Chapter 3 Finite Element Spaces of $H^1(\Omega)$

- **Definition** $\mathcal{T} = \{K_i\}$ is a subdivision of the domain Ω

$\Longleftrightarrow$ (1) K_i are open set

(2) $K_i \cap K_j = \emptyset$ if $i \neq j$

(3) $\bigcup_i \overline{K}_i = \overline{\Omega}$.

- **global smoothness**

Theorem Let Ω be a bounded domain in R^d, and $v \in C^\infty(\mathcal{T})$.

$\Longrightarrow$ "$v \in H^k(\Omega) \Longleftrightarrow v \in C^{k-1}(\Omega)$", where $k \geq 1$.

Proof $(k=1)$ For a given $v \in C^\infty(\mathcal{T})$, $\forall \varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \frac{\partial v}{\partial x_1} \varphi \, dx = \sum_K \int_K \frac{\partial v}{\partial x_1} \varphi \, dx$$

$$= -\int_\Omega v \frac{\partial \varphi}{\partial x_1} dx + \sum_K \int_{\partial K} \varphi v n_1 \, ds$$

$v \in H^1(\Omega) \Longleftrightarrow 0 = \sum_K \int_{\partial K} \varphi v n_1 \, ds = \sum_F \int_F \varphi [\![v]\!]_F n_1 \, ds$

$\Longleftrightarrow [\![v]\!]_F = 0 \quad \forall$ interior $F \Longleftrightarrow v \in C^0(\overline{\Omega})$

#

Definition (Triangulation) $\qquad \Omega \subset \mathbb{R}^2$

$\mathcal{T} = \{K\}$ is a triangulation of a polygonal domain

$\Longleftrightarrow$ (1) $\mathcal{T}$ is a subdivision of Ω,

(2) $\forall K \in \mathcal{T}$, K is a either triangle or rectangle,

(3) no vertex of any $K \in \mathcal{T}$ lies in the interior of an edge of $K' \in \mathcal{T}$.

§3.1 The Finite Element

Definition (finite element)

K — element domain

$\mathcal{P}$ — shape functions

$\mathcal{N}$ — nodal variables

$(K, \mathcal{P}, \mathcal{N})$ is a finite element

$\Longleftrightarrow$ (1) $K \subset \mathbb{R}^n$ is a bounded closed set with nonempty interior

(2) $\mathcal{P}$ is a finite-dimensional space of functions on K

(3) $\mathcal{N} = \{N_1, \cdots, N_k\}$ is a basis for $\mathcal{P}'$.

Definition (nodal basis)

$\{\varphi_1, \cdots, \varphi_k\}$ is the nodal basis of $\mathcal{P}$

$\Longleftrightarrow$ (1) $\{\varphi_1, \cdots, \varphi_k\}$ is a basis of $\mathcal{P}$

(2) $\{\varphi_1, \cdots, \varphi_k\}$ is dual to $\mathcal{N}$, i.e., $N_i(\varphi_j) = \delta_{ij}$

Example (1d Lagrange element)

$K=[a,b]$, $\mathcal{P}=P_k(K)$, $\mathcal{N}_k=\{N_0, N_1, \cdots, N_k\}$

$$N_i(v)=v\left(a+i\,\frac{b-a}{k}\right) \text{ for } i=0,1,\cdots,k$$

k=1 $N_0(v)=v(a)$, $N_1(v)=v(b)$

k=2 $N_0(v)=v(a)$, $N_1(v)=v\left(\frac{a+b}{2}\right)$, $N_2(v)=v(b)$

Lemma (3.1.4)

Let $\mathcal{P}$ be a d-dimensional vector space and let $\{N_1,\cdots,N_d\}$ be a subset of the dual space $\mathcal{P}'$.

