Chapter 2 Variational Formulations of Elliptic PDEs

HW # 4, 6, 9, 10, 11, 12, 15

§2.1 Inner Product Spaces

Let V be a linear space.

Definitions (2.1.1)

(1) $a: V \times V \to \mathbb{R}$ is a bilinear form $\iff$ $\left\{ \begin{array}{l} a(\cdot, w) \text{ and } a(v, \cdot) \\ \text{are linear on } V \end{array} \right.$

(2) $a(\cdot,\cdot)$ is symmetric $\iff$ $a(v,w) = a(w,v) \quad \forall v, w \in V$

(3) $(\cdot,\cdot): V \times V \to \mathbb{R}$ is a (real) inner product

$\iff$ (a) $(\cdot,\cdot)$ is a symmetric bilinear form

(b) $(v,v) \geq 0 \quad \forall v \in V$

(c) $(v,v) = 0 \iff v = 0$

Definition (2.1.2) $(V, (\cdot,\cdot))$ is an inner product space

$\iff$ (a) V is a linear space

(b) $(\cdot,\cdot)$ is an inner-product defined on V.

Examples (1) $V = R^n$, $(\vec{x}, \vec{y}) = \sum_{i=1}^{n} x_i y_i$

(2) $V = L^2(\Omega)$, $\Omega \subset R^n$, $(u,v)_{L^2(\Omega)} = \int_\Omega uv\,dx$

(3) $V = W_2^k(\Omega) = H^k(\Omega)$, $\Omega \subset R^n$, $(u,v)_k = \sum_{|\alpha| \le k} (D^\alpha u, D^\alpha v)$

Theorem (2.1.4) (The Schwarz inequality)

Assume that $(V, (\cdot,\cdot))$ is an inner-product space.

$\Longrightarrow$ (1) $|(u,v)| \le (u,u)^{\frac{1}{2}} (v,v)^{\frac{1}{2}}$

(2) $|(u,v)| = (u,u)^{\frac{1}{2}} (v,v)^{\frac{1}{2}} \Leftrightarrow$ u and v are linearly indep.

Proof

(1) 3 things: (u,u), (v,v), (u,v)

connection $(u,u) + (v,v) \pm 2(u,v) = (u \pm v, u \pm v) \ge 0$

$$\Rightarrow (u,v) \le \frac{1}{2}\left[(u,u) + (v,v)\right] \not\Rightarrow (1)$$

modification $(u,u) + (tv, tv) \pm 2(u, tv) = (u \pm tv, u \pm tv) \ge 0$

$$|(u,v)| = \pm(u,v) \le \frac{1}{2}\left[\frac{1}{t}(u,u) + t(v,v)\right]$$

$$= (u,u)^{\frac{1}{2}} (v,v)^{\frac{1}{2}} \quad \text{choosing } t = \frac{(u,u)^{\frac{1}{2}}}{(v,v)^{\frac{1}{2}}}$$

(2) "$\Leftarrow$" $u = \alpha v \Rightarrow |(u,v)| = (u,u)^{\frac{1}{2}} (v,v)^{\frac{1}{2}}$

"$\Rightarrow$" ~~the~~ proof of (1), $|(u,v)| = (u,v)$ or $-(u,v)$

$$\Rightarrow \pm (u,v) = (u,u)^{\frac{1}{2}} (v,v)^{\frac{1}{2}}$$

$$\Rightarrow 0 = (u \pm tv, u \pm tv) \text{ with } t = \sqrt{\frac{(u,u)}{(v,v)}} \Rightarrow u = \pm tv.$$

#

<u>Remark</u> In the proof of the Schwarz inequality,

"$(v,v) = 0 \Leftrightarrow v = 0$" is not used.

$$\Rightarrow |a(u,v)| \leq a^{\frac{1}{2}}(u,u)\, a^{\frac{1}{2}}(v,v), \text{ where } a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

is not an inner-product

<u>Proposition (2.1.9)</u> $\|v\| = \sqrt{(v,v)}$ defined a norm in $(V, (\cdot,\cdot))$

