Chapter 4 Approximation Theory

- **finite element space**

$$S_h = \left\{ v \in C^0(\bar{\Omega}) \;\middle|\; v|_K \in \mathcal{P}_k(K),\ \forall K \in \mathcal{T}_h \right\}$$

- **local interpolant**

$$I_K v = \sum_{i=0}^{N_k} N_i(v)\, \varphi_i(x) = \sum_i v(a_i)\, \varphi_i(x)$$

properties

(1) I_K is linear and (2) $I_K^2 = I_K$.

- **global interpolant** for $v \in C^0(\bar{\Omega})$

$$I_h v\big|_K = I_K v \quad \forall K \in \mathcal{T}_h.$$

- **assumption** K and $\hat{K}$ are affine equivalent.

$\Longleftrightarrow$ $\exists$ affine map $F_K : \hat{x} \in R^d \to x \in R^d$, i.e.,

$$x = F_K(\hat{x}) = B_K \hat{x} + b_K$$

s.t. $K = F_K(\hat{K})$, where B_K is non-singular.

<u>Lemma 1</u> $\forall\, v \in H^m(K)$ with $m \geq 0$

define $\hat{v} = v \circ F_K \qquad \left(\hat{v}(\hat{x}) = v\left(F_K(\hat{x})\right) = v(x) \right)$

$\Longrightarrow \hat{v} \in H^m(\hat{K})$ and $\exists\, C(m,d)$ s.t.

$$\text{(i)} \quad |\hat{v}|_{m,\hat{K}} \leq C \|B_K\|^m \left|\det(B_K)\right|^{-1/2} |v|_{m,K}$$

where $$\|B_K\| = \sup_{x \in \mathbb{R}^d} \frac{\|B_K x\|_{l_2}}{\|x\|_{l_2}}$$

similarly, $$\text{(ii)} \quad |v|_{m,K} \leq C \|B_K^{-1}\|^m \left|\det(B_K)\right|^{\frac{1}{2}} |\hat{v}|_{m,\hat{K}}, \quad \forall\, \hat{v} \in H^m(\hat{K}).$$

<u>Proof</u> $C^\infty(K)$ is dense in $H^m(K)$ $\Longrightarrow$ it suffices to prove (i) $\forall\, v \in C^\infty(K)$.

<u>m=1</u>

$$|\hat{v}|^2_{1,\hat{K}} = \int_{\hat{K}} \|\hat{\nabla}\hat{v}\|^2_{l_2} \, d\hat{x}$$

$$= \int_K \|B_K \nabla v\|^2_{l_2} \left|\det B_K^{-1}\right| dx$$

$$\leq \|B_K\|^2 \left|\det B_K\right|^{-1} |v|^2_{1,K}$$

$$\hat{v}(\hat{x}) = v(F_K(\hat{x})) = v(x)$$

$$\hat{\nabla}\hat{v} = \begin{pmatrix} \frac{\partial \hat{v}}{\partial \hat{x}_1} \\ \frac{\partial \hat{v}}{\partial \hat{x}_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial \hat{v}}{\partial x_1} \cdot \frac{\partial x_1}{\partial \hat{x}_1} + \frac{\partial \hat{v}}{\partial x_2} \cdot \frac{\partial x_2}{\partial \hat{x}_1} \\ \frac{\partial \hat{v}}{\partial x_1} \cdot \frac{\partial x_1}{\partial \hat{x}_2} + \frac{\partial \hat{v}}{\partial x_2} \cdot \frac{\partial x_2}{\partial \hat{x}_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial x_1}{\partial \hat{x}_1} & \frac{\partial x_2}{\partial \hat{x}_1} \\ \frac{\partial x_1}{\partial \hat{x}_2} & \frac{\partial x_2}{\partial \hat{x}_2} \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{v}}{\partial x_1} \\ \frac{\partial \hat{v}}{\partial x_2} \end{pmatrix} = \frac{\partial(x_1, x_2)}{\partial(\hat{x}_1, \hat{x}_2)} \nabla v = B_K \nabla v$$

$$d\hat{x} = \left| \det \frac{\partial(\hat{x}_1, \hat{x}_2)}{\partial(x_1, x_2)} \right| dx = \left|\det B_K^{-1}\right| dx$$

