Chapter 1 Sobolev Spaces

§1.1 Review of Lebesgue Integration Theory

- domain Ω: a Lebesgue-measurable subset of R^n with non-empty interior
- function $f(x)$: a real-valued function defined on Ω that is L-measurable
- the L-integral $\int_\Omega f(x)dx$, where dx denotes L-measure
- norm

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} & 1 \le p < +\infty \\ \text{ess sup}\left\{|f(x)| : x \in \Omega\right\} & p = +\infty \end{cases}$$

- the Lebesgue Space

$$\boxed{L^p(\Omega) = \left\{ f : \|f\|_{L^p(\Omega)} < +\infty \right\}}$$

- $\forall f, g \in L^p$, " $f = g \iff \|f - g\|_{L^p(\Omega)} = 0$ ".
- Minkowski's inequality $\forall f, g \in L^p(\Omega)$, $1 \le p \le \infty$

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$$

- **Hölder's Inequality** For $p, q \in [1, \infty]$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

$$f \in L^p(\Omega),\ g \in L^q(\Omega) \Rightarrow \begin{cases} fg \in L^1(\Omega), \\ \|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \end{cases}$$

- **Schwarz' inequality** $p = q = 2$.

$$\|fg\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}$$

- $L^p(\Omega)$ is a linear (vector) space.

Proof $f, g \in L^p(\Omega)$ and $\alpha, \beta \in R$

$$\xRightarrow{\text{M-ineq}} \alpha f + \beta g \in L^p(\Omega).$$ #

- **Definition of norm** Given a linear space V,

$$\|\cdot\| : V \to R \text{ is a norm} \iff \begin{cases} (1)\ \|v\| \geq 0\ \forall v \in V \\ \quad \text{and "} \|v\| = 0 \iff v = 0 \text{"} \\ (2)\ \|cv\| = |c|\,\|v\| \quad \forall c \in R \\ (3)\ \|v+w\| \leq \|v\| + \|w\| \\ \quad \forall v, w \in V \end{cases}$$

$$\boxed{\|\cdot\|_{L^p(\Omega)} \text{ is a norm}}$$

- <u>a normed vector space</u> $(V, \|\cdot\|)$
- A normed vector space V is complete (Banach space)

  $\Longleftrightarrow$ every $\underbrace{\text{Cauchy sequence}}_{\{v_n\}_{n=1}^{\infty} :\ \|v_n - v_m\| \to 0 \text{ as } n,m\to\infty .}$ has a limit in V

- $L^p(\Omega)$ for $p \in [1,\infty]$ is a Banach space.
- <u>inclusion</u> if Ω is bounded

$$\Longrightarrow \quad L^p(\Omega) \subset L^q(\Omega) \quad \text{for } p \geq q .$$

<u>Proof</u> $f \in L^p(\Omega) \xrightarrow{?} f \in L^q(\Omega)$

$$\|f\|_{L^q(\Omega)}^q = \int_\Omega 1 \cdot |f|^q dx \leq \|1\|_{L^{r'}(\Omega)} \, \| |f|^q \|_{L^r(\Omega)} \qquad \boxed{\frac{1}{r} + \frac{1}{r'} = 1}$$

$$= \left(\int_\Omega 1^{r'} dx\right)^{\frac{1}{r'}} \left(\int_\Omega |f|^{q \cdot r} dx\right)^{\frac{1}{r}}$$

Choose r such that $rq = p \xrightarrow{r \geq 1} p \geq q .$ #

§1.2 Generalized (Weak) Derivatives

- point-wise definition
$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

- functions in L-spaces $\left\{ \begin{array}{l} \bullet \text{ pointwise values are irrelevant} \\ \bullet \text{ determined by its global behavior.} \end{array} \right.$

- global defintion the duality technique using $\mathcal{D}(\Omega)$

  - multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ where $\alpha_i \geq 0$ are integers
  $$|\alpha| = \sum_{i=1}^{n} \alpha_i \quad \text{(length)}$$

  - notations $\vec{x} = (x_1, \cdots, x_n) \in \Omega \subset R^n$
  $$\vec{x}^{\,\alpha} = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad \frac{\partial}{\partial \vec{x}} = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right)$$
  $$\left(\frac{\partial}{\partial x} \right)^{\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