<u>Proof</u> (1) $\|\alpha v\| = |\alpha| \, \|v\|$

(2) $\|v\| = 0 \Rightarrow v = 0$

(3) $\|u+v\| \leq \|u\| + \|v\| \Leftrightarrow \|u+v\|^2 \leq (\|u\| + \|v\|)^2$

$$= \|u\|^2 + \|v\|^2 + 2\|u\| \, \|v\|$$

$$\|u\|^2 + \|v\|^2 + 2(u,v)$$

§2.2 Hilbert Spaces

<u>Definition (2.2.1)</u> $(V, (\cdot,\cdot))$ is a Hilbert space

$$\Longleftrightarrow \ (V, \|\cdot\|) \text{ is complete.}$$

<u>Examples</u> $\mathbb{R}^n$, $L^2(\Omega)$, $W_2^k(\Omega)$

<u>Definition (2.2.3)</u> Let H be a Hilbert space

$S \subset H$ is a subspace of $H \Longleftrightarrow \begin{cases} (1) \ S \text{ is a linear subset} \\ (2) \ S \text{ is closed under } \|\cdot\|. \end{cases}$

<u>Propersition (2.2.4)</u>

S is a subspace of $H \Longrightarrow (S, (\cdot,\cdot))$ is a Hilbert space.

<u>Examples of subspaces</u> Let H be a Hilbert space

(1) H, $\{0\}$

(2) Let $T: H \to K$ be a continuous linear map from a H-space H into a linear space K

$$\mathrm{Ker}(T) = \{ v \in H \mid Tv = 0 \} \qquad (2.x.1)$$

(3) For $x \in H$, $\underbrace{x^{\perp} = \{ v \in H \mid (v,x) = 0 \}}_{\text{is a subspace}}$ orthogonal complement

$x^\perp = \mathrm{Ker}(L_x)$ where $L_x(v) = (v, x)$ linear, bounded

(4) Let M be a subset of H

$$M^\perp = \{ v \in H \mid (x, v) = 0 \ \forall x \in M \} = \bigcap_{x \in M} x^\perp \text{ is a subspace.}$$

orthogonal complement

<u>Proposition (2.2.7)</u> Let H be a Hilbert space

(1) $\forall$ subsets $M, N \subset H$, $M \subset N \Longrightarrow N^\perp \subset M^\perp$;

(2) $\forall$ subset $M \subset H$ containing $0 \Longrightarrow M \cap M^\perp = \{0\}$;

(3) $\{0\}^\perp = H$; (4) $H^\perp = \{0\}$. (2.x.3)

<u>Proof of (2)</u>

(i) clearly $0 \in M^\perp \xRightarrow{0 \in M} 0 \in M \cap M^\perp$

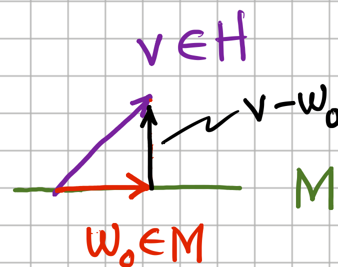
(ii) $x \in M \cap M^\perp \Longrightarrow \left\{ \begin{array}{l} x \in M \Rightarrow M^\perp \subset x^\perp \\ x \in M^\perp \end{array} \right\} \Rightarrow x \in M^\perp \subset x^\perp$

$\Rightarrow (x, x) = 0 \Rightarrow x = 0.$ #

<u>Theorem (2.2.8)</u> (Parallelogram Law) $\|\cdot\| = \sqrt{(\cdot, \cdot)}$

$$\Longrightarrow \|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right) \qquad (2.x.4)$$

§2.3 Projections onto Subspaces



Proposition (2.3.1) (Projection Theorem)

Let M be a subspace of a Hilbert space H.

$\forall v \in H, \exists w_0 \in M$ s.t. $\begin{cases} (1) & \|v - w_0\| = \inf\limits_{w \in M} \|v - w\| \quad \text{(MP)} \\ & \Updownarrow \\ (2) & v - w_0 \in M^{\perp} \quad (w_0 - v, w) = 0 \ \forall w \in M \quad \text{(VP)} \end{cases}$

Proof (1) Set $\delta = \inf\limits_{w \in M} \|v - w\|$

- $\exists \{w_n\}_{n=1}^{\infty} \subset M$ s.t. $\lim\limits_{n \to \infty} \|v - w_n\| = \delta$

this is from the definition of inf: $\exists w_n \in M$ s.t. $\begin{cases} \delta \leq \|v - w_n\| \\ \|v - w_n\| \leq \delta + \frac{1}{n} \end{cases}$

- $\{w_n\}_{n=1}^{\infty}$ is a Cauchy-sequence in M.

