Chapter 5 n-Dimensional Variational Problem

Poisson's equation $\left\langle \begin{array}{l}\text{mixed BCs}\\ \text{Neumann BCs}\end{array}\right.$

general 2^{nd}-order elliptic PDEs

plate-bending biharmonic problem

§5.1 Variational Formulation of Poisson's Equation

Poisson Equation

(5.1.1) $$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \Gamma_D \subset \partial\Omega \quad \text{Dirichlet} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \Gamma_D = \Gamma_N \quad \text{Neumann} \end{cases}$$

Assumptions
- $\partial\Omega$ is Lipschitz continuous
- ν is the outward unit normal vector to $\partial\Omega$ in $L^\infty(\Omega)^n$
- $\operatorname{mes}(\Gamma) \neq 0$

$$H^1_D(\Omega) = \left\{ v \in H^1(\Omega) \,\middle|\, v|_{\Gamma_D} = 0 \right\}$$

Proposition (5.1.4) $$\int_\Omega \nabla \cdot \vec{u}\, dx = \int_{\partial\Omega} \vec{u} \cdot \nu \, ds \qquad \forall \vec{u} \in W_1^1(\Omega)^n$$

Proof $\forall \vec{u} \in C^\infty(\bar{\Omega})^n$, (5.1.4) is true.

$\forall \vec{u} \in W_1^1(\Omega)^n \xRightarrow{C^\infty(\bar{\Omega})^n \text{ is dense in } W_1^1(\Omega)^n} \exists\, \vec{u}_k \in C^\infty(\bar{\Omega})^n$ s.t. $\vec{u}_k \to \vec{u}$ in $W_1^1(\Omega)^n$

and $\vec{u}_k \cdot \nu \to \vec{u} \cdot \nu$ in $L^1(\partial\Omega)$

(5.1.5) $$\int_\Omega \frac{\partial v}{\partial x_i} w\,dx = -\int_\Omega v \frac{\partial w}{\partial x_i}\,dx + \int_{\partial\Omega} v w \nu_i\,ds \quad \forall\, v, w \in H^1(\Omega)$$

Proof $\vec{u} = v w \vec{e}_i$ in (5.1.4).

(5.1.6) $$-\int_\Omega (\nabla\cdot\vec{u})\, v\,dx = \int_\Omega \vec{u}\cdot\nabla v\,dx - \int_{\partial\Omega} (\vec{u}\cdot\nu)\, v\,ds \quad \forall\, \vec{u} \in W_1^1(\Omega)^n,\ v \in H^1(\Omega)$$

Proof $v = u_i$ in (5.1.5) and summing over i.

Proposition (5.1.7) Let $u \in H^2(\Omega)$ solve (5.1.1) (assuming $f \in L^2(\Omega)$)

$\Rightarrow$ $u \in H^1_D(\Omega)$ satisfies $a(u,v) \equiv (\nabla u, \nabla v) = (f, v) \quad \forall\, v \in H^1_D(\Omega)$ (5.1.8)

Proposition (5.1.9) Let $f \in L^2(\Omega)$ and suppose that $u \in H^2(\Omega)$ solves (5.1.8)

$\Rightarrow$ u solves (5.1.1)

Proof $\forall\, v \in \mathcal{D}(\Omega) \subset H^1_D(\Omega)$

- $\left.\begin{array}{l} 0 = (f,v) - a(u,v) = \int_\Omega (f + \Delta u)\, v\,dx \\ \mathcal{D}(\Omega) \text{ is dense in } L^2(\Omega) \end{array}\right\} \Rightarrow f + \Delta u = 0 \text{ in } \Omega$

- $u \in H^1_D(\Omega) \Rightarrow u\big|_{\Gamma_D} = 0$

- $\forall\, v \in H^1_D(\Omega) \quad 0 = (f,v) - (\nabla u, \nabla v) = \int_\Omega (f + \Delta u)\, v\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,ds = -\int_{\partial\Omega\setminus\Gamma_D} \frac{\partial u}{\partial \nu} v\,ds$

$v\big|_{\partial\Omega\setminus\Gamma_D}$ can be chosen arbitrarily (see proof in the book)