$\phi \in C^{\infty}(\Omega)$: $D^{\alpha}\phi$, $D_x^{\alpha}\phi$, $\phi^{(\alpha)}$, $\partial_x^{\alpha}\phi$ $\quad |\alpha|$-order derivative

- support $\quad \operatorname{sppt} u = \overline{\{x : u(x) \neq 0\}}$

- u has a "compact support" w.r.t. Ω

$$\Longleftrightarrow \begin{cases} (1)\ \overline{\text{sppt}\, u} \text{ is bounded} \\ (2)\ \overline{\text{sppt}\, u} \subset \text{interior of } \Omega . \end{cases}$$

<u>Definition (1.2.1)</u> Let Ω be a domain in R^n

$$\boxed{\mathcal{D}(\Omega) = C_0^\infty(\Omega) = \left\{ u \in C^\infty(\Omega) \,\middle|\, u \text{ has a compact support in } \Omega \right\}}$$

<u>Ex. (1.2.2)</u> $\mathcal{D}(\Omega)$ is not empty.

Assume that Ω contains the closed unit ball $B_1 = \{x \mid |x| \le 1\}$

$$\Longrightarrow \phi(x) = \begin{cases} \exp\left\{\dfrac{1}{|x|^2 - 1}\right\}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \ge 1 \end{cases} \in \mathcal{D}(\Omega)$$

<u>(1) $|x| < 1$</u> $\quad \phi^{(\alpha)}(x) = P_\alpha(x)\, e^{-t}\, t^{|\alpha|+k}$ for some poly. P_α

and $t = \dfrac{1}{1-|x|^2} = \left(1 - \sum_j x_j^2\right)^{-1}$

$\dfrac{\partial t}{\partial x_i} = -\left(1 - \sum_j x_j^2\right)^{-2} \cdot (-2x_i)$

$= 2x_i\, t^2$

<u>Proof</u> <u>$|\alpha| = 0$</u>, $\phi(x) = e^{-t}$

<u>$|\alpha| = 1$</u>, $\partial_{x_i}\phi(x) = -e^{-t} \cdot \dfrac{\partial t}{\partial x_i}$

$$= (-2x_i)\, e^{-t}\, t^{|\alpha|+1}$$

$|x| \to 1 \Longleftrightarrow t \to \infty$

proof by induction

(2) $\underline{|x|>1}$ $\quad \phi^{(\alpha)}(x) = 0$

(3) $\underline{|x|=1}$

$$\lim_{|x|\to 1^-} \phi^{(\alpha)}(x) = \lim_{|x|\to 1^-} \frac{P_\alpha(x)\, t^{|\alpha|+k}}{e^t} \to 0$$

Definition (1.2.3) Given a domain $\Omega \subset R^n$

locally integrable functions

$$L^1_{loc}(\Omega) = \left\{ f \;\middle|\; f \in L^1(K) \quad \forall\, K \subset \text{interior } \Omega \right\}$$

- $C^0(\Omega) \subset L^1_{loc}(\Omega)$
- functions in $L^1_{loc}(\Omega)$ can behave badly near $\partial\Omega$

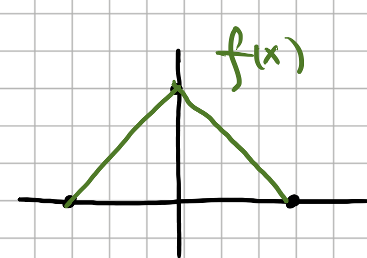
  e.g., $\exp\left\{ \exp \frac{1}{\mathrm{dist}(x,\partial\Omega)} \right\}$

Definition (1.2.4)

$f \in L^1_{loc}(\Omega)$ has a weak derivative $\iff \exists\, g \in L^1_{loc}(\Omega)$ s.t

$g = D^\alpha_w f$ $\qquad \displaystyle\int_\Omega g\,\phi\,dx = (-1)^{|\alpha|} \int_\Omega f\,\phi^{(\alpha)}\,dx, \quad \forall\, \phi \in \mathcal{D}(\Omega)$