$$
\begin{aligned}
0 < \|w_n - w_m\|^2 &= \|(w_n - v) - (w_m - v)\|^2 \\
&= 2\left[\|w_n - v\|^2 + \|w_m - v\|^2\right] - \|(w_n - v) + (w_m - v)\|^2 \\
&\leq 2\left[\left(\delta + \tfrac{1}{n}\right)^2 + \left(\delta + \tfrac{1}{m}\right)^2\right] - 4\underbrace{\left\|\tfrac{1}{2}(w_n + w_m) - v\right\|^2}_{\geq -4\delta} \\
&\to 2(\delta^2 + \delta^2) - 4\delta^2 = 0
\end{aligned}
$$

M is complete

$\Longrightarrow$ $\exists w_0 \in M$ s.t. $\lim\limits_{n \to \infty} w_n = w_0$ in M

$\Longrightarrow$ $\|v - w_0\| = \lim\limits_{n \to \infty} \|v - w_n\| = \delta.$

(2) $J(w) = \|v-w\|^2 = \|w\|^2 - 2(v,w) + \|v\|^2$

$$= a(w,w) - 2v(w) + \|v\|^2$$

(MP)

$$J(w_0) = \min_{w\in M} J(w) \iff \text{find } w_0 \in M \text{ s.t. } \underbrace{a(w_0,w)}_{=(w_0,w)} = \underbrace{v(w)}_{=(v,w)} \ \forall w \in M.$$

(VP)

#

Decomposition $H = M \oplus M^{\perp}$

For any $v \in H$, there exists a unique decomposition

$$v = w_0 + w_1, \text{ where } w_0 \in M \text{ and } w_1 = v - w_0 \in M^{\perp}.$$

Proof of uniqueness $\exists\, w_0, z_0 \in M$ and $w_1, z_1 \in M^{\perp}$ s.t.

$$v = w_0 + w_1 = z_0 + z_1 \implies w_0 - z_0 = z_1 - w_1 \in M \cap M^{\perp} = \{0\}$$

$$\implies w_0 = z_0 \text{ and } z_1 = w_1$$

#

Projection Operators

- $P_M : H \to M \iff P_M v = \begin{cases} v, & v \in M \\ w_0, & v \notin M \end{cases} \iff$ Given $v \in H$, $P_M v$ satisfies $(P_M v, w) = (v, w)\ \forall w \in M$

- $P_{M^{\perp}} : H \to M^{\perp}$

$$\implies v = P_M v + P_{M^{\perp}} v$$

Remark (1) P_M and $P_{M^\perp}$ are linear operators

(2) P_M and $P_{M^\perp}$ are projections : $P^2 = P$

§2.4 Riesz Representation Theorem

Let $(H, (\cdot,\cdot))$ be a Hilbert space

- Given $u \in H$, $L_u(v) = (u, v)$ defines a continuous linear functional

  i.e., $L_u \in H'$

- **Theorem (2.4.2) (Riesz Representation Theorem)**

(1) For any $L \in H'$, $\exists 1\, u \in H$ s.t. $L(v) = (u, v)\ \forall v \in H$.

(2) Moreover, $\|L\|_{H'} = \|u\|_H$.

Proof (1) uniqueness Assume that $\exists\, u_1, u_2 \in H$ s.t. $L(v) = (u_i, v)\ \forall v \in H$

$\Rightarrow 0 = L(v) - L(v) = (u_1 - u_2, v)\quad \forall v \in H$

$\Rightarrow u_1 = u_2$ by choosing $v = u_1 - u_2$.

existence find $u \in H$ s.t. $(u, v) = L(v)\quad \forall v \in H$

decomposition $H = M \oplus M^\perp$, where $M = \mathrm{Ker}(L) = \{v \in H \mid L(v) = 0\}$

Case 1 $M^\perp = \{0\}$ $M = H \Rightarrow$ find $u \in \overset{=M}{H}$, s.t. $(u, v) = 0\ \forall v \in H \Rightarrow \boxed{u = 0}$

Case 2 $M^{\perp} \neq \{0\}$ $\quad \exists z \in M^{\perp}$ s.t. $z \neq 0$ and $L(z) \neq 0$

$\Rightarrow \forall v \in H, \quad v = (v - \beta z) + \beta z,$ chose β s.t. $0 = L(v - \beta z) = L(v) - \beta L(z)$

