3.4

6. We need $0 = F(r) = F'(r) = F''(r)$ and $0 \neq F'''(r)$. $F'(x) = 1 + f'(x)g'(x) + f(x)g'(x)$, $F" = f''g + 2f'g' + fg''$ and $F''' = f'''g + 3f''g' + 3f'g'' + g'''$. So we need following conditions;
$g(r) = -\frac{1}{f'(r)}$, $g'(r) = \frac{f''(r)}{2(f'(r))^2}$ and $g''(r) \neq \frac{1}{3f'(r)}\left(\frac{f'''(r)}{f'(r)} - \frac{3(f''(r))^2}{2(f'(r))^2}\right)$.

7. We can start any point $a \in \mathbb{R}$. Then $|\cos a| \leq 1$. Hence from the second step, we can define $F(x) = \cos x$ on $[-1, 1]$. $\forall\ x \geq y$ on this domain, $|\cos x - \cos y| \leq \sin\xi|x - y|$, by Mean Value Thm for $\xi \in (y, x)$. Since $\lambda = \sin\xi < 1$, $|\cos x - \cos y| \leq \lambda|x - y|$. By Theorem 1, $F(x)$ has a fixed point on $[-1, 1]$. Using calculator, you will get 0.739085. (in degree mode, 0.9998477415)

12. $x = \sqrt{p + x} \Rightarrow x^2 - x - p = 0$. Since $x > 0$, $x = \frac{1 + \sqrt{1 + 4p}}{2}$.

13. Let $x_1 = \frac{1}{p}$ and $x_{n+1} = \frac{1}{p + x_n}$. We can define $F(x) = \frac{1}{p + x}$ on $[0, \infty)$. Then $F([0, \infty)) \subset [0, \infty)$. For all x and y in this domain,
$|F(x) - F(y)| = |\frac{1}{p + x} - \frac{1}{p + y}| = \frac{|x - y|}{(p + x)(p + y)} \leq \lambda|x - y|$, where $\lambda = \frac{1}{p^2} < 1$ since p, x and y are positive. Hence $F(x)$ is contractive. By Theorem 1, $F(x)$ has a fixed point. $x = \frac{1}{p + x} \Rightarrow x^2 + px - 1 = 0$. Since $x > 0$, $x = \frac{-p + \sqrt{p^2 + 4}}{2}$.

20(a). $|F(x) - F(y)| = |F'(\xi)||x - y| = |-2\xi||x - y| \leq \frac{|x - y|}{2}$ since $\xi \in [-\frac{1}{4}, \frac{1}{4}]$. Hence $F(x)$ is contractive. But $F(0) = 3 \notin [-\frac{1}{4}, \frac{1}{4}]$. So this $F(x)$ doesn't satisfy the condition for Theorem 1. Hence Theorem 1 doesn't need to be hold for this $F(x)$.

(b). $|F(x) - F(y)| = |-\frac{x}{2} + \frac{y}{2}| \leq \frac{|x - y|}{2}$. Hence $F(x)$ is contractive. But $F(1) = -\frac{1}{2} \notin [-2, -1] \cup [1, 2]$. So this $F(x)$ doesn't satisfy the condition for Theorem 1. Hence Theorem 1 doesn't need to be hold for this $F(x)$.

40. Let $F(x) = \frac{x(x^2 + 3R)}{3x^2 + R}$. Then $F'(x) = \frac{3(x^2 - R)^2}{(3x^2 + R)^2}$, $F''(x) = \frac{48Rx(x^2 - R)}{(3x^2 + R)^3}$, and $F'''(x) = \frac{-48R(9x^4 - 18Rx^2 + R^2)}{(3x^2 + R)^4}$. Hence $0 = F'(\sqrt{R}) = F''(\sqrt{R})$ and $F'''(\sqrt{R}) = \frac{3}{2R} \neq 0$.