"$\{N_1,\cdots,N_d\}$ is a basis for $\mathcal{P}'$" $\Longleftrightarrow$ "Given $v\in\mathcal{P}$ with $N_i v=0$ for $i=1,\cdots,d$ $\Longrightarrow v\equiv 0$"

$\Updownarrow$

$\forall\, L\in\mathcal{P}'$, $\exists\, \alpha_i$ s.t. $L=\sum_{i=1}^{d}\alpha_i N_i$

Proof Let $\{\varphi_1,\cdots,\varphi_d\}$ be a basis of $\mathcal{P}$.

"$\Longrightarrow$" $v\in\mathcal{P} \Longrightarrow v=\sum_{j=1}^{d}\beta_j\varphi_j \xrightarrow{N_i v=0} 0=N_i(v)=\sum_{j=1}^{d}\beta_j N_i(\varphi_j)$

$$\Longrightarrow C\vec{\beta}=\vec{0}, \text{ where } C=\left(N_i(\varphi_j)\right)_{d\times d}$$

$$\Longrightarrow \text{"}\vec{\beta}=\vec{0}\text{"} \Longleftrightarrow C \text{ is invertible}$$

" $\Leftarrow$ " Let $y_j = L(\varphi_j) = \sum_{i=1}^{d} \alpha_i N_i(\varphi_j)$ for $j = 1, \cdots, d$

$\Leftrightarrow \vec{y} = B\vec{\alpha}$, where $B = \left(N_i(\varphi_j)\right)_{d \times d} = C^t$

$\Rightarrow$ " $\exists \vec{\alpha} \Leftrightarrow B$ is invertible "

<u>Definition (3.1.8)</u>

$\mathcal{N}$ determines $\mathcal{P} \Leftrightarrow$ " $\psi \in \mathcal{P}$ with $N(\psi) = 0 \; \forall N \in \mathcal{P}'$

$\Rightarrow \psi \equiv 0$ "

<u>Lemma (3.1.10)</u>

Let P be a polynomial of degree $k \geq 1$ that vanishes on a hyperplane L.

$\Rightarrow P = LQ$, where Q is a polynomial of degree $(k-1)$.

<u>Proof</u> hyperplane L : $\left\{ \vec{x} \in R^n \,\middle|\, L(\vec{x}) = 0 \right\}$

§3.2 Triangular Finite Elements

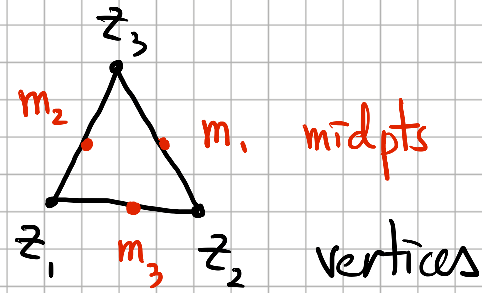
K is a triangle, $\mathcal{P}_k = \{$ poly. in 2 variables of degree $\leq k \}$

$\dim \mathcal{P}_k = \frac{1}{2}(k+1)(k+2)$

The Lagrange Element

$\underline{k=1}$ $\quad \mathcal{P}_1 = \text{span}\{1, x, y\}$

$$\mathcal{N}_1 = \{N_1, N_2, N_3\}$$



• <u>linear Lagrange element</u>

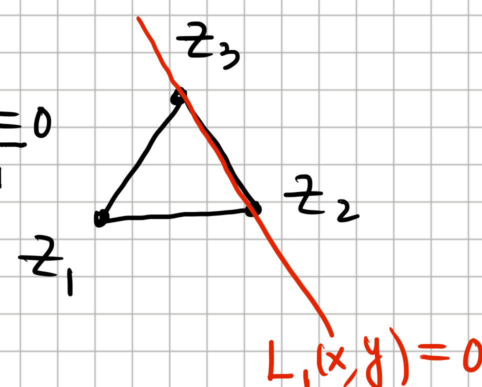
$$N_i(v) = z_i \quad \text{for } i = 1, 2, 3.$$

(1) <u>$\mathcal{N}_1$ determines $\mathcal{P}_1$</u>

<u>Lemma</u> $\forall\, v \in \mathcal{P}_1,\ N_i(v) = v(z_i) = 0 \Longrightarrow v(x,y) \equiv 0$ on K.

