as in the preceding section) we define

$$\text{dist}(f, G) = \inf_{g \in G} \|f - g\|$$

In Corollary 1, the choice of words suggests that the exponential polynomial described there is unique. To verify this, suppose that $\sum_{k=0}^{N-1} a_k E_k$ is an exponential polynomial that interpolates f at $x_0, x_1, \ldots, x_{N-1}$ (where $x_j = 2\pi j/N$). Then

$$\sum_{k=0}^{N-1} a_k E_k(x_j) = f(x_j) \qquad (0 \le j \le N-1)$$

If we multiply both sides of this equation by $E_n(-x_j)$ and sum with respect to j, the result is

$$\sum_{k=0}^{N-1} a_k \sum_{j=0}^{N-1} E_k(x_j)E_n(-x_j) = \sum_{j=0}^{N-1} f(x_j)E_n(-x_j)$$

By Equation (12), this implies that

$$\sum_{k=0}^{N-1} a_k \langle E_k, E_n\rangle_N = \langle f, E_n\rangle_N$$

Since $\langle E_k, E_n\rangle_N = \delta_{kn}$, we conclude that

$$a_n = \langle f, E_n\rangle_N = c_n$$

PROBLEMS 6.12

1. Using the notation of Theorem 3 and Corollary 1, show that if an exponential polynomial $g(x) = \sum_{k=0}^{N-1} a_k E_k(x)$ assumes the value 0 at each node x_j, then the coefficients a_k are all 0.
2. (Continuation) Use the result of the preceding problem to give another proof that the interpolating function in Corollary 1 is unique.
3. Prove that $E_k E_n = E_{k+n}$ and that $\overline{E}_k = E_{-k}$.
4. Prove that if f and g are functions such that

$$f(x_j) = \langle g, E_j\rangle_n \qquad (x_j = 2\pi j/n)$$

then $g(x_j) = n\langle f, E_j\rangle_n$.

5. By taking real and imaginary parts in a suitable exponential equation, prove that

$$\frac{1}{n}\sum_{j=0}^{n-1} \cos\frac{2\pi jk}{n} = \begin{cases} 1 & \text{if } k \text{ divides } n \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{n}\sum_{j=0}^{n-1} \sin\frac{2\pi jk}{n} = 0$$

6. Show that the inner product defined in Equation (12) satisfies the three Properties **1**, **2**, and **3** following that equation. Why is $\|\cdot\|_N$ not a norm?