$\Rightarrow 0 = \int_{\partial\Omega\setminus\Gamma_D} \frac{\partial u}{\partial \nu} v\,ds \Rightarrow \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega\setminus\Gamma_D} = 0$ #

§5.2 VF of the Pure Neumann Problem

$$(5.2.1)\quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

Remarks (1) the solution is unique up to an additive constant, $u + C$

(2) the solution exists if $\int_\Omega f\, dx = 0$

solution space $\hat{H}^1(\Omega) = \left\{ v \in H^1(\Omega) \,\middle|\, \int_\Omega v\, dx = 0 \right\}$, $\quad \bar{g} = \frac{1}{|\Omega|} \int_\Omega g\, dx$

Prop. (5.2.5) Let $u \in H^2(\Omega)$ solve Poisson's equation $(5.2.1)$ $\left(f \in L^2(\Omega) \text{ and } \bar{f} = 0 \right)$

$\Rightarrow u - \bar{u} \in \hat{H}^1(\Omega)$ satisfies

$$a(u,v) = (\nabla u, \nabla v) = (f, v) \quad \forall v \in \hat{H}^1(\Omega)$$

Prop. (5.2.6) Let $f \in L^2(\Omega)$ and suppose that $u \in H^2(\Omega) \cap \hat{H}^1(\Omega)$ s.t. $a(u,v) = (f,v)$

$$\Rightarrow \begin{cases} -\Delta u = \tilde{f} = f - \bar{f} & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \qquad \left(\bar{\tilde{f}} = 0 \right)$$

Proof $\forall v \in \hat{H}^1(\Omega) \Rightarrow \bar{v} = 0 \Rightarrow (f, v) = (\tilde{f}, v)$

$$\forall v \in \tilde{\mathcal{D}}(\Omega) = \left\{ v \in \mathcal{D}(\Omega) \,\middle|\, \bar{v} = 0 \right\}$$

$$0 = a(u,v) - (\tilde{f}, v) = -\int_\Omega (\Delta u + \tilde{f})\, v\, dx$$

$$\overline{\tilde{\mathcal{D}}(\Omega)}^{L^2(\Omega)} \underset{(ex.5.x.1)}{=} L^2_0(\Omega) = \left\{ f \in L^2(\Omega) \,\middle|\, \int_\Omega f\, dx = 0 \right\} \Longrightarrow \Delta u + \tilde{f} = \overset{\text{const}}{c} \quad \text{in } \Omega$$

$$c = \bar{c} = \overline{\Delta u + \tilde{f}} = \overline{\Delta u} = \frac{1}{|\Omega|}\int_\Omega \Delta u \, dx = \frac{1}{|\Omega|}\int_{\partial\Omega} \frac{\partial u}{\partial \nu} ds$$

$$\Rightarrow -\Delta u = \tilde{f} + c \Rightarrow 0 = (\nabla u, \nabla v) - (\tilde{f}, v) = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, ds \quad \forall v \in \hat{H}^1(\Omega)$$

Since $v|_{\partial\Omega}$ may be chosen arbitrarily

$$\Rightarrow \boxed{\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0}$$

$$\Rightarrow c = \frac{1}{|\Omega|}\int_{\partial\Omega} \frac{\partial u}{\partial \nu} ds = 0$$

$$\Rightarrow \boxed{-\Delta u = \tilde{f} \quad \text{in } \Omega}$$

[let $w \in H^1(\Omega)$ be arbitrary
$\phi \in \mathcal{D}(\Omega)$ and $\bar{\phi} = 1$
$v = w - \bar{w}\phi$
$\Rightarrow \bar{v} = 0$ and $v|_{\partial\Omega} = w|_{\partial\Omega}$]

$$\bar{v} = \bar{w} - \overline{\bar{w}\phi} = \bar{w} - \bar{w}$$

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§5.3 Coercivity of the Variational Problem

Deny-Lions Lemma

$$\inf_{p \in \mathcal{P}_k} \|v - p\|_{H^{k+1}(\Omega)} \leq C\,|v|_{H^{k+1}(\Omega)} \quad \forall v \in H^{k+1}(\Omega)$$

$$\geq C\left[\,|v|^2_{H^{k+1}(\Omega)} + \sum_{|\alpha|\leq k}\left(\int_\Omega D^\alpha v \, dx\right)^2\right]^{\frac{1}{2}}$$