$\Rightarrow v - \beta z \in M$ and $\beta z \in M^{\perp}$

$\Longrightarrow L(v) = L(v - \beta z) + L(\beta z) = \beta L(z)$

$\overset{?}{\|}$

$(u, v) = (u, v - \beta z) + (u, \beta z) = (u, \beta z)$ if $u \in M^{\perp}$

$$\Rightarrow (u, z) = L(z) \xrightarrow{u = \alpha z} \alpha(z, z) = L(z) \Rightarrow u = \frac{L(z)}{\|z\|_H^2} z$$

(2) $$u = \frac{L(z)}{\|z\|_H^2} z \Longrightarrow \|u\|_H = \frac{|L(z)|}{\|z\|_H}$$

$$\|L\|_{H'} = \sup_{v \in H} \frac{|L(v)|}{\|v\|_H}, \quad |L(v)| = |(u, v)| \leq \|u\|_H \|v\|_H$$

$$\leq \|u\|_H = \frac{|L(z)|}{\|z\|_H} \leq \|L\|_{H'}$$

$$\Rightarrow \|L\|_{H'} = \|u\|_H.$$ #

Remark $M^{\perp}$ is one dimension.

Assume that $z_1, z_2 \in M^{\perp}$ and $L(z_i) \neq 0$

$\Rightarrow \forall v \in H, \quad L(v - \beta_i z_i) = 0$ by choosing $\beta_i = \dfrac{L(v)}{L(z_i)}$

$$\Rightarrow L(\beta_1 z_1 - \beta_2 z_2) = L\big((\beta_1 z_1 - v) + (v - \beta_2 z_2)\big) = 0$$

$\Rightarrow \beta_1 z_1 - \beta_2 z_2 \in M$

$z_i \in M^\perp \Rightarrow \beta_1 z_1 - \beta_2 z_2 \in M^\perp$

$\Bigg\} \Rightarrow \beta_1 z_1 = \beta_2 z_2$

$\Rightarrow z_1, z_2$ — linearly dep.

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Remark The Riesz Rep Thrm $\Rightarrow \exists$ a natural isometry

$$\tau : L_u \in H' \longrightarrow u \in H$$

$$\Rightarrow H' \cong H \Rightarrow H^{-m}(\Omega) \cong H^m(\Omega)$$

§2.5 Formulation of Symmetric Variational Problems

Example (2.5.1)

- $H^1(0,1) = W_2^1(0,1)$ is a Hilbert space under the inner product

$$(u,v)_{H^1} = \int_0^1 uv\,dx + \int_0^1 u'v'\,dx .$$

- $V = \{v \in H^1(0,1) : v(0) = 0\}$ is a subspace of $H^1(0,1)$
- $a(u,v) = \int_0^1 u'v'\,dx$ is not an inner product in $H^1(0,1)$ since $a(1,1) = 0$

Definition (2.5.2) (Continuity and Coercivity)

Let $a(\cdot,\cdot)$ be a bilinear form on a normed linear space H.

(1) $a(\cdot,\cdot)$ is bounded $\Leftrightarrow \forall u, v \in H$, $\exists$ a const. $C>0$, s.t. $|a(u,v)| \le C \|u\|_H \|v\|_H$

(2) $a(\cdot,\cdot)$ is coercive on $V \subset H \Leftrightarrow \forall v \in H$, $\exists\, \alpha > 0$, s.t. $a(v,v) \ge \alpha \|v\|_H^2$

Propersition (2.5.3) Let H be a Hilbert space, and $V \subset H$ be a subspace.

Assume that (1) $a(\cdot,\cdot)$ is sym. and bilinear

(2) $a(\cdot,\cdot)$ is continuous on H and coercive on V.

$$\Longrightarrow \left(V, a(\cdot,\cdot)\right) \text{ is a Hilbert space.}$$

Proof (1) $a(\cdot,\cdot)$ is an inner product on V

(2) $\left(V, \|\cdot\|_E = a^{\frac{1}{2}}(\cdot,\cdot)\right)$ is complete.

$$\alpha \|v\|_H^2 \leq \|v\|_E^2 \leq C \|v\|_H^2 \quad \forall v \in V$$

$\{v_n\}$ is a C-sequence in $V \Rightarrow \{v_n\}$ is a C-sequence in H

$\Rightarrow \exists\, v \in H$, s.t., $\|v_n - v\|_H \to 0$

$$\Rightarrow \quad \|v_n - v\|_E^2 \leq C \|v_n - v\|_H^2 \to 0 \overset{V\text{-complete}}{\Longrightarrow} v \in V. \qquad \#$$

Assumptions (2.5.4)

(1) $(H, (\cdot,\cdot))$ is a H-space

(2) V is a (closed) subspace of H

(3) $a(\cdot,\cdot)$ is a bounded, sym. bilinear form that is coercive on V

- symmetric variational formulation (SVF)

Given $F \in V'$, find $u \in V$ s.t. $a(u,v) = F(v) \quad \forall v \in V$.

