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LEAST-SQUARES METHODS FOR INCOMPRESSIBLE NEWTONIAN FLUID FLOW: LINEAR STATIONARY PROBLEMS*

ZHIQIANG CAI†, BARRY LEE‡, AND PING WANG§

Abstract. This paper develops and analyzes two least-squares methods for the numerical solution of linear, stationary incompressible Newtonian fluid flow in two and three dimensions. Both approaches use the L^2 norm to define least-squares functionals. One is based on the stress-velocity formulation (see section 3.2), and it applies to general boundary conditions. The other is based on an equivalent formulation for the *pseudostress* and velocity (see section 4.2), and it applies to pure velocity Dirichlet boundary conditions. The velocity gradient and vorticity can be obtained algebraically from this new tensor variable. It is shown that the homogeneous least-squares functionals are elliptic and continuous in the $H(\text{div}; \Omega)^d \times H^1(\Omega)^d$ norm. This immediately implies optimal error estimates for conforming finite element approximations. As well, it admits optimal multigrid solution methods if Raviart–Thomas finite element spaces are used to approximate the stress or the pseudostress tensor.

Key words. least-squares method, mixed finite element method, Navier–Stokes, Stokes, incompressible Newtonian flow

AMS subject classifications. 65M60, 65M15

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1. Introduction.

For incompressible Newtonian fluid flow with homogeneous density, the primitive physical equations are the conservation of momentum and the constitutive law. The constitutive law relates the stress tensor to the deformation rate tensor and pressure, and it states the incompressibility condition. It is a first-order partial differential system for the physical variables stress, velocity, and pressure. By differentiating and eliminating the stress, one obtains the well-known second-order incompressible Navier–Stokes equations in the velocity-pressure formulation. A tremendous amount of computational research has been done on this second-order partial differential system (see, e.g, mathematical books [17, 18]), but these equations may still be one of the most challenging problems in computational fluid mechanics and computational mathematics.

In recent years there has been substantial interest in the use of least-squares principles for the numerical approximation of Newtonian fluid flow problems (see, e.g., the survey paper [5], the monograph [21], and references therein). In particular, there are many research articles in both the mathematics and engineering communities on least-squares methods for the stationary Stokes equations (see [5]). Specifically, least-squares methods based on five first-order partial differential systems have been proposed, analyzed, implemented, and tested. These five first-order systems are formulations for variables (i) velocity, vorticity, and pressure [5, 21], (ii) velocity, pres-

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†Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067 (zcai@math.purdue.edu). The research of this author was sponsored in part by the National Science Foundation under grant INT-0139053 and by the Korea Research Foundation under grant KRF-2002-015-C00014.
‡Center for Applied Scientific Computing, Lawrence Livermore National Lab, Livermore, CA 94551 (lee123@llnl.gov).
§DNT Computing Applications Division, Lawrence Livermore National Lab, Livermore, CA 94551 (wang32@llnl.gov).

sure, and "stress" [4], (iii) velocity, velocity gradient, and pressure [11], (iv) velocity, velocity gradient, and pressure with additional constraints [11], and (v) constrained velocity gradient and pressure [15]. The new "stress" variable in (ii) is actually the deformation rate tensor and not the physical stress. Least-squares methods based on the first three formulations employ either discrete inverse norms (see [10, 7]) or mesh-weighted L^2 norms (see [3]) in order to achieve optimal finite element approximations. The inverse norm approach is very expensive due to its discrete inverse norm evaluations, and fast multigrid solvers are still a missing ingredient for the mesh-weighted L^2 norm approaches. If the original problem is sufficiently smooth, methods based on the last two formulations are equivalent to the H^1 norm. Such equivalence implies optimal finite element approximations and optimal convergence of multigrid solvers. But the smoothness requirement is restrictive.

A common feature of all these formulations is that they do not involve the primitive physical equations. Based on the velocity-pressure formulation of the Stokes equations, they are derived by introducing new variables such as vorticity in (i), "stress" in (ii), velocity gradient in (iii) and (iv), and constrained velocity gradient in (v). Some of these new variables have physical meanings, but they are not original physical quantities of interest.