Ex. $f(x) = 1 - |x|$, $\quad x \in \Omega = [-1, 1]$



$$\int_{-1}^{1} f\phi' dx = \int_{-1}^{0} f\phi' dx + \int_{0}^{1} f\phi' dx$$

$$= -\int_{-1}^{0} f'\phi\, dx + [f\phi']_{-1}^{0} - \int_{0}^{1} f'\phi\, dx + [f\phi']_{0}^{1}$$

$$= -\left[\int_{-1}^{0} (1)\phi\, dx + \int_{0}^{1} (-1)\phi\, dx\right] \implies D_w^1 f = \begin{cases} 1, & x<1 \\ -1, & x>1 \end{cases}.$$

$$\int_{-1}^{1} f\phi'' dx = \int_{-1}^{0} f\phi'' + \int_{0}^{1} f\phi'' dx$$

$$= -\left[\int_{-1}^{0} f'\phi' dx + \int_{0}^{1} f'\phi'\right] + [f\phi'']_{-1}^{0} + [f\phi'']_{0}^{1}$$

$$= \left[\int_{-1}^{0} f''\phi\, dx + \int_{0}^{1} f''\phi\, dx\right] - \left\{ \underbrace{[f'\phi]_{-1}^{0} + [f'\phi]_{0}^{1}}_{-[f'(0^-) - f'(0^+)]\phi(0) \neq 0} \right\}$$

$\Rightarrow D_w^2 f$ DNE.

<u>Remark of this example</u>

(1) $D_w^\alpha f = D^\alpha f$ at regular pt.

(2) $f \in C^0(\Omega) \implies D_w^1 f$ exists (depending on the dimension n)

<u>Ex. (1.2.6)</u> $f(x) = |x|^\beta$ is defined on $\Omega = \{\vec{x} \in R^n \mid |x| < 1\}$.

$\forall \varphi \in \mathcal{D}(\Omega)$, for $|\alpha| = 1$, let $\alpha = (0, \cdots, \overset{k}{1}, \cdots 0)$

$$\int_\Omega f(x)\varphi^{(\alpha)}(x)\,dx = -\int_\Omega \frac{\partial}{\partial x_k} f(x)\, \varphi(x)\,dx$$

$$= -\beta \int_\Omega |x|^{\beta-1} \frac{x^\alpha}{|x|} \varphi(x)\,dx \overset{?}{<} +\infty$$

$$\frac{\partial}{\partial x_k}|x|^\beta = \beta|x|^{\beta-1}\frac{\partial|x|}{\partial x_k} = \beta|x|^{\beta-1}\frac{x^\alpha}{|x|}$$

If $\int_0^1 r^{\beta-1} \cdot r^{n-1} dr < +\infty$

$$\underset{\displaystyle \int_0^1 r^{\beta+n-2}\,dr}{\parallel} = \begin{cases} E, & \beta > 1-n \implies D_w^\alpha |x|^\beta = \beta|x|^{\beta-1}\dfrac{x^\alpha}{|x|} \\ DEN, & \beta < 1-n \end{cases}$$

Proposition $\varphi \in C^{|\alpha|}(\Omega) \Longrightarrow D_w^\alpha \varphi$ exists and $D_w^\alpha \varphi = D^\alpha \varphi$.

HW #1, 3, 11, 13

§1.3 Sobolev Norms and Associated Spaces

- assumptions

$k \geq 0$ integer, $f \in L^1_{loc}(\Omega)$, $D_w^\alpha f$ exist $\forall\ |\alpha| \leq k$.

$|\alpha| = 0, 1, \cdots, k$

- Sobolev norms

$$\|f\|_{k,p,\Omega} = \|f\|_{W_p^k(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D_w^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, & p \in [1, \infty) \\ \max_{|\alpha| \leq k} \|D_w^\alpha f\|_{L^\infty(\Omega)} & p = \infty \end{cases}$$

- Sobolev spaces

$$W_p^k(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \;\middle|\; \|f\|_{W_p^k(\Omega)} < +\infty \right\}$$

Lipschitz function

Proposition (1) $W_\infty^1(\Omega) = Lip(\Omega) \equiv \left\{ f \in L^\infty(\Omega) \;\middle|\; \|f\|_{Lip(\Omega)} < +\infty \right\}$

where $\|f\|_{Lip(\Omega)} = \|f\|_\infty + \sup_{x \neq y \in \Omega} |f(x) - f(y)| \big/ |x-y|$

(2) $W^k_\infty(\Omega) = \left\{ f \in C^{k-1}(\Omega) \,\middle|\, f^{(\alpha)} \in Lip(\Omega) \ \forall |\alpha| \le k-1 \right\}$

proof. see Ex. 1.x.14–15.