<u>Theorem (2.5.6)</u> Under the assumptions (1)–(3),

(SVF) has a unique solution.

<u>Proof</u> • $(V, a(\cdot,\cdot))$ is a H-space

• $F \in (V, \|\cdot\|_H)' \Longrightarrow F \in (V, a(\cdot,\cdot))'$

• RRT $\Longrightarrow \exists\, u \in V$ s.t. $F(v) = a(u,v)\ \forall v \in V$. #

• <u>Galerkin Approximation</u> Given $F \in V'$, a finite-dimensional subspace $V_h \subset V$,

find $u_h \in V_h$ s.t. $a(u_h, v) = F(v)\ \ \forall v \in V_h$.

<u>Theorem (2.5.8)</u> Under the assumptions (1)–(3),

(GA) has a unique solution.

<u>Proof</u> $(V_h, a(\cdot,\cdot))$ is a H-space and $F|_{V_h} \in V_h'$. #

<u>Fundamental Orthogonality</u> (2.5.9)

$$a(u - u_h, v) = 0 \quad \forall v \in V_h.$$

<u>Corollary</u> (2.5.10)

$$\|u - u_h\|_E = \min_{v \in V_h} \|u - v\|_E.$$

<u>Proof</u> $\|u-u_h\|_E^2 = a(u-u_h, u-u_h) = a(u-u_h, u-v)\ \ \forall v \in V_h$

$$\leq \|u-u_h\|_E \, \|u-v\|_E$$

$$\Rightarrow \|u-u_h\|_E \leq \|u-v\|_E \quad \forall v \in V_h$$

$$\Rightarrow \|u-u_h\|_E \leq \min_{v\in V_h} \|u-v\|_E \leq \|u-u_h\|_E \qquad \#$$

- <u>Minimization Problem</u> Given $F \in V'$,

  find $u \in V$ s.t. $J(u;F) = \inf_{v \in V} J(v;F)$,

  where $J(v;F) = \frac{1}{2} a(v,v) - F(v)$.

<u>Theorem</u> Under the assumptions (1)−(3),

(MP) has a unique solution.

<u>Proof</u> RRT $\Rightarrow \exists\, \tau_F \in V$ s.t. $F(v) = a(\tau_F, v) \;\; \forall v \in V$

$$\Rightarrow J(v;F) = \frac{1}{2} a(v,v) - a(\tau_F, v)$$

$$= \frac{1}{2} a(v-\tau_F, v-\tau_F) - \frac{1}{2} a(\tau_F, \tau_F). \qquad \#$$

- <u>Ritz Approximation</u>

  Find $u_h \in V_h$ s.t. $J(u_h;F) = \min_{v \in V_h} J(v;F)$

<u>Theorem</u> Under the assumptions (1)−(3),

(RA) has a unique solution.

Theorem $\|u-u_h\|_E = \min_{v\in V_h} \|u-v\|_E$

Proof $\|u-u_h\|_E^2 = 2\left(J(u_h;F) - J(u;F)\right)$

$\leq 2\left(J(v;F)-J(u;F)\right) = \|u-v\|_E^2$ #

§2.6 Formulation of Non-symmetric Variational Problems

Assumptions

(1) $(H, (\cdot,\cdot))$ is a H-space,

(2) V is a closed subspace of H,

(3) $a(\cdot,\cdot)$ is a bilinear form on V, (but not necessarily sym)

(4) $a(\cdot,\cdot)$ is continuous and coercive on V.

- **Non-sym Variational Problem**

  Given $F\in V'$, find $u\in V$ s.t. $a(u,v) = F(v)$ $\forall v\in V$.

- **Galerkin Approximation**

  Given $F\in V'$, a finite-dim subspace $V_h \subset V$

  find $u_h\in V_h$ s.t. $a(u_h, v) = F(v)$ $\forall v\in V_h$.