The first objective of this paper is to develop a new least-squares method that does not have the above mentioned drawbacks and that computes the original physical quantities directly. For linear, stationary problems of incompressible Newtonian fluid flow, our least-squares method is based directly on the primitive first-order partial differential system: the stress-velocity-pressure formulation, without introducing any new variables nor any new equations. We define the least-squares functional by applying a L^2 norm least-squares principle to this first-order system. It is shown that the homogeneous least-squares functional is elliptic and continuous in the $H(\text{div}; \Omega)^d$ norm for the stress, the $H^1(\Omega)^d$ norm for the velocity, and the L^2 norm for the pressure. This immediately implies optimal error estimates for conforming finite element approximations in $H(\text{div}; \Omega)^d \times H^1(\Omega)^d \times L^2(\Omega)$. It also admits optimal multigrid solution methods if Raviart–Thomas finite element spaces are used to approximate the stress tensor. Both discretization accuracy and multigrid convergence rates are uniform in the viscosity parameter.

Since the pressure can be represented in terms of the normal stress and since the stress is an independent variable in the first-order system, the pressure can be eliminated from the first-order system. By replacing the pressure with the normal stress, we derive the stress-velocity formulation for incompressible Newtonian fluid flow. We can then define the corresponding least-squares method and show identical numerical properties to those of the stress-velocity-pressure formulation, since the stress-velocity formulation is a special case of the stress-velocity-pressure formulation. It is important to note that, mathematically, the stress-velocity formulation for linear, stationary problems of incompressible Newtonian fluid flow is the limiting case of the stress-displacement formulation for elastic problems when $2\mu = \nu$. This indicates that this paper, together with [14], develops a unified least-squares approach for both elastic solids and incompressible Newtonian fluids with respect to spatial discretization and fast solution solvers, even though the variables and materials have different physical meanings. Hence, our method can be extended to problems coupling elastic deformation with fluid flow.

Many applications in incompressible Newtonian fluid flow do not have traction boundary conditions. It is then not necessary to use the stress as an independent variable. This is especially true because the stress does not contain any information on the vorticity that is a physical quantity of great interest in fluid mechanics. Thus, for pure velocity Dirichlet boundary conditions, we define a new independent variable, *pseudostress*, in terms of the velocity gradient and pressure, and then derive an equivalent first-order system containing the pseudostress and velocity. The pressure, the velocity gradient, and, hence, the vorticity are expressed in terms of the pseudostress. The L^2 norm least-squares functional based on this first-order system is again shown to be elliptic and continuous in the $H(\text{div};\,\Omega)^d \times H^1(\Omega)^d$ norm. Hence, Raviart–Thomas finite elements for the pseudostress and standard continuous piecewise polynomials for the velocity yield optimal approximation, and the resulting algebraic equations can be solved with optimal multigrid methods.

For completeness, we also study inverse norm least-squares functionals and show that their homogeneous forms are elliptic and continuous in appropriate Hilbert spaces. These functionals can be used to develop discrete inverse norm least-squares methods (see, e.g., [6]). Also, for many applications, it is convenient to impose boundary conditions weakly through boundary functionals. Such functionals are also studied in this paper (see section 4.5). (See [23] for the computational feasibility of methods based on these types of functionals.)

Least-squares methods developed in this paper for linear, stationary problems can be easily extended to nonlinear incompressible Newtonian flows, at least in principle. One can simply include an appropriate form of the nonlinear convection term in the residual of the momentum equations. Possible choices for this form can (1) involve only the velocity or (2) involve the (pseudo-) stress which replaces the velocity gradient. Mathematical analysis for least-squares methods applied to nonlinear problems is much more difficult, but it still can be established using the abstract theory of [9]. Formulations of our methods can be easily extended to incompressible non-Newtonian flows as well: only a simple modification is needed in the constitutive equation.

An outline of the paper is as follows. In section 2, the stress-velocity-pressure formulation for incompressible Newtonian fluid flow problems and the corresponding linear, stationary problems are introduced, as well as some notation and the Stokes equations. In section 3, least-squares functionals based on the stress-velocity-pressure and stress-velocity formulations are developed, their ellipticity and continuity are established, and finite element approximations and multigrid solvers are discussed. In section 4, least-squares methods for pure Dirichlet boundary conditions are developed.

