Homework Problems (Brenner & Scott)

Chapter 0 #2, 3, 8

\#2. Derive variational formulation of BVP $\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases}$

\#3. Derive variational formulation of BVP $\begin{cases} -u'' = f & \text{in } (0,1) \\ u'(0) = 0, \ u'(1) = 0 \end{cases}$

Explain why (BVP) and (VP) are not well-posed.

\#8. Prove that (BVP) $\begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = 0, \ u'(1) = 0 \end{cases}$ has a solution $u \in C^2([0,1])$ provided $f \in C^0([0,1])$.

(Hint: $u(x) = \int_0^x \left(\int_s^1 f(t)\,dt \right) ds$)

Chapter 1 #1, 3

\#1. Suppose that Ω is bounded and that $1 \leq p \leq q \leq \infty$.

Prove that $L^q(\Omega) \subset L^p(\Omega)$.

\#3. Suppose that Ω is bounded and that $\lim\limits_{j \to \infty} \|f_j - f\|_{L^p(\Omega)}$.

Prove $\lim\limits_{j \to \infty} \int_\Omega f_j(x)\,dx = \int_\Omega f(x)\,dx$.

#4 Assume that $a(\cdot,\cdot)$ is symmetric bilinear and that $f(\cdot)$ is linear. Let V be a Hilbert space and $S \subset V$ be a subset. If $J(u) = \min_{v \in V} J(v)$, prove that

$$a(u-v, u-v) = 2\big(J(v) - J(u)\big) \quad \forall v \in S,$$

where $J(v) = \frac{1}{2} a(v,v) - f(v)$.

#5 Let $\Delta: 0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $I = (0,1)$.

Let $S_0^{-1}(\Delta) = \left\{ v \in L^2(I) \,\middle|\, v|_{[x_i, x_{i+1}]} \in P_0(I_i) \right\}$.

Prove that for any $u(x) \in C^1[0,1]$, we have

$$\min_{v \in S_0^{-1}(\Delta)} \|u - v\| \le ch \|u'\| \quad \text{for both } L^2(I) \text{ and } L^\infty(I) \text{ norms.}$$

Homework (Computer Project : fixed, adaptive, and free meshes)

Let $u(x) = x^r$ be a function defined on $[0,1]$ with $r \in (0,2)$.

Use the continuous piecewise linear polynomials defined on the uniform, adaptive, and free meshes to solve

(1) the best least-squares approximation : find $u_n \in S_1^0(\Delta_n)$ s.t.

$$\|u - u_n\| = \min_{v \in S_1^0(\Delta_n)} \|u - v\|, \quad \text{where } \|v\| = \sqrt{\int_0^1 v^2 dx}$$

(2) the Poisson equation : find $u_n \in S_1^0(\Delta_n)$ s.t.

$$\begin{cases} -u''(x) = f(x) & \text{in } I = (0,1) \\ u(0) = 0 \text{ and } u'(1) = r \end{cases}$$

where $f(x) = r(1-r)\, x^{r-2}$.

- For $r = 0.1, 0.4, 0.8, 1.2, 1.6$, examine approximation accuracies using these three approaches for a fixed number of DoF.

- Given a prescribed tolerance $\varepsilon > 0$, examine the number of DoF $n(\varepsilon)$ s.t.

  (1) $\|u - u_n\| \le \varepsilon \|u\|$

  (2) $\|u - u_n\|_E \le \varepsilon \|u\|_E$.

$$\|v\|_E = \sqrt{\int_0^1 (v')^2 dx}$$