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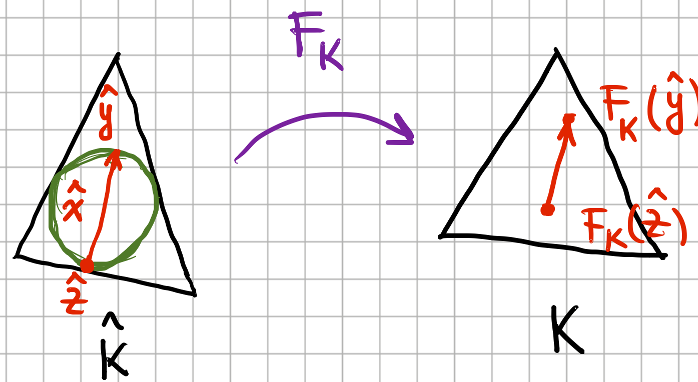
Lemma 2 Assume that K and $\hat{K}$ are affine equivalent.

$$\Longrightarrow \|B_K\| \leq \frac{h_K}{\rho_{\hat{K}}} \quad \text{and} \quad \|B_K^{-1}\| \leq \frac{h_{\hat{K}}}{\rho_K},$$

where $h_K = \operatorname{diam}(K)$ and $\rho_K = \sup\{\operatorname{diam}(S) \mid S \text{ is a ball contained in } K\}$.

Proof For $\hat{x} \in \mathbb{R}^d$ with $\|\hat{x}\|_{l_2} = \rho_{\hat{K}}$

$\Longrightarrow \exists\, \hat{y}, \hat{z} \in \hat{K}$, s.t. $\hat{x} = \hat{y} - \hat{z}$



$$\Longrightarrow \|B_K\| = \sup_{\hat{x} \in \mathbb{R}^d} \frac{\|B_K \hat{x}\|}{\|\hat{x}\|}$$

$$= \sup \frac{\|B_K(\hat{y} - \hat{z})\|}{\rho_{\hat{K}}} = \sup \frac{\|\overbrace{F_K(\hat{y})}^{\in K} - \overbrace{F_K(\hat{z})}^{\in K}\|}{\rho_{\hat{K}}} \leq \frac{h_K}{\rho_K}. \qquad \#$$

Theorem For any $v \in H^{k+1}(\Omega)$ with $k \geq 1$, $\exists$ const. $C = C(\hat{K}, I_{\hat{K}}, k, m, d)$ s.t.

$$|v - I_K v|_{m,K} \leq C \frac{h_K^{k+1}}{\rho_K^m} |v|_{k+1,K} \quad (0 \leq m \leq k+1).$$

Proof $H^{k+1}(K) \hookrightarrow C^0(\bar{K}) \Longrightarrow I_K$ is well defined in $H^{k+1}(K)$.

$$\widehat{v - I_K v} = (v - I_K v) \circ F_K(\hat{x}) = \hat{v} - \widehat{I_K v} = \hat{v} - \sum_i v(a_i)\, \hat{\varphi}_i(x)$$

$$= \hat{v} - \sum_i v(F_K(\hat{a}_i))\, \hat{\varphi}_i$$

$$= \hat{v} - I_{\hat{K}} \hat{v}$$

$$|v - I_K v|_{m,K} \leq C \|B_K^{-1}\|^m |\det B_K|^{1/2} \, |\hat{v} - I_{\hat{K}}\hat{v}|_{m,\hat{K}}$$

$$\| \qquad |(I - I_{\hat{K}})(\hat{v}+\hat{p})|_{m,\hat{K}} \quad \forall \hat{p} \in \mathcal{P}_k$$

$$\lesssim \|B_K^{-1}\|^m |\det B_K|^{\frac{1}{2}} \|I - I_{\hat{K}}\|_{\mathcal{L}(H^{k+1}(\hat{K}), H^m(\hat{K}))} \inf_{\hat{p}\in\mathcal{P}_k} \|\hat{v}+\hat{p}\|_{k+1,\hat{K}}$$