1.1. Notation. We use the standard notation and definitions for the Sobolev spaces $H^s(\Omega)^d$ and $H^s(\partial\Omega)^d$ for $s \ge 0$. The standard associated inner products are denoted by $(\cdot,\,\cdot)_{s,\Omega}$ and $(\cdot,\,\cdot)_{s,\partial\Omega}$, and their respective norms are denoted by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\partial\Omega}$. (We suppress the superscript d because the dependence on dimension will be clear by context. We also omit the subscript Ω from the inner product and norm designation when there is no risk of confusion.) For $s = 0$, $H^s(\Omega)^d$ coincides with $L^2(\Omega)^d$. In this case, the inner product and norm will be denoted by $\|\cdot\|$ and $(\cdot,\,\cdot)$, respectively. Set $H^1_D(\Omega) := \{q \in H^1(\Omega) \,:\, q = 0 \text{ on } \Gamma_D\}$. We denote the duals of $H^1_D(\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ by $H^{-1}_D(\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ with norms defined by

$$\|\phi\|_{-1,\,D} = \sup_{0 \ne \psi \in H^1_D(\Omega)} \frac{(\phi,\,\psi)}{\|\psi\|_1} \quad \text{and} \quad \|\phi\|_{-1/2,\,\partial\Omega} = \sup_{0 \ne \psi \in H^{\frac{1}{2}}(\partial\Omega)} \frac{(\phi,\,\psi)}{\|\psi\|_{1/2,\partial\Omega}}.$$

When $D = \partial\Omega$, we denote the dual of $H^1_0(\Omega) = H^1_D(\Omega)$ and its norm by $H^{-1}_0(\Omega)$ and $\|\cdot\|_{-1,\,0}$, respectively. When D is empty, the dual of $H^1(\Omega)$ and its norm are denoted by the respective $H^{-1}(\Omega)$ and $\|\cdot\|_{-1}$. Also, we denote the product space $\prod_{i=1}^{d} H_D^{-1}(\Omega)$ with the standard product norm by $H_D^{-1}(\Omega)^d$. Finally, set

$$H(\mathrm{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 \,:\, \nabla\cdot\mathbf{v} \in L^2(\Omega)\},$$

which is a Hilbert space under the norm

$$\|\mathbf{v}\|_{H(\mathrm{div};\,\Omega)} = \left(\|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2\right)^{\frac{1}{2}},$$

and define the subspace

$$H_N(\mathrm{div};\Omega) = \{\mathbf{v} \in H(\mathrm{div};\Omega) \,:\, \mathbf{n}\cdot\mathbf{v} = 0\}.$$

2. Mathematical equations for incompressible Newtonian fluid flow. Let Ω be a bounded, open, connected subset of $\Re^d$ ($d = 2$ or 3) with a Lipschitz continuous boundary $\partial\Omega$. Denote the outward unit vector normal to the boundary by $\mathbf{n} = (n_1, \ldots, n_d)^t$. We partition the boundary of Ω into two open subsets Γ_D and Γ_N such that $\partial\Omega = \bar\Gamma_D \cup \bar\Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, we will assume that Γ_D is not empty (i.e., $\mathrm{mes}\,(\Gamma_D) \neq 0$).

For a second-order tensor $\boldsymbol{\tau} = (\tau_{ij})_{d\times d}$, define its divergence and normal by

$$\nabla\cdot\boldsymbol{\tau} = \begin{pmatrix} \partial\tau_{11}/\partial x_1 + \cdots + \partial\tau_{1d}/\partial x_d \\ \vdots \\ \partial\tau_{d1}/\partial x_1 + \cdots + \partial\tau_{dd}/\partial x_d \end{pmatrix} \quad \text{and} \quad \mathbf{n}\cdot\boldsymbol{\tau} = \begin{pmatrix} n_1\tau_{11} + \cdots + n_d\tau_{1d} \\ \vdots \\ n_1\tau_{d1} + \cdots + n_d\tau_{dd} \end{pmatrix},$$

respectively. That is, the divergence and normal operators apply to each row of the tensor. Also denote the matrix trace operator by tr:

$$\mathrm{tr}\,\boldsymbol{\tau} = \tau_{11} + \cdots + \tau_{dd}.$$

Let $\mathbf{f} = (f_1, \ldots, f_d)^t$ be a given external body force defined in Ω and $\mathbf{g} = (g_1, \ldots, g_d)^t$ be a given external surface traction applied on Γ_N. Let $\mathbf{u}(\mathbf{x}, t) = (u_1, \ldots, u_d)^t$ be the velocity vector field of a particle of fluid that is moving through $\mathbf{x}$ at time t, and let $\boldsymbol{\sigma} = (\sigma_{ij})_{d\times d}$ be the stress tensor field. Without loss of generality, we assume that the homogeneous density is one. Then conservation of momentum implies both symmetry of the stress tensor and the local relation