(3) $W^1_1(\Omega) = \{$absolute continuous functions on $\Omega\}$ when $\Omega \subset \mathbb{R}$

<u>Theorem</u> $W^k_p(\Omega)$ is a Banach space.

<u>Proof</u> (1) $W^k_p(\Omega)$ is a normed vector space

(2) $W^k_p(\Omega)$ is complete.

<u>Proof of (2)</u> Let $\{v_j\}_{j=1}^\infty$ be a Cauchy sequence in $W^k_p(\Omega)$.

i.e., $\|v_j - v_i\|^p_{W^k_p(\Omega)} \to 0$ as $i, j \to \infty$

$$\| \qquad \sum_{|\alpha|\le k} \|D^\alpha_w(v_j - v_i)\|^p_{L^p(\Omega)}$$

$\Longrightarrow \{D^\alpha_w v_j\}$ is a C-sequence in $L^p(\Omega)$ $\forall |\alpha| \le k$

$\xrightarrow{L^p(\Omega) \text{ is complete}} \exists\, v^\alpha \in L^p(\Omega)$ s.t. $\|D^\alpha_w v_j - v^\alpha\|_{L^p(\Omega)} \to 0$ $\forall |\alpha| \le k$

$\xRightarrow{\alpha=(0,\cdots,0)} \lim\limits_{j\to\infty} v_j = v = v^{(0,\cdots,0)}$

? $\Downarrow$?

$L^p(\Omega) \ni v^\alpha \overset{?}{=} D^\alpha_w v$

$$\int_\Omega \left(D^\alpha_w v_j - v^\alpha\right) \varphi\, dx \to 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$$

$$\left|\int_\Omega (D_\omega^\alpha v_j - v^\alpha)\varphi\, dx\right| \le \|D_\omega^\alpha v_j - v^\alpha\|_{L^p(\Omega)} \|\varphi\|_{L^q(\Omega)} \to 0$$

$$\int_\Omega v^\alpha \varphi\, dx = \lim_{j\to\infty} \int_\Omega D_\omega^\alpha v_j\, \varphi\, dx = \lim_{j\to\infty} (-1)^{|\alpha|} \int_\Omega v_j\, \varphi^{(\alpha)} dx$$

$$= (-1)^{|\alpha|} \int_\Omega \left(\lim_{j\to\infty} v_j\right) \varphi^{(\alpha)} dx = (-1)^{|\alpha|} \int_\Omega v\, \varphi^{(\alpha)} dx$$

$$\Longrightarrow \quad v^\alpha = v^{(\alpha_1, \alpha_2, \cdots, \alpha_n)} = D_\omega^\alpha v \in L^p(\Omega)$$

$$\Longrightarrow \quad v \in W_p^k(\Omega). \qquad \#$$

- a different definition

$$\overline{C^k(\Omega)}^{\|\cdot\|_{W_p^k(\Omega)}} = \begin{cases} W_p^k(\Omega), & p \in [1,\infty) \\ C^k(\Omega) \neq W_\infty^k(\Omega), & p = \infty \end{cases}$$

Theorem For any open set $\Omega \subset R^n$

$$\Longrightarrow \boxed{C^\infty(\Omega) \cap W_p^k(\Omega) \text{ is dense in } W_p^k(\Omega) \text{ for all } p < +\infty.}$$

Proof ($\Omega = R^n$) $\quad \forall f \in L^p(\Omega) = W_p^0(\Omega)$ $\quad f^\varepsilon(x)$: mollification of $f(x)$

Let $\quad f_\varepsilon(x) = \int_\Omega f(y)\, \varphi^\varepsilon(x-y)\, dy = f * \varphi^\varepsilon(x)$

where $\varphi^{\varepsilon}(x-y)=\varepsilon^{-n}\varphi\left(\frac{x-y}{\varepsilon}\right)$, $\varphi\in\mathcal{D}(\Omega)$ and $\int_{\Omega}\varphi=1$.