Poincaré Inequality (I) $\quad \forall v \in \hat{H}^1(\Omega) \quad$ (i.e., $\bar{v} = 0$)

$$\|v\|^2_{H^1(\Omega)} \leq C\,|v|^2_{H^1(\Omega)}$$

Proof: Deny-Lions Lemma with $k=0 \Rightarrow \inf_{c\in\mathbb{R}} \|v - c\|_{H^1(\Omega)} \leq C\,|v|_{H^1(\Omega)} \quad \forall v \in \hat{H}^1(\Omega)$

$$\Rightarrow \inf_{c\in\mathbb{R}} \|v - c\|_{L^2(\Omega)} = \|v - \bar{v}\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega)} \leq C\,|v|_{H^1(\Omega)}.$$

$$\inf_{c\in\mathbb{R}}\|v-c\|^2_{L^2(\Omega)} = \inf_c\left[\|v\|^2 - 2c\bar{v}|\Omega| + c^2|\Omega|\right] \Rightarrow c = \bar{v}$$

Poincaré Inequality (II) $\forall\, v \in H^1_D(\Omega) = H^1(\Omega) \cap \{v|_{\Gamma_D} = 0\}$

$$(*) \qquad \|v\|_{H^1(\Omega)} \le c\,|v|_{H^1(\Omega)} . \qquad a(v,v) = |v|^2_{H^1(\Omega)}$$

Proof (compactness argument) Assume that $(*)$ is not true.

$$\Rightarrow \exists \{v_j\}_{j=1}^{\infty} \subset H^1_D(\Omega) \text{ s.t. } \begin{cases} \|v_j\|_1 = 1 \\ |v_j|_1 \le \frac{1}{j} \to 0 \text{ as } j \to \infty \end{cases}$$

① $$|\bar{v}_j| = \frac{1}{|\Omega|}\left|\int_\Omega v_j\,dx\right| \le \frac{1}{|\Omega|}\left(\int_\Omega 1^2 dx\right)^{\frac{1}{2}} \|v_j\|$$

$$\le |\Omega|^{-\frac{1}{2}} \|v_j\|_1 = |\Omega|^{-\frac{1}{2}}$$

$\Rightarrow \{\bar{v}_j\}$ is a bounded sequence

$\Rightarrow \exists$ a subsequence $\{\bar{v}_{j_\ell}\}_\ell$ s.t. $\bar{v}_{j_\ell} \to r_0$ as $\ell \to \infty$

② $$\|v_{j_\ell} - r_0\|_1 \le \|v_{j_\ell} - \bar{v}_{j_\ell}\|_1 + \|\bar{v}_{j_\ell} - r_0\|_1$$

PI (I) $$\le c\,|v_{j_\ell} - \bar{v}_{j_\ell}|_1 + \|\bar{v}_{j_\ell} - r_0\|_1 = c\,|v_{j_\ell}|_1 + \|\bar{v}_{j_\ell} - r_0\|_1 \to 0 \text{ as } \ell \to \infty$$

$$v_{j_\ell} \to r_0 \text{ in } H^1(\Omega) \Longrightarrow v_{j_\ell}\big|_{\Gamma_D} \to r_0 \text{ in } L^2(\Gamma) \quad \text{(trace thrm)}$$

③ $$\left.\begin{array}{l} v_{j_\ell} \in H^1_D(\Omega) \Rightarrow v_{j_\ell}\big|_{\Gamma_D} = 0 \\ \text{meas}(\Gamma_D) \ne 0 \end{array}\right\} \Rightarrow r_0 = 0$$

$$\left.\begin{array}{l} v_{j_\ell} \to r_0 = 0 \text{ in } H^1(\Omega) \\ \|v_{j_\ell}\|_1 = 1 \end{array}\right\} \Rightarrow 0 = \lim_{\ell\to\infty} \|v_{j_\ell}\|_1 = 1 \quad \text{contradiction.}$$