$$\text{(2.1)} \qquad \left\{ \begin{array}{rcll} \frac{D\mathbf{u}}{Dt} - \nabla\cdot\boldsymbol{\sigma} & = & \mathbf{f} & \text{in} \quad \Omega, \\ \mathbf{n}\cdot\boldsymbol{\sigma} & = & \mathbf{g} & \text{on} \quad \Gamma_N, \end{array} \right.$$

where $\frac{D}{Dt}$ is the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla = \frac{\partial}{\partial t} + \sum_{i=1}^{d} u_i \frac{\partial}{\partial x_i}.$$

In this paper, we restrict ourselves to linear, stationary problems, i.e., problems where the momentum equation in (2.1) is of the form

$$\text{(2.2)} \qquad -\nabla\cdot\boldsymbol{\sigma} = \mathbf{f}.$$

Let ν be the viscosity parameter, p the pressure, and

$$\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla\,\mathbf{u} + (\nabla\,\mathbf{u})^t\right)$$

the deformation rate tensor, where $\nabla\mathbf{u}$ is the velocity gradient tensor with entries $(\nabla\mathbf{u})_{ij} = \partial u_i/\partial x_j$. Then the constitutive law for incompressible Newtonian fluids is

$$\begin{cases} \boldsymbol{\sigma} = \nu\,\boldsymbol{\epsilon}(\mathbf{u}) - p\,I & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega. \end{cases} \tag{2.3}$$

The second equation in (2.3) is the incompressibility condition. Without loss of generality, we assume that $\nu = 1$, since otherwise $\mathbf{u}$ can be rescaled to $\nu\,\mathbf{u}$. Now, combining (2.2) and (2.3), we have the stress-velocity-pressure formulation for incompressible Newtonian fluid flow:

$$\begin{cases} -\nabla\cdot\boldsymbol{\sigma} = \mathbf{f} & \text{in } \Omega, \\ \boldsymbol{\sigma} + p\,I - \boldsymbol{\epsilon}(\mathbf{u}) = \mathbf{0} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

Differentiating and eliminating the stress in the above system leads to the well-known incompressible Stokes equations:

$$\begin{cases} -\nabla\cdot\boldsymbol{\epsilon}(\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega. \end{cases} \tag{2.5}$$

3. General boundary conditions. For simplicity, we assume that the boundary conditions are homogeneous:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{0} \quad \text{on } \Gamma_N. \tag{3.1}$$

When Γ_N is nonempty, because of the traction boundary condition, it is natural and necessary to have the stress be the independent variable. Hence, we study least-squares functionals based on formulations for stress-velocity-pressure (section 3.1) and for stress-velocity (section 3.2). Our primary goal in this section is to establish continuity and ellipticity for these least-squares functionals in appropriate Hilbert spaces. The least-squares finite element method based on the stress-velocity formulation is described in section 3.3.

3.1. Least-squares functionals based on the stress-velocity-pressure formulation. The first-order system (2.4), together with boundary conditions (3.1), is the stress-velocity-pressure formulation for linear, stationary incompressible Newtonian flow. Taking the trace of the second equation in (2.4) and using the fact that

$$\operatorname{tr}\boldsymbol{\epsilon}(\mathbf{u}) = \nabla\cdot\mathbf{u} = 0,$$

we have the following important relation between the pressure and normal stress:

$$\operatorname{tr}\boldsymbol{\sigma} + d\,p = 0. \tag{3.2}$$

Before defining least-squares functionals, let us first describe solution spaces. When $\Gamma_D = \partial\Omega$, Stokes system (2.5) and (3.1) have a unique solution, provided that

$$\int_\Omega p\,dx = 0. \tag{3.3}$$

Together with (3.2), this implies

$$\int_\Omega \mathrm{tr}\boldsymbol{\sigma}\, dx = 0.$$

Therefore, we are at liberty to impose these conditions on the stress and pressure. Thus, define the spaces

$$\mathbf{X}_N = \begin{cases} H_N(\mathrm{div};\,\Omega)^d & \text{if } \Gamma_N \neq \emptyset, \\ \mathbf{X}_0 \equiv \left\{ \boldsymbol{\tau} \in H(\mathrm{div};\,\Omega)^d \mid \int_\Omega \mathrm{tr}\,\boldsymbol{\tau}\, dx = 0 \right\} & \text{otherwise} \end{cases}$$

and

$$L^2_N(\Omega) = \begin{cases} L^2(\Omega) & \text{if } \Gamma_N \neq \emptyset, \\ L^2_0(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_\Omega q\, dx = 0 \right\} & \text{otherwise.} \end{cases}$$