By the dominated convergence theorem,

" $f_j(x)\to f(x)$ a.e. in Ω and $|f_j(x)|\le g(x)$ $\forall j$, $x\in\Omega$, $g\in L^1(\Omega)$ $\Longrightarrow$ $\begin{cases}\int_{\Omega}|f_j-f|\,dx\to 0\\ \text{and } \int_{\Omega}f_j-\int_{\Omega}f\to 0.\end{cases}$

(1) $f_\varepsilon\in C^\infty(\Omega)$

(2) " $f\in L^p(\Omega)\Longrightarrow f_\varepsilon\to f$ as $\varepsilon\to 0$ in $L^p(\Omega)$ "

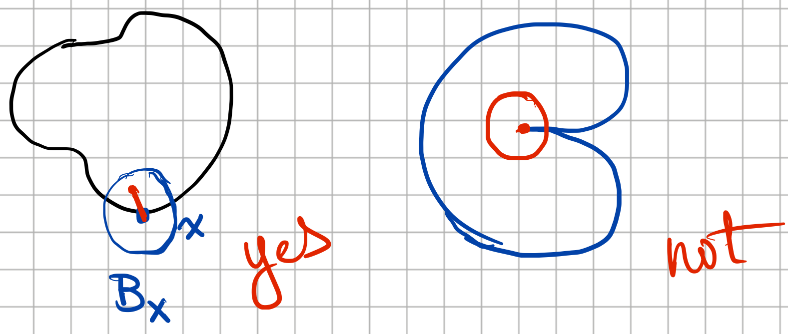
(3) " $f\in W_k^p(\Omega)\Longrightarrow f_\varepsilon\to f$ as $\varepsilon\to 0$ in $W_k^p(\Omega)$ ". #

<u>Remark</u> $\boxed{C^\infty(\bar{\Omega}) \text{ is dense in } W_p^k(\Omega)}$

provided that Ω satisfies the segment condition:

$\forall x\in\partial\Omega$, $\exists$ an open ball B_x containing x and $\vec{n}_x\neq\vec{0}$ s.t.

$\forall z\in\bar{\Omega}\cap B_x$, the segment $\{\vec{z}+t\vec{n}_x \mid t\in(0,1)\}\subset\Omega$.



- Sobolev semi-norm $k \geq 0$ integer, $f \in W_p^k(\Omega)$

$$|f|_{W_p^k(\Omega)} = \begin{cases} \left(\sum_{|\alpha|=k} \|D_w^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, & p \in [1,\infty) \\ \max_{|\alpha|=k} \|D_w^\alpha f\|_{L^\infty(\Omega)} & p = \infty \end{cases}.$$

§1.4 Inclusion Relations and Sobolev's Inequalities

Proposition (1.4.1) For any domain $\Omega \subset \mathbb{R}^n$,

$0 \leq k \leq m$ are integers, $\forall p \in [1,\infty]$

$$\Longrightarrow \quad W_p^m(\Omega) \subset W_p^k(\Omega)$$

Proposition (1.4.2) $\Omega \subset \mathbb{R}^n$ is bounded, $k \geq 0$ integer, $1 \leq p \leq q \leq \infty$

$$\Longrightarrow \quad W_q^k(\Omega) \subset W_p^k(\Omega)$$

Remark $\exists$ $k < m$ and $p > q$ s.t.

$$W_q^m(\Omega) \subset W_p^k(\Omega).$$

Theorem (1.4.5) Assume that Ω has a Lipschitz boundary.