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Poincaré Inequality in $\mathring{W}^1_p(\Omega)$

$$\|v\|_{W^1_p(\Omega)} \leq C\,|v|_{W^1_p(\Omega)}.$$

The Trace Thrm

$$\|v\|_{L^p(\partial\Omega)} \leq C\,\|v\|^{1-\frac{1}{p}}_{L^p(\Omega)}\,\|v\|^{\frac{1}{p}}_{W^1_p(\Omega)} \qquad \forall v \in W^1_p(\Omega).$$

§5.4 Variational Approximation of Poisson's Equation

$\mathcal{T}_h$ — a regular triangulation of Ω (assume that points where BCs chang are vertices)

$V_h \subset V = H^1_D(\Omega)$ or $\hat{H}^1(\Omega)$ — a (conforming) finite element space satisfying

$$\inf_{v\in V^h} \|u-v\|_1 \leq Ch^{m-1}|u|_m$$

I^h — a global interpolator satisfying

- $I^h v \in C^0(\bar{\Omega}) \quad \forall v \in V$
- $I^h(V \cap C^k(\bar{\Omega})) \subset V_h$
- $\|u - I^h u\|_1 \leq Ch^{m-1}|u|_m$

Approximation Find $u_h \in V_h$ s.t. $a(u_h, v) = (f, v) \quad \forall v \in V_h$.

Thrm (error estimation)

$$\|u-u_h\|_1 \leq Ch^{m-1}|u|_m$$

$$\|u-u_h\|_0 \leq Ch^m|u|_m \text{ if } \begin{cases} -\Delta w = u-u_h \text{ in } \Omega \\ w = 0 \text{ on } \Gamma_D \\ \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_N \end{cases} \text{ satisfies } |w|_2 \leq C|e|_0.$$

Inhomogeneous BCs

$$\begin{cases} u = g_D & \text{on } \Gamma_D \\ \frac{\partial u}{\partial \nu} = g_N & \text{on } \Gamma_N \end{cases}$$

- extending g_D into Ω s.t. $g_D \in H^1(\Omega)$
- $g_N \in L^2(\Gamma_N)$

(VP) Find u such that $u - g_D \in H^1_D(\Omega)$ and that

$$a(u,v) \equiv (\nabla u, \nabla v) = f(v) \equiv (f, v) + \langle g_N, v\rangle_{\Gamma_N} \quad \forall v \in H^1_D(\Omega)$$

(FEA) Find u_h s.t. $u_h - I^h g_D \in V_h$ and that

$$a(u_h, v) = f(v) \quad \forall v \in V_h.$$

Ref Bartel-Carstensen-Dolzmann, Inhomogeneous Dirichlet conditions in a priori and a posteriori finite element error analysis, Numer. Math., 99 (2004), 1-24.

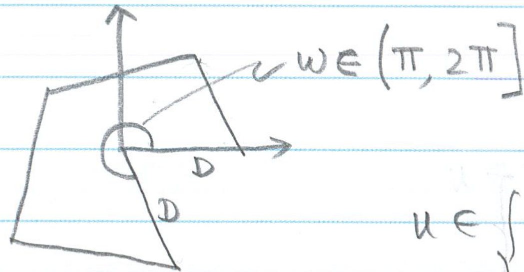
§5.5 Elliptic Regularity Estimates

the validity of $|v|_2 \le C \|\Delta v\|$

(1) Ω has a smooth boundary and $\Gamma = \emptyset$ or $\partial\Omega$

(2) $(n=2)$ Ω is convex or $\Gamma = \emptyset$ or $\partial\Omega$

in R^2



$$u \in \begin{cases} H^{1+\frac{\pi}{\omega}-\varepsilon}(\Omega) & D/D \text{ or } N/N \\ H^{1+\frac{\pi}{2\omega}-\varepsilon}(\Omega) & D/N \text{ or } N/D,\ \omega \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ H^{1+\frac{3\pi}{2\omega}-\varepsilon}(\Omega) & D/N \text{ or } N/D,\ \omega \in \left(\frac{3\pi}{2}, 2\pi\right] \end{cases}$$