<u>Proof</u>

$$\left.\begin{array}{l} v\big|_{L_1} = a + bs \text{ is linear} \\ v(z_2) = v(z_3) = 0 \end{array}\right\} \Rightarrow v\big|_{L_1} \equiv 0$$

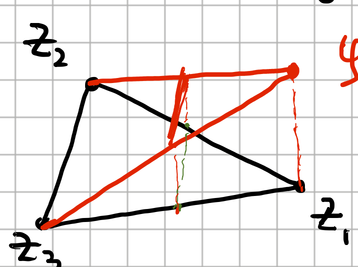


$$\Longrightarrow v(x,y) = c\, L_1(x,y)$$

$$\xrightarrow{v(z_1)=0} c = 0 \Longrightarrow v \equiv 0 \text{ on } K. \qquad \#$$

(2) <u>nodal basis functions</u> $\{\varphi_i\}_{i=1}^{3}$

$$N_j(\varphi_i) = \varphi_i(z_j) = \delta_{ij}$$



$\varphi_1(x,y) = \lambda_1(x,y)$

$$\varphi_1(z_2) = \varphi_1(z_3) = 0 \Longrightarrow \varphi_1\big|_{L_1} \equiv 0$$

$$\Longrightarrow \varphi_1 = c\, L_1(x,y)$$

$$\xrightarrow{\varphi_1(z_1)=1} \quad \varphi_1(x,y) = \frac{L_1(x,y)}{L_1(z_1)} \equiv \lambda_1(x,y)$$

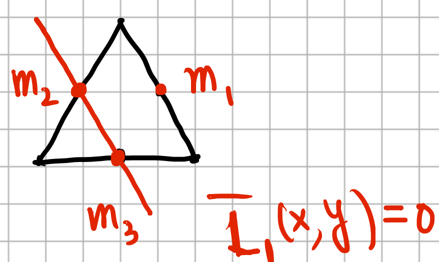
<u>Barycentric center</u> $\quad \lambda_i(z_j) = \delta_{ij} \Longrightarrow \lambda_i(x,y) = L_i(x,y)/L_i(z_i)$

<u>Partition of unity</u> $\quad \lambda_1(x,y) + \lambda_2(x,y) + \lambda_3(x,y) \equiv 1$ on K.

(3) global smoothness $C^0(\bar{\Omega})$

- <u>Crouzeix-Raviart element</u>

$$N_i(v) = v(m_i)$$



(1) $\mathcal{N}_1$ determines $\mathcal{P}_1$

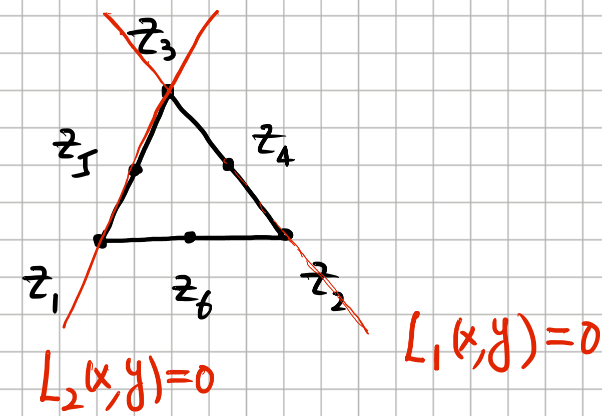
(2) nodal basis function $\varphi_i(x,y) = \dfrac{\bar{L}_i(x,y)}{\bar{L}_i(m_i)}$