$\Rightarrow \forall$ integer $k \geq 0$, $\forall\, p \in [1, \infty]$,

$\exists$ an extension mapping $E: W_p^k(\Omega) \to W_p^k(R^n)$ s.t.

$$Ev\big|_{\Omega} = v \quad \forall\, v \in W_p^k(\Omega) \text{ and } \|v\|_{W_p^k(R^n)} \leq C \|v\|_{W_p^k(\Omega)}.$$

Theorem (1.4.6) (Sobolev's Inequality)

Assume that $\Omega \subset R^n$ has a L-boundary and that integer $k > 0$ and real number $p \in [1, \infty)$ satisfy

$$\begin{cases} k \geq n, & \text{when } p = 1 \\ k > \dfrac{n}{p}, & \text{when } p > 1. \end{cases}$$

$\Longrightarrow \forall\, u \in W_p^k(\Omega)$, $\exists$ a constant $C > 0$ s.t.

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{W_p^k(\Omega)}.$$

[continuous imbedding : $L^\infty(\Omega) \hookrightarrow W_p^k(\Omega)$: $\left\langle \begin{array}{l} L^\infty(\Omega) \subset W_p^k(\Omega) \\ \|u\|_{L^\infty(\Omega)} \leq C \|u\|_{W_p^k} \end{array} \right.$

Moreover, $\exists$ a continuous function in the $L^\infty(\Omega)$ equivalent class of u.

Corollary (1.4.7) $\Omega \subset \mathbb{R}^n$ has a L-boundary

Assume that integers $k > m > 0$ and $p \in [1, \infty)$ satisfy

$$\begin{cases} k-m \geq n, & \text{when } p=1 \\ k-m > \frac{n}{p}, & \text{when } p>1 \end{cases}$$

$\Longrightarrow$ $\forall\, u \in W_p^k(\Omega)$, $\exists$ const. $C>0$ s.t.

$$\|u\|_{W_\infty^m(\Omega)} \leq C \|u\|_{W_p^k(\Omega)}.$$

Moreover, $\exists$ a $C^m(\Omega)$ function in the $L^p(\Omega)$ equivalence class of u.

§1.5 Review of Chapter 0

- solution space $\quad V = \{ v \in W_2^1(\Omega) \mid v(0) = 0 \}$, $\quad \Omega = (0,1)$
- Sobolev's inequality $\quad k = 1 > \frac{1}{2} = \frac{n}{p} \Longrightarrow$ pointwise values are well

$$L^\infty(\Omega) \hookrightarrow W_2^1(\Omega)$$

$$L^\infty(\Omega) \hookrightarrow W_1^1(\Omega) \qquad (k = 1 \geq n = 1)$$

- $S = \{ v \in C^0(\Omega) \mid v|_{I_i} \in P_1(I_i) \}$ continuous piecewise linear

  $D_\omega^1 v$ is piecewise constant

  $\Longrightarrow \quad S \subset W_\infty^1(\Omega) \subset W_2^1(\Omega)$

- (0.3.4) $\exists\, \varepsilon > 0$ s.t. $\inf\limits_{v \in S} \|w - v\|_E \leq \varepsilon |w|_{W_2^2(\Omega)} = \varepsilon \|w''\|_{L^2(\Omega)}$

- $a(v,v) = \int_0^1 (v')^2 dx = 0 \Longrightarrow v(x) \equiv 0$

Proof Poincaré's inequality : $\|v\|^2_{L^2(\Omega)} \leq C\, a(v,v), \forall v \in V$

using $v(x) = \int_0^x D_w^1 v(s) ds$ (Ex #16)

§1.6 Trace Theorems

domain $\Omega \subset R^n$ and boundary $\partial\Omega$ is a manifold in R^{n-1}

Theorem (1.6.6) Assume that Ω has a L-boundary and $p \in [1, \infty]$

$\Longrightarrow \exists$ a constant s.t. $\forall u \in W_p^1(\Omega)$

$$\|u\|_{L^p(\partial\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1-\frac{1}{p}} \|u\|_{W_p^1(\Omega)}^{\frac{1}{p}}.$$

Notations

$$\overset{\circ}{W}{}^k_p(\Omega) = \left\{ v \in W^k_p(\Omega) \;\middle|\; v^{(\alpha)}\big|_{\partial\Omega} = 0 \text{ in } L^p(\partial\Omega) \quad \forall\, |\alpha| \le k \right\}$$

