HW1

Problem 1 (S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods, p. 1-2)**.** Give weak formulations of the two-point boundary value problem

a) $-u'' + u = f$ in $(0, 1)$

b) $u(0) = u(1) = 0$.

If u is the solution and v is any (sufficiently regular) function such that $v(0) = v(1) = 0$, then integration by parts yields

$$\begin{aligned}(f, v) &:= \int_0^1 f(x)v(x)\,dx \\ &= \int_0^1 -u''(x)v(x) + u(x)v(x)\,dx \\ &= \int_0^1 u(x)v(x) + u'(x)v'(x)\,dx =: a(u, v).\end{aligned}$$

Let us define

$$V = \{v \in L^2(0, 1) : a(v, v) < \infty \text{ and } v(0) = v(1) = 0\}.$$

Then we can say that the solution u is characterized by

$$u \in V \text{ such that } a(u, v) = (f, v) \quad \forall v \in V,$$

which is called the variational or weak formulation of the problem.

Problem 2 (Student submission)**.** Explain what is wrong in both the variational setting (VP) and the classical setting (BVP) for the problem

$$-u'' = f \quad \text{with } u'(0) = u'(1) = 0.$$

That is, explain in both contexts why this problem is not well-posed.

a) There exists at least one solution.

b) There exists at most one solution.

c) The solution depends continuously on the data.

1. (BVP): If u is a solution, $u + c$ for some constant c is also a solution since

   $$-(u+c)'' = -u'' = f \text{ and } (u+c)'(0) = u'(0) = 0 = u'(1) = (u+c)'(1).$$

   Hence, b) does not hold.

2. If u is a solution and v is any (sufficiently regular) function, then integration by parts yields

   $$\begin{aligned}(f, v) &:= \int_0^1 f(x)v(x)\,dx \\ &= \int_0^1 -u''(x)v(x)\,dx \\ &= \int_0^1 u'(x)v'(x)\,dx =: a(u, v).\end{aligned}$$

   Let us define
   $$V = \{v \in L^2(0,1) : a(v,v) < \infty\}.$$
   Then we have the variational formulation of the problem

   $$u \in V \text{ such that } a(u, v) = (f, v) \quad \forall v \in V.$$

   If u is a solution, then $u + c$ is also a solution since

   $$\int_0^1 (u+c)'(x)v'(x)\,dx = \int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

   Hence, b) does not hold.

Problem 3. Prove that

$$-u'' = f \text{ in } (0,1) \text{ with } u(0) = u'(1) = 0$$

has a solution $u \in C^2([0,1])$ provided $f \in C^0([0,1])$. (Hint: write

$$u(x) = \int_0^x \left(\int_s^1 f(t)\, dt \right) ds$$

and verify the equations.)
Since $f \in C^0([0,1])$, we have

$$g(s) = \int_s^1 f(t)\, dt \in C^1([0,1]).$$

Similarly,

$$u(x) = \int_0^x \left(\int_s^1 f(t)\, dt \right) ds \in C^1([0,1]).$$

Hence, $u(x) \in C^2([0,1])$. Moreover,

1. $$u(0) = \int_0^0 \left(\int_s^1 f(t)\, dt \right) ds = 0$$

2. $$u'(1) = \int_1^1 f(t)\, dt = 0$$

3. $$\begin{aligned} -u''(x) &= -(u'(x))' \\ &= -\left(\int_x^1 f(t)\, dt \right)' \\ &= -(-f(x)) = f(x) \end{aligned}$$

Problem 4 (Royden, Halsey Lawrence and Fitzpatrick, Patrick)**.** Suppose that Ω is bounded and that $1 \leq p < q \leq \infty$. Prove that $L^q(\Omega) \subset L^p(\Omega)$. (Hint: use Hölder's inequality.) Give examples to show that the inclusion is strict if $p < q$ and false if Ω is not bounded.

1. Assume $q < \infty$. Define $r = q/p > 1$ and let s be the conjugate of r $(1 = 1/r + 1/s)$. Let f belong to $L^q(\Omega)$. Observe that f^p belongs to $L^r(\Omega)$ and $g = \chi_\Omega$ ($g(x) = 1$ in Ω and 0 otherwise) belongs to $L^s(\Omega)$ since $m(\Omega) < \infty$ (area of Ω). Apply Hölder's inequality . Then

$$\int_\Omega |f|^p = \int_\Omega |f|^p \cdot g \leq \|f\|^p_{L^q(\Omega)} \cdot \left[\int_\Omega |g|^s\right]^{1/s} = \|f\|^p_{L^q(\Omega)}[m(\Omega)]^{1/s}.$$

   Take the $1/p$ power of each side.

2. Assume $q = \infty$ and let f belong to $L^\infty(\Omega)$. Then

$$\int_\Omega |f|^p \leq \int_\Omega \|f\|^p_\infty = \|f\|^p_\infty m(\Omega) < \infty.$$

3. In general, for Ω of finite measure and $1 \leq p < q \leq \infty$, $L^q(\Omega)$ is a proper subspace of $L^p(\Omega)$. For instance, let $\Omega = (0, 1]$ and f be defined by $f(x) = x^\alpha$ for $0 < x \leq 1$, where $-1/p < \alpha \leq -1/q$. Then $f \in L^p(\Omega) \setminus L^q(\Omega)$.

4. For $\Omega = (0, \infty)$ and f defined by

$$f(x) = \frac{x^{-1/2}}{1 + |\ln x|} \text{ for } x > 0,$$

   f belongs to $L^p(\Omega)$ if and only if $p = 2$.

Problem 5. Suppose that Ω is bounded and that $f_j \to f$ in $L^p(\Omega)$. Using Hölder's inequality prove that

$$\int_\Omega f_j(x)\,dx \to \int_\Omega f(x)\,dx \text{ as } j \to \infty.$$

By the linearity and triangle inequality of integration,

$$\begin{aligned}
\left|\int_\Omega f_j(x)\,dx - \int_\Omega f(x)\,dx\right| &= \left|\int_\Omega f_j(x) - f(x)\,dx\right| \\
&\le \int_\Omega |f_j(x) - f(x)|\,dx \\
&= \int_\Omega |(f_j(x) - f(x))\chi_\Omega|\,dx \\
&\le \|f_j - f\|_{L^p(\Omega)} [m(\Omega)]^{1/q} \to 0
\end{aligned}$$

where $1 = 1/p + 1/q$.