<u>$k=2$</u> (quadratic Lagrange element)

$$\mathcal{P}_2 = \text{span}\{1, x, y, x^2, xy, y^2\}$$

$$\mathcal{N}_2 = \{N_1, N_2, \cdots, N_6\}$$

$$N_i(v) = v(z_i)$$



(1) $\mathcal{N}_2$ determines $\mathcal{P}_2$

" $\forall v \in \mathcal{P}_2$, $v(z_i)=0 \Longrightarrow v(x,y)\equiv 0$ on K "

Proof

$$\left.\begin{array}{l} v|_{L_1} = a+bs+cs^2 \\ v(z_2)=v(z_3)=v(z_4)=0 \end{array}\right\} \Longrightarrow v|_{L_1}\equiv 0 \Longrightarrow v = Q_1(x,y)\,L_1(x,y)$$

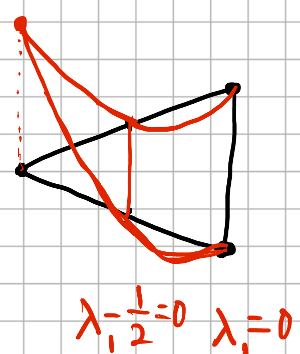
$$Q_1 \in \mathcal{P}_1(K)$$

$v|_{L_2} \equiv 0 = L_1(x,y)Q_1(x,y) \Longrightarrow Q_1(x,y)|_{L_2} = 0$ except possible at z_3

$$\Longrightarrow Q_1(x,y)|_{L_2} \equiv 0 \Longrightarrow Q_1(x,y) = c\,L_2(x,y)$$

$$\Longrightarrow v = c\,L_1(x,y)\,L_2(x,y) \xrightarrow{v(z_6)=0} c=0 \Longrightarrow v(x,y)\equiv 0 \text{ on } K. \quad \#$$

(2) nodal basis functions

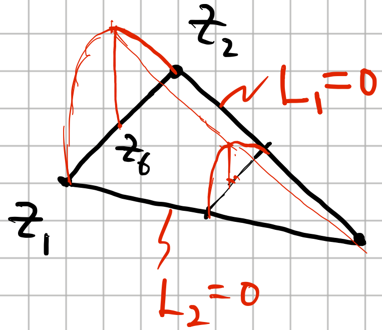


$$\varphi_1(x,y) = c\,\lambda_1(x,y)\left(2\lambda_1(x,y)-1\right)$$

$$\varphi_1(z_1) = 1$$

$$\Longrightarrow \varphi_1(x,y) = \lambda_1(2\lambda_1-1)$$

$$\varphi_2(x,y) = \lambda_2(2\lambda_2-1), \quad \varphi_3(x,y) = \lambda_3(2\lambda_3-1)$$



$$\left.\begin{aligned} \varphi_6 &= c_1\lambda_1\lambda_2 \\ \varphi_6(z_6) &= 1 = c\lambda_1(z_6)\lambda_2(z_6) = \frac{c}{4} \end{aligned}\right\} \Rightarrow \varphi_6(x,y) = 4\lambda_1\lambda_2$$

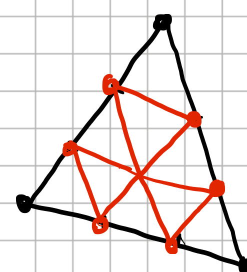
$$\varphi_5 = 4\lambda_2\lambda_3, \quad \varphi_6 = 4\lambda_1\lambda_3$$

<u>$k=3$</u> (cubic Lagrange element)

$$\mathcal{P}_3 = \mathrm{span}\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$$

$$\mathcal{N}_3 = \{N_1, \dots, N_{10}\}$$

$$N_i(v) = v(z_i)$$



3 vertices

6 edge nodes

1 interior node

<u>$\mathcal{N}_3$ determines $\mathcal{P}_3$</u>

$$v(z_i) = 0 \Longrightarrow v = cL_1L_2L_3 \xRightarrow{v(z_{10})=0} v(x,y) \equiv 0 \text{ on } K.$$