• Example

$$(1)\quad \begin{cases} -u'' + u' + u = f \text{ in } (0,1) \\ u'(0)=0,\ u'(1)=0 \end{cases}$$

$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,dx$ is coercive on $V = H^1(0,1)$.

$$a(v,v) = \frac{1}{2}\int_0^1 \left[(v'+v)^2 + \left(v'^2 + v^2\right) \right] dx \geq \frac{1}{2}\|v\|_{H^1}^2$$

$$(2)\quad \begin{cases} -u'' + ku' + u = f \text{ in } (0,1) \\ u'(0)=0,\ u'(1)=0 \end{cases}$$

$a(u,v) = \int_0^1 [u'v' + ku'v + uv]\,dx$ is not coercive for large k.

§2.7 The Lax-Milgram Theorem

Theorem (Lax-Milgram)

Given a Hilbert space $(V, (\cdot,\cdot))$, a continuous and coercive bilinear form $a(\cdot,\cdot)$, and a linear form $F \in V'$,

$\Longrightarrow \exists 1\ u \in V$ s.t. $a(u,v) = F(v) \quad \forall v \in V$.

Lemma (2.7.2) (Contraction Mapping Principle)

Let V be a Banach space. Let $T: V \to V$ be a map satisfying

$\exists M \in [0,1)$ s.t. $\|T(v_1)-T(v_2)\| \leq M\|v_1-v_2\| \quad \forall v_1, v_2 \in V$

$\Longrightarrow \exists 1 \; u \in V$ s.t. $u = T(u)$.

Remark (1) The contraction mapping T has a unique fixed point u.

Proof uniqueness Assume that $\exists u_i \in V$ s.t. $T(u_i) = u_i$, $i=1,2$.

$$\Longrightarrow \|u_1-u_2\| = \|T(u_1)-T(u_2)\| \leq M\|u_1-u_2\| < \|u_1-u_2\|.$$

existence pick $u_0 \in V$, compute $u_{k+1} = T(u_k)$

- $\{u_k\}$ is a C-sequence.

$$\forall N > n, \quad \|u_N - u_n\| = \left\| \sum_{k=n}^{N-1} (u_{k+1}-u_k) \right\|$$

$$\leq \sum_{k=n}^{N-1} \|u_{k+1}-u_k\| = \sum_{k=n}^{N-1} \|T(u_k)-T(u_{k-1})\|$$

$$\leq \sum_{k=n}^{N-1} M^{k-1} \|u_1-u_0\| \leq \frac{M^n}{1-M}\|u_1-u_0\| = \frac{M^n}{1-M}\|T(u_0)-u_0\|$$

- V is complete $\Longrightarrow \lim_{n\to\infty} u_n = u \in V$

$$\Longrightarrow u = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} T(u_n) = T\left(\lim_{n\to\infty} u_n\right) = T(u).$$

#

Proof of Lax-Milgram Thrm

- converting $\begin{cases} \text{Find } u \in V \text{ s.t.} \\ a(u,v) = F(v) \; \forall v \in V \end{cases}$ into $Au = F$ in V'

Define $A : V \to V'$ by

$\forall u \in V$, $Au \in V'$ given by $Au(v) = a(u,v)$ $\forall v \in V$.

$\Rightarrow Au(v) = F(v)$ $\forall v \in V \Rightarrow Au = v$ in V'

- A is linear and bounded

$$\|A\|_{L(V,V')} = \sup_{v \in V} \frac{\|Av\|_{V'}}{\|v\|_V} = \sup_{w \in V} \frac{|Av(w)|}{\|w\|_V} \qquad |Av(w)| = |a(v,w)| \le C\|v\|_V \|w\|_V$$

$\le C$.

- $Au = F$ in V' $\xrightarrow{RRT}$ $\boxed{\tau A u = \tau F \text{ in } V}$

where $\tau : V' \to V$ is isometric mapping

$$\begin{cases} \forall \varphi \in V', \ \varphi(v) = (\tau\varphi, v) \\ \quad \text{and } \|\tau\varphi\|_V = \|\varphi\|_{V'} \end{cases}.$$

- $\boxed{u = u - \rho(\tau A u - \tau F) \equiv Tu}$

$\boxed{T \text{ is a contraction mapping}}$, i.e., $\exists M \in [0,1)$ s.t. $\|Tv_1 - Tv_2\|_V \le M\|v_1 - v_2\|_V$,

