Name: ____________________

PUID#: ____________________

Midterm – Math 514 (10/17/2013)

SHOW ALL RELEVANT WORK!!!

1. (10pts) Find (1) general solution of the following difference equation

$$x_{n+2} - 3x_{n+1} + 2x_n = 0 \quad \text{for} \quad n \geq 1,$$

and (2) the solution with $x_1 = 0$ and $x_2 = 1$.

-4 $x_n = ?$

-3 wrong a or b

(1) $0 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$

$\Rightarrow \lambda = 1, 2$

$x_n = a + b\,2^n$

(2) $0 = x_1 = a + 2b$

$1 = x_2 = a + 4b$

$\Longrightarrow b = \frac{1}{2}, a = -1$

$x_n = -1 + \frac{1}{2} 2^n$

$= 2^{n-1} - 1$

2. (10pts) For any $x_0 > 1$, the sequence defined recursively by $x_{n+1} = 2^{n+1}\left\{\sqrt{1 + 2^{-n}x_n} - 1\right\}$ converges to $\ln(x_0 + 1)$. Arrange this formulation in a way that avoids loss of significance.

$$x_{n+1} = 2^{n+1} \frac{(1 + 2^{-n}x_n) - 1}{\sqrt{1+2^{-n}x_n} + 1}$$

$$= \frac{2x_n}{\sqrt{1+2^{-n}x_n} + 1}$$

3. (10pts) If the bisection method is used starting with the interval [2, 3], how many steps must be taken to compute a root with absolute accuracy $< 10^{-8}$?

$$|r-c_n| \le \frac{b_0-a_0}{2^{n+1}} = \frac{1}{2^{n+1}} < 10^{-8}$$

$$2^{n+1} > 10^{8}$$

$$n+1 > 8\log_2 10$$

$$n > 8\log_2 10 - 1$$

4. (10pts) Suppose that r is a double root of the function f, i.e., $f(r) = f'(r) = 0$ and $f''(r) \neq 0$. Show that if f'' is continuous, then in Newton's method we shall have

$$x_{n+1} - r \approx (x_n - r)/2 \qquad \text{(linear convergence).}$$

Proof $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \implies e_{n+1} = x_{n+1} - r = e_n - \frac{f(x_n)}{f'(x_n)}$

$$f(x_n) = f(r) + f'(r)(x_n-r) + \frac{f''(r)}{2!}(x_n-r)^2 + \cdots$$

$$\approx \frac{f''(r)}{2!}(x_n-r)^2$$

$$f'(x_n) = f'(r) + f''(r)(x_n-r) + \cdots$$

$$\approx f''(r)(x_n-r)$$

$$\implies \frac{f(x_n)}{f'(x_n)} \approx \frac{1}{2}(x_n-r) = \frac{1}{2}e_n$$

$$\implies e_{n+1} \approx e_n - \frac{1}{2}e_n = \frac{1}{2}e_n$$

5. (10pts) Starting with $(0,0,1)$, carry out one iteration of Newton's method for nonlinear system on

$$\begin{cases} xy - z^2 &= 1 \\ xyz - x^2 + y^2 &= 2, \\ e^x - e^y + z &= 3. \end{cases}$$

Explain your results.

$$\vec{x}_1 = \vec{x}_0 - \left[\vec{f}'(\vec{x}_0)\right]^{-1} \vec{f}(\vec{x}_0), \quad \vec{x}_0 = (0,0,1)$$

$$\vec{f}'(\vec{x}_0) = \begin{bmatrix} y & x & -2z \\ yz-2x & xz+2y & xy \\ e^x & -e^y & 1 \end{bmatrix}_{\vec{x}=\vec{x}_0} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \text{ singular}$$

3

3

$$\vec{f}(\vec{x}_0) = -2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Since $\vec{f}'(\vec{x}_0)$ is singular, $\vec{x}_1$ cannot be computed.
One should use a different $\vec{x}_0$.

6. (10pts) If the secant method is applied to the function $f(x) = x^2 - 2$ with $x_0 = 0$ and $x_1 = 1$, what is x_2?

$$x_2 = x_1 - \frac{f(x_1)}{\frac{f(x_1)-f(x_0)}{x_1 - x_0}} = x_1 - f(x_1)\cdot\frac{x_1 - x_0}{f(x_1)-f(x_0)}$$

$$= 1 - (-1)\frac{1-0}{(-1)-(-2)} = 2$$

7. (10pts) Write the Lagrange and Newton interpolating polynomials for the following data

x	2	0	3
y	11	7	28

$$P_2(x) = 11\,\frac{(x-0)(x-3)}{(2-0)(2-3)} + 7\,\frac{(x-2)(x-3)}{(0-2)(0-3)} + 28\,\frac{(x-2)(x-0)}{(3-2)(3-0)}$$

$$= -\frac{11}{2}x(x-3) + \frac{7}{6}(x-2)(x-3) + \frac{28}{3}(x-2)x$$

x	y		
2	11	2	5
0	7	7	
3	28		

$$P_2(x) = 11 + 2(x-2) + 5(x-2)x$$

8. (10pts) For $j = k-1,\, k$, let $p_j(x)$ be the polynomials of degree $\le j$ such that

$$p_j(x_i) = y_i \quad \text{for} \quad i = 0, 1, \cdots, j.$$

Prove that $p_k(x) = p_{k-1}(x)$ for all x if and only if $p_{k-1}(x_k) = y_k$.

