1.2.

8. ① $e^h = 1 + h + \frac{h^2}{2!} + \cdots = 1 + O(h) = 1 + o(1)$

② $(1-h^4)^{-1} = 1 + h^4 + h^8 + \cdots = 1 + O(h^4) = 1 + o(h^3)$

③ $\cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \cdots = 1 + O(h^2) = 1 + o(h)$

④ $1 + \sin(h^3) = 1 + h^3 - \frac{h^9}{3!} + \cdots = 1 + O(h^3) = 1 + o(h^2)$

9. $\frac{1+h-e^h}{h^2} = -\frac{1}{2} - \frac{h}{6} - \frac{h^2}{24} \cdots \to -\frac{1}{2}$ as $h \to 0$

$= -\frac{1}{2} + O(h) = -\frac{1}{2} + o(h^0)$

Hence $\alpha = 1$ and $\beta = 0$ are the best values.

22. Let $x_n = \frac{1+a\theta^n}{1+a\theta^{n-1}}$. $x_n \to 1$ as $n \to \infty$

Then $\left|\frac{x_{n+1}-1}{x_n-1}\right| = \left|\theta \frac{1+a\theta^{n-1}}{1+a\theta^n}\right| \to \theta$ as $n \to \infty$

Hence we can choose N such that

if $n \geq N$, $\left|\theta \frac{1+a\theta^{n-1}}{1+a\theta^n}\right| < \frac{\theta}{2} + \frac{1}{2}$.

(since $\theta = \frac{\theta}{2} + \frac{\theta}{2} < \frac{\theta}{2} + \frac{1}{2}$)

Let $c = \frac{\theta}{2} + \frac{1}{2} < 1$.

Hene $\frac{1+a\theta^n}{1+a\theta^{n-1}}$ converges to 1 linearly

1.3.

10. By the definition of E,

$x = [x_1, x_2, \cdots]$ where $x_i \in \mathbb{R}$ if $1 \le i \le r$

$x_i = 0$ if $i \geq r+1$

11. (a) characteristic equation : $x^3 - 3x^2 + 4 = 0$

Hence $x = -1$, 2 (double)

Basis : $[-1, 1, -1, 1, \cdots, (-1)^n, \cdots]$

$[2, 4, 8, 16, \cdots, 2^n, \cdots]$

$[1, 4, 12, 32, \cdots, n2^{n-1}, \cdots]$

(b) characteristic equation : $2x^6 - 9x^5 + 12x^4 - 4x^3 = 0$

Hence $x = 0$ (triple), $\frac{1}{2}$ (simple), 2 (double)

Basis : $[1, 0, 0, \cdots]$

$[0, 1, 0, \cdots]$

$[0, 0, 1, 0, 0 \cdots]$

$[\frac{1}{2}, \frac{1}{4}, \cdots, (\frac{1}{2})^n, \cdots]$

$[2, 4, \cdots, 2^n, \cdots]$

$[1, 4, 12, \cdots, n2^{n-1}, \cdots]$

13.

(a) $x_{n+1} = n x_n$

$$x_2 = 1 \cdot x_1$$

$$x_3 = 2 \cdot x_2$$

$$\vdots$$

$$\times)\ \ x_{n+1} = n \cdot x_n$$

$$x_2 x_3 \cdots x_{n+1} = 1 \cdot 2 \cdots n \cdot x_1 \cdot x_2 x_3 \cdots x_n$$

$$x_{n+1} = n!\, x_1.$$

(b) $x_{n+1} - x_n = n.$

$$x_2 - x_1 = 1$$

$$x_3 - x_2 = 2$$

$$\vdots$$

$$+)\ \ x_{n+1} - x_n = n$$

$$x_{n+1} - x_1 = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\therefore\ x_{n+1} = x_1 + \frac{n(n+1)}{2}$$

(c) $x_{n+1} - x_n = 2.$

$$x_2 - x_1 = 2$$

$$x_3 - x_2 = 2$$

$$\vdots$$

$$+)\ \ x_{n+1} - x_n = 2$$

$$x_{n+1} - x_1 = 2n$$

$$\therefore\ x_{n+1} = x_1 + 2n.$$

14. $x = [x_1, x_2, \cdots]$

$$(I+\Delta)(x) = I(x) + \Delta(x)$$

$$= [x_1, x_2, \cdots] + [x_2 - x_1, x_3 - x_2, \cdots]$$

$$= [x_2, x_3, \cdots] = Ex$$

$$\therefore \quad I + \Delta = E.$$

Let $P(x)$ be a polynomial of degree m,

By the Taylor's Thm, we can write

$$p(x) = \sum_{i=0}^{m} \frac{p^{(i)}(a)}{i!} (x-a)^i$$

Plugging in $x = E$ and $a = I$, we get

$$P(E) = \sum_{i=0}^{m} \frac{p^{(i)}(I)}{i!} (E - I)^i$$

$$= \sum_{i=0}^{m} \frac{p^{(i)}(I)}{i!} \Delta^i$$

27. characterist equation : $x^2 - 2x + 2 = 0$

Roots $x = \dfrac{1 \pm \sqrt{3}}{2}$

$$\therefore \quad x_n = \alpha (1+\sqrt{3})^n + \beta (1-\sqrt{3})^n.$$

$$\left.\begin{array}{l} \alpha(1+\sqrt{3}) + \beta(1-\sqrt{3}) = 1 \\ \alpha(1+\sqrt{3})^2 + \beta(1-\sqrt{3})^2 = 1-\sqrt{3} \end{array}\right\} \Rightarrow \begin{array}{l} \alpha = 0 \\ \beta = 1. \end{array}$$