$\forall v_1, v_2 \in V$, let $v = v_1 - v_2$

$$\|Tv_1 - Tv_2\|^2 = \|v - \rho\tau A v\|^2$$
$$= \|v\|^2 - 2\rho\underset{\substack{\| \\ a(v,v)}}{(\tau A v, v)} + \rho^2 \underset{\substack{\| \\ a(v,\tau A v)}}{(\tau A v, \tau A v)}$$

$$\leq \|v\|^2 - 2\rho\alpha\|v\|^2 + \rho^2 C\|v\|\,\|\tau A v\|$$

$\|Av\| \leq C\|v\|$

$$\leq \left(1 - 2\rho\alpha + C\rho^2\right)\|v\|^2$$

Set $1 - 2\rho\alpha + C\rho^2 = M^2 < 1$

$$\Rightarrow \rho\left(C\rho - 2\alpha\right) < 0 \Leftrightarrow \begin{cases} \rho > 0 \\ \rho < \frac{2\alpha}{C} \end{cases}$$

$\Rightarrow$ T is a contraction mapping

$\Rightarrow \exists 1\ u \in V$ s.t. $\tau A u = \tau F$ in V. #

§2.8 Galerkin Approximation

Assumptions $(H, (\cdot,\cdot))$ is a Hilbert space

(1) V is a (closed) subspace of H

(2) $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a bilinear form

(3) $a(\cdot,\cdot)$ is continuous on V and coercive on V.

(VP) Given $F \in V'$, find $u \in V$ s.t.

$$a(u,v) = F(v) \ \forall v \in V.$$

Galerkin Approximation Let V_h be a finite dimensional subspace of V

Given $F \in V'$, find $u_h \in V_h$ s.t.

$$a(u_h, v) = F(v) \quad \forall v \in V_h.$$

the error equation $\quad a(u - u_h, v) = 0 \quad \forall v \in V_h.$

Céa's Lemma

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V .$$

§2.9 Higher-dimensional Examples

- diffusion problem

$$\begin{cases} -\nabla \cdot (A \nabla u) = f \text{ in } \Omega \\ u\big|_{\Gamma_D} = g_D \text{ and } A\nabla u \cdot \vec{n}\big|_{\Gamma_N} = g_N \end{cases}$$

- convection-diffusion-reaction problem

$$\begin{cases} -\nabla \cdot (A \nabla u) + \vec{b} \cdot \nabla u + cu = f \text{ in } \Omega \\ u\big|_{\Gamma_D} = g_D \text{ and } A\nabla u \cdot \vec{n}\big|_{\Gamma_N} = g_N . \end{cases}$$

• linear elasticity

$$\begin{cases} -\nabla\cdot\sigma = \vec{f} \quad \text{in } \Omega \\ \sigma = C_\lambda \varepsilon(\vec{u}) = 2\mu\,\varepsilon(\vec{u}) + \lambda \operatorname{tr}\varepsilon(\vec{u})\,\delta \end{cases}$$

B.C. $\quad \vec{u}\big|_{\Gamma_D} = \vec{0}$ and $\sigma\vec{n}\big|_{\Gamma_N} = \vec{g}$.

§2.10 Minimization Problems

Ciarlet - Chapter 1

Let V be a normed vector space

$a(\cdot,\cdot): V\times V \to \mathbb{R}$ is a continuous bilinear form

$f: V\to \mathbb{R}$ is a continuous linear form

and $U\in V$ is a non-empty set.

(MP) Find $u\in U$ s.t.

$$J(u) = \inf_{v\in U} J(v), \quad \text{where } J(v) = \frac{1}{2}a(v,v) - f(v).$$

Theorem 1.1.1

Assume that (1) V is complete

(2) U is a closed convex subset of V

(3) $a(\cdot,\cdot)$ is sym and coercive

$\Longrightarrow$ (MP) has one and only one solution.

<u>Proof</u> $a(\cdot,\cdot)$ is an inner product over V

$\Longrightarrow$ $(V, a(\cdot,\cdot))$ is a Hilbert space

that is equivalent to $(V, \|\cdot\|_V)$

$$\xrightarrow{\text{RRT}} f(v) = a(\tau f, v), \text{ where } \tau: V' \to V \text{ with}$$

$$\|\tau\varphi\|_V = \|\varphi\|_{V'}, \ \forall \varphi \in V'$$

$$\Longrightarrow J(v) = \frac{1}{2}a(v,v) - (\tau f, v)$$

$$= \frac{1}{2}a(v-\tau f, v-\tau f) - \frac{1}{2}a(\tau f, \tau f)$$

$$\xrightarrow{\text{Projection Thrm}} \exists 1\ u \in U \text{ s.t. } a(u-\tau f, u-\tau f) = \inf_{v \in U} a(v-\tau f, v-\tau f)$$