<u>general k</u>

$$\mathcal{P}_k = \{\text{polynomials of degree} \le k\}$$

$$\mathcal{N}_k = \{N_1, \dots, N_{m(k)}\}, \quad m(k) = \dim \mathcal{P}_k = \frac{1}{2}(k+2)(k+1)$$

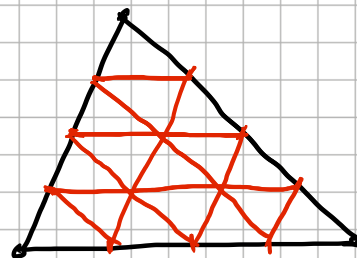
$N_i(v) = v(z_i)$ — nodal values

3 vertices

$3(k-1)$ edge nodes

$\dim \mathcal{P}_{k-3} = \frac{1}{2}(k-2)(k-1)$ interior nodes

$$\dim \mathcal{P}_k = \frac{1}{2}(k+2)(k+1)$$



<u>$\mathcal{N}_k$ determines $\mathcal{P}_k$</u>

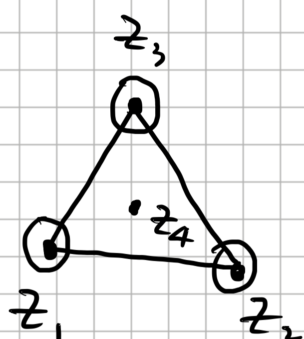
$$v = L_1 L_2 L_3 Q, \quad Q \in \mathcal{P}_{k-3}$$

$\Longrightarrow$ $Q = 0$ at all interior nodes

$\Longrightarrow$ $Q \equiv 0$ $\Longrightarrow$ $v \equiv 0$ on K. #

The Hermite Element

$\underline{k=3}$ (cubic Hermite) $\mathcal{P} = \mathcal{P}_3$

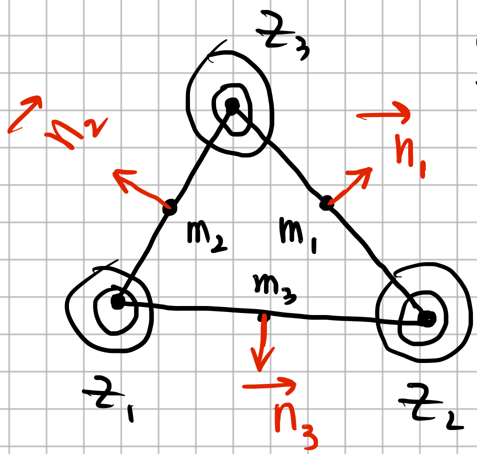


$\mathcal{N}$: • nodal value $\begin{cases} \text{vertices} & 3 \\ \text{interior} & 1 \end{cases}$

∘ gradient at vertices $2 \times 3 = 6$

$$\forall v \in \mathcal{P}_3, \; v\big|_{L_i} \equiv 0 \Longrightarrow v = c_1 L_1 L_2 L_3 \xrightarrow{v(z_4)=0} c = 0 \Rightarrow v \equiv 0 \text{ on } K.$$

The Argyris Element $C^1(\Omega)$



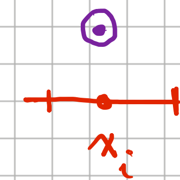
$\mathcal{P} = \mathcal{P}_5$

$\dim \mathcal{P}_5 = 21$

$\mathcal{N}$:
- • value 3×1
- ◦ gradient 3×2
- ◎ 2^{nd}-order der. 3×3
- → normal der. 3×1