(Deny-Lions lemma)

$$\leq C(I_{\hat{K}}, \hat{K}) \|B_K^{-1}\|^m |\det B_K|^{\frac{1}{2}} |\hat{v}|_{k+1,\hat{K}}$$

$$\lesssim \|B_K^{-1}\|^m \|B_K\|^{k+1} |v|_{k+1,K}$$

$$\lesssim \left(\frac{h_{\hat{K}}}{\rho_K}\right)^m \left(\frac{h_K}{\rho_{\hat{K}}}\right)^{k+1} |v|_{k+1,K} \lesssim \frac{h_K^{k+1}}{\rho_K^m} |v|_{k+1,K}$$

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<u>Remark</u> (1) $\forall\, v \in H^{l+1}(\Omega)$ with $1 \leq l \leq k$

$$|v - I_K v|_{m,K} \lesssim \frac{h_K^{l+1}}{\rho_K^m} |v|_{l+1,K} \quad \text{for } m = 0, 1, \cdots, l+1.$$

(2) $\forall\, v \in W_p^{l+1}(K)$ with $1 \leq l \leq k$ and $p \in [1, +\infty)$

$$|v - I_K v|_{m,p,K} \lesssim \frac{h_K^{l+1}}{\rho_K^m} |v|_{l+1,p,K} \quad \text{for } m = 0, 1, \cdots, l+1.$$

(3) $1 \leq l \leq k$

$$|v - I_K v|_{m,\infty,K} \lesssim \begin{cases} \dfrac{h_K^{l+1}}{\rho_K^m} |K|^{-\frac{1}{2}} |v|_{l+1,K}, & 0 \leq m < l+1-\frac{d}{2} \\[2ex] \dfrac{h_K^{l+1}}{\rho_K^m} |v|_{l+1,\infty,K}, & 0 \leq m \leq l+1. \end{cases}$$

Definition (Regular Triangulation)

A family of triangulation $\{\mathcal{T}_h\}_{h>0}$ is regular

$$\Longleftrightarrow \quad \max_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K} \le \sigma \quad \forall\, h>0.$$

Theorem Let $\mathcal{T}_h$ be regular and $m=0,1$; $k\ge 1$.

$\forall\, v\in H^{l+1}(\Omega)$ with $1\le l\le k$

$$\Longrightarrow \left|v - I_K v\right|_{m,\Omega} \le C h^{l+1-m} |v|_{l+1,\Omega}, \quad h = \max_{K\in\mathcal{T}_h} h_K.$$

Inverse Estimate

Theorem Let $\mathcal{T}_h$ be a regular triangulation of Ω.

$\forall\, v\in S_h$, $\forall\, 0\le m\le t$, $\exists\, C>0$, s.t.

$$|v|_{t,\Omega} \le C h^{m-t} \|v\|_m.$$

Clément-type Interpolation $\quad S_h = \{v\in C^0(\bar\Omega) \mid v|_K\in \mathcal{P}_1(K),\ \forall K\in\mathcal{T}_h\} = \operatorname{span}\{\varphi_i\}$

$\forall\, v\in L^2(\Omega)$, $\quad I_h v = \sum_i v_i \varphi_i(x)$, $\quad v_i = \frac{1}{|\omega_i|}\int_{\omega_i} v\,dx$

$\omega_i = \operatorname{suppt}\{\varphi_i\}$.