Lemma $\forall\ |\alpha| = 2,\quad \|D^\alpha v\|_{L^2(R^n)} \le \|\Delta v\|_{L^2(R^n)}$

Proof Fourier transformation $\hat{v}(\vec{\xi}) = \int_{R^n} v(\vec{x})\, e^{-i\vec{x}\cdot\vec{\xi}}\, d\vec{x}$

$$\widehat{D^\alpha v}(\vec{\xi}) = (i\vec{\xi})^\alpha \hat{v}(\vec{\xi}) = -\frac{(i\vec{\xi})^\alpha}{|\vec{\xi}|^2}\,\widehat{\Delta v}(\vec{\xi})$$

$|\alpha| = 2$ $$\frac{\vec{\xi}^\alpha}{|\vec{\xi}|^2} = \frac{\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}}{\xi_1^2 + \cdots + \xi_n^2} \le 1$$

$$\Longrightarrow \quad \|\widehat{D^\alpha v}(\vec{\xi})\|_{L^2(R^n)} \le \|\widehat{\Delta v}(\vec{\xi})\|_{L^2(R^n)}$$

$$\|\widehat{D^\alpha v}(\vec{\xi})\|_{L^2(R^n)} = \|D^\alpha v\|_{L^2(R^n)}, \qquad \|\widehat{\Delta v}(\vec{\xi})\|_{L^2(R^n)} = \|\Delta v\|_{L^2(R^n)}$$

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§5.6 General 2^{nd}-order Elliptic Operators

Diffusion Problem

$$\begin{cases} \mathcal{L}_s u \equiv -\operatorname{div}(A\nabla u) = -\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \nu \cdot (A\nabla u) = 0 & \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D \end{cases}$$

(VP) Find $u \in H_D^1(\Omega) = \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \}$ s.t.

$$a_s(u,v) \equiv (A\nabla u, \nabla v) = (f, v) \quad \forall\, v \in H_D^1(\Omega)$$

Assumptions (1) $a_{ij} \in L^\infty(\Omega)$

(2) $A = (a_{ij})_{n\times n}$ is uniformly elliptic $\iff \exists\, \alpha > 0$ s.t. $\xi^t A \xi \geq \alpha\, \xi^t \xi$ a.e. in Ω $\forall \xi \in \mathbb{R}^n$

Lemma 1 $a_s(v,v) \geq \alpha |v|_1^2 \quad \forall\, v \in H^1(\Omega)$.

Convection-Diffusion Problem

$$\begin{cases} \mathcal{L}_N u \equiv \mathcal{L}_s u + Xu = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \nu \cdot (A\nabla u) = 0 & \text{on } \Gamma_N \end{cases} \qquad X_u = \vec{b}\cdot\nabla u + b_0 u$$

(VP) Find $u \in H_D^1(\Omega)$ s.t.

$$a_N(u,v) \equiv a_s(u,v) + (Xu, v) = (f, v) \quad \forall\, v \in H_D^1(\Omega)$$

Thrm (Gårding Inequality) Assume that $b_i \in L^\infty(\Omega)$.

$\Longrightarrow \exists\ K > 0$ s.t.
$$a_N(v,v) + K\|v\|^2 \geq \frac{\alpha}{2}\|v\|_1^2 \quad \forall\, v \in H^1(\Omega)$$

Proof
$$\begin{aligned} a_N(v,v) &= a_s(v,v) + (Xv, v) \\ &\geq \alpha |v|_1^2 - C_1(|v|_1 + \|v\|)\|v\| \\ &\geq \alpha \|v\|_1^2 - C_2\|v\|_1\|v\| \\ &\geq \frac{\alpha}{2}\|v\|_1^2 - K\|v\|^2 \end{aligned}$$

$\left[\, |(Xv,v)| \leq C_1 |v|_1 \|v\| + C_2\|v\|^2 \,\right]$

$\left[\, C_2\|v\|_1\|v\| \leq \frac{\alpha}{2}\|v\|_1^2 + \frac{C_2^2}{2\alpha}\|v\|^2 \,\right]$

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§5.7 Variational Approximation of General Elliptic Problems

Assumptions (1) $|a(u,v)| \leq C_1 \|u\|_1 \|v\|_1 \quad \forall\, u,v \in H^1(\Omega)$