21

$C^1(I)$ element in $\mathbb{R}^1$



$p(x_i^-) = p(x_i^+)$

$p'(x_i^-) = p'(x_i^+)$

$\Rightarrow$ 4 at each $[x_{i-1}, x_i]$

$\Rightarrow$ cubic polynomial

$\mathcal{N}$ determines $\mathcal{P}$

Proof $\forall v \in \mathcal{P}_5,\ v|_{L_i} \equiv 0 \Longrightarrow v = L_1 L_2 L_3 Q,\quad Q \in \mathcal{P}_2$

$$0 = \partial_{L_1}\left(\partial_{L_2} v\right)(z_3) = Q(z_3)\, L_3(z_3) \frac{\partial L_1}{\partial L_2}\cdot\frac{\partial L_2}{\partial L_1} \Longrightarrow \boxed{Q(z_3) = 0} \Longrightarrow \begin{cases} Q(z_1) = 0 \\ Q(z_2) = 0 \end{cases}$$

$\dfrac{\partial L_i(x,y)}{\partial L_i} \equiv 0$ since $L_i(x,y) = $ constant along the L_i

$L_1(z_3) = 0 = L_2(z_3)$, $L_3(z_3) \neq 0$, $\dfrac{\partial L_1}{\partial L_2} \neq 0$, $\dfrac{\partial L_2}{\partial L_1} \neq 0$

$$\frac{\partial v}{\partial L_2} = L_2 \frac{\partial}{\partial L_2}(L_1 L_3 Q) \Longrightarrow \left.\frac{\partial^2 v}{\partial L_1 \partial L_2}\right|_{z_3} = \frac{\partial L_2}{\partial L_1}\left.\frac{\partial}{\partial L_2}(L_1 L_3 Q)\right|_{z_3} + \underbrace{L_2}_{0}\left.\frac{\partial^2}{\partial L_1 \partial L_2}(L_1 L_3 Q)\right|_{z_3}$$

$$\frac{\partial}{\partial L_2}(L_1 L_3 Q)(z_3) = L_3 Q \left.\frac{\partial L_1}{\partial L_2}\right|_{z_3}$$

$$\xrightarrow{L_1(m_1) = 0}\quad 0 = \frac{\partial v}{\partial \vec{n}_1}(m_1) = \frac{\partial L_1}{\partial \vec{n}_1} L_2(m_1) L_3(m_1) Q(m_1) \Longrightarrow Q(m_1) = 0$$

$$\Longrightarrow Q(m_2) = Q(m_3) = 0$$

$$\Longrightarrow Q \equiv 0 \longrightarrow v \equiv 0 \text{ on } K.$$

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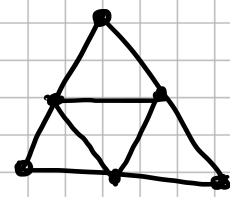
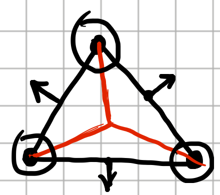
The Composite Elements

(1) C^0 macro piecewise linear element



(2) Hsieh - Clough - Tocher element $C^1(\Omega)$

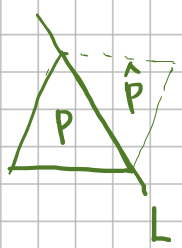


$T = \bigcup_{i=1}^{3} K_i$

$v|_{K_i} \in \mathcal{P}_3$ piecewise cubic

- • nodal values at vertices 3
- o gradient at vertices 2×3
- → normal der. at midpts 3

dim = 12



$p \in \mathcal{P}_3$

$p|_L = \hat{p}|_L \in P_3^1 - 4$

$\frac{\partial p}{\partial n}\Big|_L = \frac{\partial \hat{p}}{\partial n}\Big|_L \in P_2^1 - 3$

7 constraints

motivation

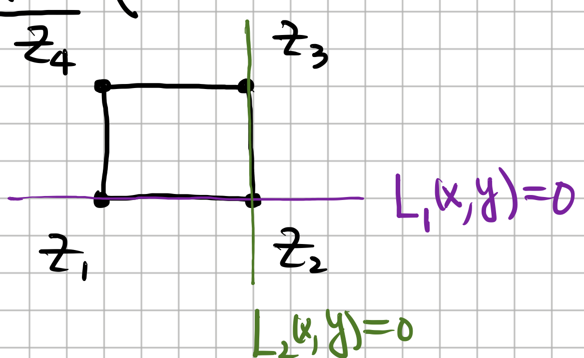
$v(x,y) = \left(\sum_i v_x^i \varphi_i(x)\right)\left(\sum_j v_y^j \psi_j(y)\right)$