Deny-Lions Lemma $\quad \forall v \in H^{k+1}(K),\ \exists C(k,K)$ s.t.

$$\inf_{p \in \mathcal{P}_k} \|v+p\|_{k+1,K} \leq C\,|v|_{k+1,K}.$$

Proof For a fixed $v \in H^{k+1}(K)$, $\exists 1\ q \in \mathcal{P}_k$ s.t.

$$\int_K D^\alpha q\, dx = -\int_K D^\alpha v\, dx \quad \forall\ |\alpha| \leq k$$

since $\dim \mathcal{P}_k = \#\{\alpha = (\alpha_1, \cdots, \alpha_d) \mid \alpha_i - \text{integer},\ |\alpha| \leq k\}$

Now, it suffices to prove that $\exists\ C > 0$ s.t.

$$\|v\|_{k+1,K} \leq C \left(|v|^2_{k+1,K} + \sum_{|\alpha| \leq k} \left(\int_K D^\alpha v\, dx \right)^2 \right)^{\frac{1}{2}} \quad \forall\ v \in H^{k+1}(K). \quad (*)$$

$$\Rightarrow \inf_{p \in \mathcal{P}_k} \|v+p\|_{k+1,K} \leq \|v+q\|_{k+1,K} \lesssim \left(|v+q|^2_{k+1,K} + 0 \right)^{\frac{1}{2}} = |v|_{k+1,K}.$$

Proof of $(*)$ (compactness argument)

Assume that $(*)$ is wrong.

$$\Rightarrow \exists \{v_j\}_{j=1}^{\infty} \subset H^{k+1}(\Omega) \text{ s.t. } \begin{cases} \|v_j\|_{k+1,K} = 1 \\ \text{but } |v_j|^2_{k+1,K} + \sum_{|\alpha| \leq k} \left(\int_K D^\alpha v_j\, dx \right)^2 < \frac{1}{j}. \end{cases}$$

[$X \hookrightarrow Y$ is compact : (1) $X \subset Y$

(2) $\|v\|_Y \leq c\,\|v\|_X$

(3) any bounded set in X is precompact in Y

(every bounded sequence in X has a subsequence that is Cauchy in Y.)

$$\left\{\begin{array}{c} H^{k+1}(\Omega) \hookrightarrow H^{k}(\Omega) \\ \|v_j\|_{k+1,K} = 1 \end{array}\right\} \Rightarrow \exists \{v_{j_\ell}\} \text{ that converges in } H^k(\Omega).$$

$$\Rightarrow \|v_{j_\ell} - v_{j_{\ell'}}\|_{k+1,K}^2 = |v_{j_\ell} - v_{j_{\ell'}}|_{k+1,K}^2 + \|v_{j_\ell} - v_{j_{\ell'}}\|_{k,K}^2$$

$$\leq 2\left(|v_{j_\ell}|_{k+1,K}^2 + |v_{j_{\ell'}}|_{k+1,K}^2\right) + \|v_{j_\ell} - v_{j_{\ell'}}\|_{k,K}^2 \to 0$$

$\Rightarrow \{v_{j_\ell}\}$ is a Cauchy sequence in $H^{k+1}(K)$

$\Rightarrow \exists\, w \in H^{k+1}(K)$ s.t. $\|v_{j_\ell} - w\|_{k+1,K} \to 0$

$$\Rightarrow \|w\|_{k+1,K} = \lim_{\ell\to\infty} \|v_{j_\ell}\|_{k+1,K} = 1$$

On the other hand,

$$0 = \lim_{\ell\to\infty}\left[|v_{j_\ell}|_{k+1,K}^2 + \sum_{|\alpha|\leq k}\left(\int_K D^\alpha v_{j_\ell}\,dx\right)^2\right]$$

$$= |w|_{k+1,K}^2 + \sum_{|\alpha|\leq k}\left(\int_K D^\alpha w\,dx\right)^2$$

$$\Rightarrow \left\{\begin{array}{l} D^\alpha w = 0,\ |\alpha| = k+1 \quad \Rightarrow w \in \mathcal{P}_k \\ \int_K D^\alpha w = 0,\ |\alpha| \leq k \end{array}\right\} \Rightarrow w \equiv 0$$

This is contraditory with $\|w\|_{k+1,K} = 1$. #