(2) $\exists\, K \in R$ s.t. $a(v,v) + K\|v\|^2 \geq \alpha\|v\|_1^2 \quad \forall\, v \in H^1(\Omega)$

(3) the following problems have unique solutions

(VP) $\left[\begin{array}{l}\text{Find } u \in H_D^1(\Omega) \text{ s.t.} \\ \quad a(u,v) = (f,v)\ \forall v \in H_D^1(\Omega)\end{array}\right.$ (AVP) $\left[\begin{array}{l}\text{Find } w \in H_D^1(\Omega) \text{ s.t.} \\ \quad a(v,w) = (f,v)\ \forall v \in H_D^1(\Omega)\end{array}\right.$

(4) $|u|_2 \leq C\|f\|$ and $|w|_2 \leq C\|f\|$

(5) $V_h \subset H_D^1(\Omega)$
$$\inf_{v \in V_h} \|u - v\|_1 \leq C h |u|_2 \quad \forall\, u \in H^2(\Omega).$$

(FEA) Find $u_h \in V_h$ s.t.

$$a(u_h, v) = (f, v) \quad \forall v \in V_h.$$

<u>the error equation</u> $\quad a(u-u_h, v) = 0 \quad \forall v \in V_h.$

<u>Thrm</u> $\exists h_0 > 0$ s.t. $\forall h \leq h_0$ s.t. (FEA) has a unique solution u_h s.t.

$$\|u-u_h\|_1 \leq C \inf_{v \in V_h} \|u-v\|_1 \quad \text{and} \quad \|u-u_h\| \leq Ch \|u-u_h\|_1 .$$

<u>Proof</u> $\quad e_h = u - u_h$

$$\begin{aligned} \alpha \|u-u_h\|_1^2 &\leq a(e_h, e_h) + K \|e_h\|^2 \\ &= a(e_h, u-v) + K \|e_h\|^2 \quad \forall v \in V_h \\ &\leq C \|e_h\|_1 \|u-v\|_1 + K \|e_h\|^2 \quad \forall v \in V_h \end{aligned}$$

<u>the adjoint problem</u> Find $w \in H_D^1(\Omega)$ s.t. $a(v, w) = (u-u_h, v) \quad \forall v \in H_D^1(\Omega)$

$$\begin{aligned} \|e_h\|^2 &= a(e_h, w) = a(e_h, w - I_h w) \\ &\leq C \|e_h\|_1 \|w - I_h w\|_1 \leq Ch \|e_h\|_1 |w|_2 \leq Ch \|e_h\|_1 \|e_h\| \end{aligned}$$

$$\Rightarrow \quad \|e_h\| \leq Ch \|e_h\|_1$$

$$\Rightarrow \alpha \|e_h\|_1^2 \leq C \|e_h\|_1 \|u-v\|_1 + Ch^2 \|e_h\|_1^2$$

$$\Rightarrow (\alpha - Ch^2) \|e_h\|_1 \leq C \|u-v\|_1 \quad \forall v \in V_h \xrightarrow{\text{sufficiently small } h} \|e_h\|_1 \leq C \inf_{v \in V_h} \|u-v\|_1$$

uniqueness & existence

$$\left.\begin{aligned} &f = 0 \Rightarrow u = 0 \\ &\|u-u_h\|_1 \leq C \|u-v\|_1 \quad \forall v \in V_h \end{aligned}\right\} \Rightarrow \|u_h\|_1 \leq C \|v\|_1 \quad \forall v \in V_h$$

$$\xrightarrow{v=0} u_h = 0 \Rightarrow \text{uniqueness} \Leftrightarrow \text{existence.}$$