$= \sum_{i,j} v_x^i v_y^j \varphi_i(x)\psi_j(y)$

Rectangular Element (tensor product element)

$$Q_k = \left\{ p(x)q(y) \;\middle|\; p, q \in \mathcal{P}_k^1 \right\},$$ where $\mathcal{P}_k^1$ — poly. of degree $\leq k$ of one variable

$k=1$ (bi-linear element)



$Q_1 = \text{span}\{1, x, y, xy\}$

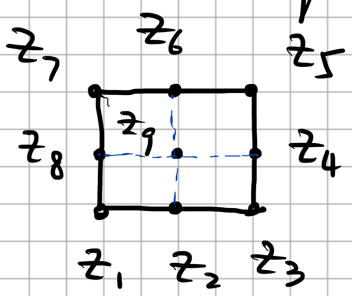
$\mathcal{N}_1$ — nodal values at vertices

$\mathcal{N}_1$ determines Q_1

$v(z_i) = 0 \ \forall v \in Q_1$

$\Rightarrow v = c L_1 L_2 \xrightarrow{v(z_4)=0} c = 0 \Rightarrow v \equiv 0$ on K.

$k=2$ (bi-quadratic element)



$$Q_2 = \text{span}\left\{1, x, y, xy, x^2, y^2, x^2y, xy^2, x^2y^2\right\}$$

$\mathcal{N}_2$ – nodal values at z_i.

Lemma $v \in \mathcal{P}_2 \Longrightarrow \exists\, c_i$ s.t. $v(z_9) = \sum_{i=1}^{8} c_i v(z_i)$

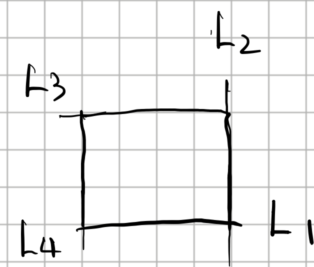
Proof $\forall v \in \mathcal{P}_2$, $v(x,y) = \sum_{i=1}^{6} v(z_i)\varphi_i(x,y) \Rightarrow v(z_9) = \sum_{i=1}^{8} c_i v(z_i)$ #

$c_i = \varphi_i(z_9)$, $i=1,\cdots,6$, $c_7 = c_8 = 0$

The Serendipity element

$$\mathcal{P} = \left\{ v \in Q_2 \;\middle|\; \sum_{i=1}^{8} c_i v(z_i) - v(z_9) = 0 \right\}$$

$\mathcal{N}$ = nodal values at z_i for $i=1,\cdots,8$



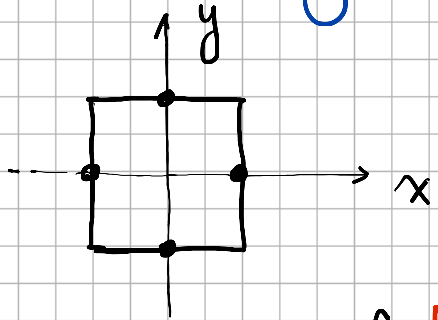
$\mathcal{N}$ determines $\mathcal{P}$

$$v(z_i) = 0 \text{ for } i=1,\cdots,8 \Longrightarrow v = c L_1 L_2 L_3 L_4$$

$$\Rightarrow 0 = \sum_{i=1}^{8} c_i v(z_i) = v(z_9) = c\, L_1(z_9) L_2(z_9) L_3(z_9) L_4(z_9)$$