Proof (1)

"$\Rightarrow$" $P_k(x) = P_{k-1}(x)\ \forall x$

$\Rightarrow P_k(x_k) = y_k = P_{k-1}(x_k)$

"$\Leftarrow$" Given $P_{k-1}(x_k) = y_k$

$\Rightarrow P_{k-1}(x)$ and $P_k(x)$ interpolate f at $x_0, x_1, \cdots, x_k$

and they are poly. of degree $\le k$

by the uniqueness of the interpolation

$P_{k-1}(x) = P_k(x)$

Proof (2)

"$\Leftarrow$" Given $P_{k-1}(x_k) = y_k$

$\Rightarrow P_k(x) - P_{k-1}(x)$ has $k+1$ zeros: $x_0, x_1, \cdots, x_k$

but $P_k(x) - P_{k-1}(x)$ is a poly of degree $\le k$

$\Rightarrow P_k(x) - P_{k-1}(x) = 0\ \forall x$

Proof (3) By Newton's formula

$$P_k(x) = P_{k-1}(x) + c_k \prod_{j=0}^{k-1}(x - x_j)$$

$$\Rightarrow c_k = \frac{P_k(x_k) - P_{k-1}(x_k)}{\prod_{j=0}^{k-1}(x_k - x_j)} = 0$$

$$\Leftrightarrow P_k(x_k) - P_{k-1}(x_k) = 0$$

$$\Leftrightarrow P_k(x) = P_{k-1}(x).$$

9. (10pts) Show that if u is any function that interpolates f at $x_0, x_1, \cdots, x_{n-1}$, and if v is any function that interpolates f at $x_1, x_2, \cdots, x_n$, then the function

$$g(x) = \{(x_n - x)u(x) + (x - x_0)v(x)\}/(x_n - x_0)$$

interpolates f at $x_0, x_1, \cdots, x_n$.

Proof $\quad u(x_i) = f(x_i) \quad$ for $i = 0, 1, \cdots, n-1$

$v(x_i) = f(x_i) \quad$ for $i = 1, 2, \cdots, n$

For $j = 1, \cdots, n-1$

$$g(x_j) = \frac{(x_n - x_j)\,u(x_j) + (x_j - x_0)\,v(x_j)}{x_n - x_0}$$

$$= \frac{(x_n - x_j) + (x_j - x_0)}{x_n - x_0} f(x_j) = f(x_j)$$

$$g(x_0) = \frac{(x_n - x_0)\,u(x_0)}{x_n - x_0} = u(x_0) = f(x_0)$$

$$g(x_n) = \frac{(x_n - x_0)\,v(x_n)}{x_n - x_0} = v(x_n) = f(x_n)$$

10. (10pts) Choose one of the following two problems:

(a) Starting with $x_0 = 7$, show that the following fixed point iteration

$$x_{n+1} = f(x_n) = \frac{1}{2(1+x_n^2)} \quad \text{for} \quad n = 1, 2, \ldots$$

converges.

(b) Prove that the sequence generated by the iteration $x_{n+1} = F(x_n)$ will converge if $|F'(x)| \leq \lambda < 1$ on the interval $[x_0 - \rho, x_0 + \rho]$, where $\rho = |F(x_0) - x_0|/(1 - \lambda)$.

Proof (a) $x_0 = 7$, $x_1 = \frac{1}{2(1+49)} = \frac{1}{100}$, $x_2 = \frac{1}{2(1+10^{-4})} < \frac{1}{2}$

if $x_n \in [0, \frac{1}{2}]$, then $x_{n+1} = \frac{1}{2(1+x_n^2)} > 0$ 2

and $x_{n+1} = \frac{1}{2(1+x_n^2)} < \frac{1}{2}$

$\Rightarrow x_{n+1} \in [0, \frac{1}{2}] \Rightarrow$ if $x \in [0, \frac{1}{2}]$, then $F(x) \in [0, \frac{1}{2}]$ 3

$\forall x, y \in [0, \frac{1}{2}]$

$$|f(x) - f(y)| = \frac{1}{2}\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| = \frac{1}{2}\frac{|y+x|}{(1+x^2)(1+y^2)}|y-x|$$

$$\leq \frac{1}{2}(y+x)|y-x| \leq \frac{1}{2}|y-x|$$ 3

$\Rightarrow f(x)$ is a contractive mapping

$\Rightarrow \{x_n\}$ converges 2

Proof (b) $\forall x \in [x_0 - \rho, x_0 + \rho]$

$$|F(x) - x_0| = |(F(x_0) - x_0) + F'(\xi)(x - x_0)|$$

$$\leq |F(x_0) - x_0| + \lambda |x - x_0|$$

$$\leq \rho(1-\lambda) + \lambda \rho = \rho$$

$\Longrightarrow F(x) \in [x_0 - \rho, x_0 + \rho]$ 3

$\forall x, y \in [x_0 - \rho, x_0 + \rho]$

$$|F(x) - F(y)| = |F'(\xi)|\,|x - y|$$

$$\leq \lambda |x - y|$$ 3

$\Longrightarrow$ F is a contractive mapping.

$\Longrightarrow \{x_{n+1} = F(x_n)\}$ converges. 4