$$\Longrightarrow \exists 1\ u \in U \text{ s.t. } J(u) = \inf_{v \in U} J(v).$$

#

• <u>charaterization</u>

$J(u) = \inf_{v \in U} J(v) \Longleftrightarrow$

(1) U is a closed convex set

$\exists 1\ u \in U$ s.t. $a(u, v-u) \geq f(v-u) \ \forall v \in U$

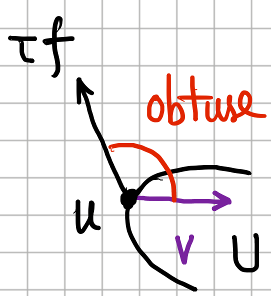
$a(u, v-u) - f(v-u) = J'(u)(v-u) \geq 0 \ \forall v \in U$

(2) U is a convex cone

$\exists 1\ u \in U$ s.t. $\begin{cases} a(u,v) \geq f(v) & \forall v \in U \\ a(u,u) = f(u) \end{cases}$

$$\begin{cases} (3)\ U \text{ is a closed subspace} \\ \exists 1\ u\in U \text{ s.t. } a(u,v)=f(v) \quad \forall v\in U. \end{cases}$$

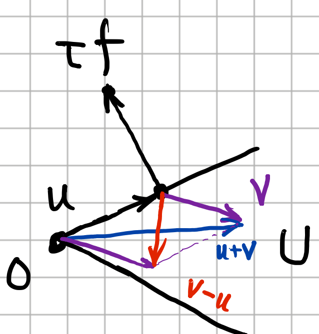
<u>Proof</u> (1) $\|u-\tau f\|_a = \inf_{v\in U} \|v-\tau f\|_a$



$$\Longleftrightarrow \quad a(\tau f-u,\ v-u) \le 0 \quad \forall v\in U$$

$$\Longleftrightarrow \quad a(u,v-u) \ge a(\tau f, v-u) = f(v-u) \quad \forall v\in U$$

(2) U is a closed convex cone with vertex 0.



$$a(u,v-u) \ge f(v-u) \xRightarrow{v=u+w} a(u,w) \ge f(w)$$

$$\Downarrow\ v=0$$

$$-a(u,u) \ge -f(u) \quad \Rightarrow \quad a(u,u)=f(u)$$

(3) U is a closed subspace

$$\begin{cases} a(u,v) \ge f(v) \quad \forall v\in U \\ a(u,u)=f(u) \end{cases} \xRightarrow{v=-w} -a(u,w) \ge -f(w) \quad \forall w\in U$$

$$\Downarrow$$

$$a(u,v) \le f(v) \quad \forall v\in U$$

$$\Longrightarrow \quad a(u,v) = f(v) \quad \forall v\in U.$$

#

<u>Projection Theorem</u> (Kinderlehrer & Stampacchia page 8-10)

Let H be a Hilbert space, U is a closed convex set of H, and $\eta\in H$.

$\Rightarrow$ " $\exists 1 \; y \in U$ s.t. $\|\eta - y\| = \inf_{x \in U} \|\eta - x\|$

$\iff (y, x-y) \geq (\eta, x-y) \quad \forall x \in U.$ "

<u>Proof</u> " $\Rightarrow$ " $\forall x \in U, \; \forall t \in [0,1]$

$\varphi(t) = \|\eta - ((1-t)y + tx)\|^2$ attains its minimum at $t=0$.

$= \|(\eta - y) - t(x-y)\|^2$

$= \|\eta - y\|^2 - 2t(\eta - y, x-y) + t^2\|x-y\|^2$

$\Rightarrow 0 \overset{?}{\leq} \varphi'(0) = -2(\eta - y, x-y) \Rightarrow (\eta - y, x-y) \leq 0 \quad \forall x \in U.$

" $\Leftarrow$ " $0 \leq (y-\eta, x-y) = (y-\eta, (x-\eta) + (\eta - y))$

$= -\|\eta - y\|^2 + (y-\eta, x-\eta)$

$\Rightarrow \|\eta - y\|^2 \leq (y-\eta, x-\eta) \leq \|y-\eta\| \, \|x-\eta\|$

$\Rightarrow \|\eta - y\| \leq \|x-\eta\|$

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