$$\Rightarrow c = 0 \Rightarrow v \equiv 0 \text{ on } K$$ #

The Rectangular non-conforming element

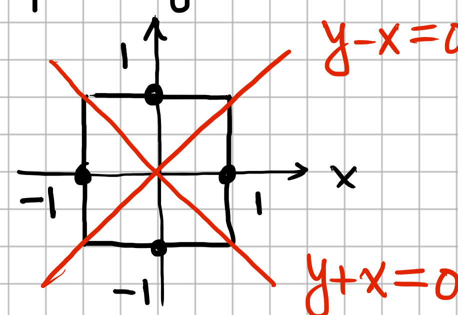


$\mathcal{P} = Q_1 = \mathrm{span}\{1, x, y, xy\}$

$\mathcal{N}$ — nodal values at midpts

$\mathcal{N}$ does not determine Q_1

Proof xy vanishes at all 4 midpts.



$$\mathcal{P} = \{1, x, y, (y-x)(y+x)\}.$$

$\mathcal{N}$ determines $\bar{\mathcal{P}}$

Proof Let $v(x,y) = c_0 + c_1 x + c_2 y + c_3 (y^2 - x^2)$

at $(\pm 1, 0)$ $\quad 0 = v(\pm 1, 0) = c_0 \pm c_1 - c_3 \Longrightarrow \begin{cases} c_0 = c_3 \\ c_1 = 0 \end{cases}$

at $(0, \pm 1)$ $\quad 0 = v(0, \pm 1) = c_0 \pm c_2 + c_3 \Longrightarrow \begin{cases} c_0 = -c_3 \\ c_2 = 0 \end{cases}$

$\Longrightarrow c_i = 0 \Rightarrow v \equiv 0$ on K. #

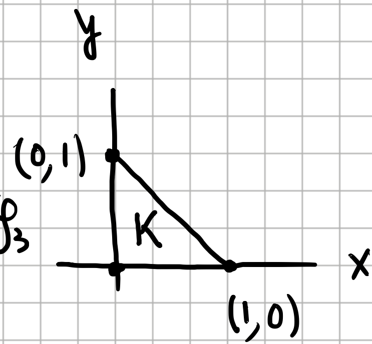
§3.3 The Interpolant

- local interpolant for finite element $(K, \mathcal{P}, \mathcal{N})$

$$I_K v(x) = \sum_{i=1}^{k} N_i(v)\, \varphi_i(x), \qquad x \in K$$

where $\mathcal{P} = \text{span}\{\varphi_i\}_{i=1}^{k}$.

e.g. $f(x,y) = e^{xy}$, $\mathcal{P}_1 = \text{span}\{\varphi_i\}_{i=1}^{3}$



$$I_K f = f(0,0)\varphi_1(x,y) + f(1,0)\varphi_2 + f(0,1)\varphi_3$$

$$= (1-x-y) + x + y = 1$$

Properties

(1) I_K is linear

(2) $N_i(I_K f) = N_i(f)$ for $i = 1, \cdots, k$

$$\overset{\|}{N_i}\left(\sum_j N_j(f)\varphi_j\right) = \sum_j N_j(f)\, N_i(\varphi_j) = N_i(f)$$

(3) $I_K(f) = f \quad \forall f \in \mathcal{P}$. $\Longrightarrow I_K^2 = I_K$, i.e., I_K is a projection

- global interpolant

$$I_{\mathcal{T}} f\big|_K = I_K f \qquad \forall K \in \mathcal{T}.$$

§3.4 Neural Nets

$$\mathcal{M}_N(\vec{\theta}) = \left\{ c_0 + \sum_i c_i \, \sigma_k\left(\vec{\omega}_i \cdot \vec{x} - b_i\right) : \ c_i, b_i \in R, \ \vec{\omega}_i \in S^{d-1} \right\}$$

$$\sigma_k(t) = \max\left\{0, t^k\right\} = \begin{cases} t^k, & t>0 \\ 0, & t<0. \end{cases}$$