Proof of Theorem in the case that $p=2$ and $\Omega \subset \mathbb{R}^2$ is a unit disk

$$\forall\, u \in W^1_2(\Omega) \implies u\big|_{\partial\Omega} \in L^2(\partial\Omega) \text{ and } \|u\|_{L^2(\partial\Omega)} \lesssim \|u\|^{\frac{1}{2}}_{L^2(\Omega)} \|u\|^{\frac{1}{2}}_{W^1_2(\Omega)}$$

Proof (1) $\boxed{u \in C^1(\bar\Omega)}$, $\quad \Omega = \left\{ (r,\theta) \,\middle|\, 0 \le r < 1,\ \theta \in [0, 2\pi] \right\}$

$$u^2\big|_{\partial\Omega} = u^2(1,\theta) = \int_0^1 \partial_r \left(r^2 u^2(r,\theta) \right) dr = 2\int_0^1 \left(u\, \nabla u \cdot \binom{x}{y} + r u^2 \right) r\,dr$$

$$\le 2\int_0^1 \left(|u|\, |\nabla u| + u^2 \right) r\,dr$$

$$\implies \int_{\partial\Omega} u^2\,ds = \int_0^{2\pi} u^2(1,\theta)\,d\theta \le 2\int_\Omega \left(|u|\,|\nabla u| + u^2 \right) dx\,dy$$

$$\le 2\sqrt{2}\, \|u\|_{L^2(\Omega)} \|u\|_{W^1_2(\Omega)}.$$

(2) $\boxed{u \in W^1_2(\Omega)}$ $\quad \boxed{C^1(\bar\Omega) \text{ is dense in } W^1_2(\Omega)}$

$\implies \exists$ a sequence $\{u_j\} \subset C^1(\bar\Omega)$ s.t. $\|u - u_j\|_{W^1_2(\Omega)} \to 0$

(a) $\{u_j\}$ is a C-sequence in $L^2(\partial\Omega)$

$$\forall k, j, \quad \|u_k - u_j\|_{L^2(\partial\Omega)} \lesssim \|u_k - u_j\|^{\frac{1}{2}}_{L^2(\Omega)} \|u_k - u_j\|^{\frac{1}{2}}_{W_2^1(\Omega)}$$

$$\leq \|u_k - u_j\|_{W_2^1(\Omega)}$$

$$\leq \|u_k - u\|_{W_2^1(\Omega)} + \|u - u_j\|_{W_2^1(\Omega)} \to 0$$

(a) $\xrightarrow{L^2(\partial\Omega) \text{ is complete}} \exists\, v \in L^2(\partial\Omega)$ s.t. $\|v - u_j\|_{L^2(\partial\Omega)} \to 0$

(b) v is independent of the choice of sequences.

Assume that $\{v_j\} \neq \{u_j\}$ in $C^1(\bar{\Omega})$ and that $\|u - v_j\|_{W_2^1(\Omega)} \to 0$

$$\|v - v_j\|_{L^2(\partial\Omega)} \leq \|v - u_j\|_{L^2(\partial\Omega)} + \|u_j - v_j\|_{L^2(\partial\Omega)}$$

$$\lesssim \|v - u_j\|_{L^2(\partial\Omega)} + \|u_j - v_j\|_{W_2^1(\Omega)}$$

$$\leq \|v - u_j\|_{L^2(\partial\Omega)} + \|u_j - u\|_{W_2^1(\Omega)} + \|u - v_j\|_{W_2^1(\Omega)} \to 0$$

$$\Longrightarrow \|u_j\|_{L^2(\partial\Omega)} \lesssim \|u_j\|^{\frac{1}{2}}_{L^2(\Omega)} \|u_j\|^{\frac{1}{2}}_{W_2^1(\Omega)}$$

$$\xrightarrow{\overline{\lim}_{j\to\infty}} \underbrace{\|v\|_{L^2(\partial\Omega)}}_{u|_{\partial\Omega}} \lesssim \|u\|^{\frac{1}{2}}_{L^2(\Omega)} \|u\|^{\frac{1}{2}}_{W_2^1(\Omega)}$$

#

§1.7 Negative Norms and Duality

For $k < 0$, $p \in [1, \infty]$, let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$

$$W_p^k(\Omega) = \left(W_q^{-k}(\Omega) \right)'$$