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§5.8 Negative Norm Estimate

Assumptions (1) $\|u\|_{s+2} \le c\,\|f\|_s$ for some $s \ge 0$

(2) $\inf_{v \in V_h} \|u-v\|_1 \le c\,h^{s+1}\,\|u\|_{s+2}$

Thrm $\exists\, h_0 > 0$ s.t. $\forall\, h \le h_0$

$$\Longrightarrow \|u-u_h\|_{-s} \le c\,h^{s+1}\,\|u-u_h\|_1$$

Proof the adjoint problem $w \in H^1_D(\Omega)$ s.t. $a(v,w) = (v,\varphi)$ $\forall\, v \in H^1_D(\Omega)$, $\forall\, \varphi \in H^s(\Omega)$

$$\overset{v=u-u_h}{\Longrightarrow} (u-u_h,\varphi) = a(u-u_h, w) = a(u-u_h, w-I_h w)$$

$$\le C\,\|u-u_h\|_1\,\|w-I_h w\|_1$$

$$\le c\,h^{s+1}\,\|u-u_h\|_1\,\|w\|_{s+2}$$

$$\le C\,h^{s+1}\,\|u-u_h\|_1\,\|\varphi\|_s$$

$$\Longrightarrow \|u-u_h\|_{-s} = \sup_{\varphi \in H^s(\Omega)} \frac{(u-u_h,\varphi)}{\|\varphi\|_s} \le c\,h^{s+1}\,\|u-u_h\|_1$$

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HW #8, 9, 10, 12, 14, 15, 16

§5.9 The Plate-Bending Biharmonic Problem

Variational Problem Given $F \in H^2(\Omega)'$, find $u \in V \subset H^2(\Omega)$ with $V \cap \mathcal{P}_1 = \phi$ s.t.

$$a(u,v) = F(v) \quad \forall\, v \in V.$$

where $a(u,v) = \int_\Omega \Delta u \Delta v - (1-\nu)\left(2u_{xx}v_{yy} + 2u_{yy}v_{xx} - 4u_{xy}v_{xy}\right) dx dy$, $F(v) = \int_\Omega f v \, dx dy$

ν: Poisson ratio in $[0, \frac{1}{2}]$

$$\begin{cases} \Delta^2 u = f \\ u = \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \quad \text{if } V = V^c = \left\{ v \in H^2(\Omega) \,\middle|\, v = \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \right\} \\ \text{or} \\ u = 0 \text{ and } \Delta u + (1-\nu)\underbrace{u_{tt}}_{2^{nd}\text{-order tangential der.}} = 0 \text{ on } \partial\Omega \quad \text{if } V = V^{ss} = \left\{ v \in H^2(\Omega) \,\middle|\, v = 0 \text{ on } \partial\Omega \right\} \end{cases}$$

$\cancel{V^c \quad \int_\Omega f v \, dx = \int_\Omega (\Delta u)(\Delta v) \, dx - \int_{\partial\Omega} (\Delta u) \frac{\partial v}{\partial n} ds + \int_{\partial\Omega} v \frac{\partial(\Delta u)}{\partial n} ds}$

Lemma (Gårding Ineq.) $\forall\, \nu \in (-3, 1)$, $\exists\, \alpha > 0$, $K < +\infty$, s.t.

$$a(v,v) + K\|v\|^2 \geq \alpha \|v\|_2^2 \quad \forall\, v \in H^2(\Omega)$$

Lemma $\exists\, \alpha > 0$ s.t.

$$a(v,v) \geq \alpha \|v\|_2^2 \quad \forall\, v \in V \subset H^2(\Omega) \text{ with } V \cap \mathcal{P}_1 = \phi.$$

Proof $a(v,v) = \int_\Omega \nu(v_{xx} + v_{yy})^2 + (1-\nu)\left[(v_{xx} - v_{yy})^2 + 4v_{xy}^2\right]$

$$\geq \min\{\nu, 1-\nu\} \int_\Omega \underbrace{(v_{xx}+v_{yy})^2 + (v_{xx}-v_{yy})^2 + 4v_{xy}^2}_{= 2(v_{xx}^2 + v_{yy}^2) + 4v_{xy}^2}$$

$$\geq 2\min\{\nu, 1-\nu\} |v|_2^2 \geq \alpha \|v\|_2^2 \quad \forall\, v \in V/\mathcal{P}_1.$$

Thrm (1) VP has a unique sol. in $V/\mathcal{P}_1$

(2) $\|u - u_h\|_2 \leq C \inf_{v \in V_h} \|u - v\|_2 \leq C h^{k-1} \|v\|_{k+1}$.