Then for $\mathbf{f} \in L^2(\Omega)^d$ we define the following least-squares functionals:

$$G_{-1}(\boldsymbol{\sigma},\, \mathbf{u},\, p\,;\, \mathbf{f}) = \|\nabla\cdot\boldsymbol{\sigma} + \mathbf{f}\|^2_{-1,D} + \|\boldsymbol{\sigma} + p\, I - \boldsymbol{\epsilon}(\mathbf{u})\|^2 + \|\nabla\cdot\mathbf{u}\|^2 \tag{3.4}$$

and

$$G(\boldsymbol{\sigma},\, \mathbf{u},\, p\,;\, \mathbf{f}) = \|\nabla\cdot\boldsymbol{\sigma} + \mathbf{f}\|^2 + \|\boldsymbol{\sigma} + p\, I - \boldsymbol{\epsilon}(\mathbf{u})\|^2 + \|\nabla\cdot\mathbf{u}\|^2 \tag{3.5}$$

for $(\boldsymbol{\sigma},\, \mathbf{u},\, p) \in \mathcal{V} \equiv \mathbf{X}_N \times H^1_D(\Omega)^d \times L^2_N(\Omega)$. We will first establish uniform boundedness and ellipticity (i.e., equivalence) of the homogeneous functionals $G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{0})$ and $G(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{0})$ in terms of the respective functionals $M_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q)$ and $M(\boldsymbol{\tau},\, \mathbf{v},\, q)$ defined on $\mathcal{V}$ by

$$M_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q) = \|\mathbf{v}\|_1^2 + \|q\|^2 + \|\boldsymbol{\tau}\|^2$$

and

$$M(\boldsymbol{\tau},\, \mathbf{v},\, q) = \|\mathbf{v}\|_1^2 + \|q\|^2 + \|\boldsymbol{\tau}\|^2 + \|\nabla\cdot\boldsymbol{\tau}\|^2.$$

To accomplish this, let $\mathcal{A}_\lambda : R^{d\times d} \longrightarrow R^{d\times d}$ be a linear map defined by

$$\mathcal{A}_\lambda\, \boldsymbol{\tau} = \boldsymbol{\tau} - \frac{\lambda}{d\lambda + 2\mu}(\mathrm{tr}\,\boldsymbol{\tau})\, I \quad \forall\, \boldsymbol{\tau} \in R^{d\times d}.$$

The $\mathcal{A}_\lambda$ is the compliance tensor of fourth order, a terminology from elasticity. Parameters λ and μ are material constants for both solids and fluids. We will use the following fundamental inequality for the trace of $\mathbf{X}_N$:

$$\|\mathrm{tr}\,\boldsymbol{\tau}\| \le C\, \left(\sqrt{(\mathcal{A}_\lambda\, \boldsymbol{\tau},\, \boldsymbol{\tau})} + \|\nabla\cdot\boldsymbol{\tau}\|_{-1,D} \right) \quad \forall\, \boldsymbol{\tau} \in \mathbf{X}_N, \tag{3.6}$$

where C is a positive constant independent of λ. This inequality was proved in [1] for two dimensions and Dirichlet boundary conditions (i.e., $d = 2$ and $\Gamma_N = \emptyset$) and in [13] for both two and three dimensions and general boundary conditions. When λ approaches ∞, the limit of the linear map $\mathcal{A}_\lambda$ is

$$\mathcal{A}_\infty\, \boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{d}\,(\mathrm{tr}\,\boldsymbol{\tau})\, I : R^{d\times d} \longrightarrow R^{d\times d}.$$

Note that $\mathcal{A}_\infty$ is not an invertible map. A simple calculation gives

$$\begin{cases} (\mathcal{A}_\infty \boldsymbol{\tau},\, \boldsymbol{\tau}) = \|\boldsymbol{\tau}\|^2 - \frac{1}{d}\,\|\mathrm{tr}\,\boldsymbol{\tau}\|^2 = \|\mathcal{A}_\infty \boldsymbol{\tau}\|^2, \\ (\mathcal{A}_\lambda \boldsymbol{\tau},\, \boldsymbol{\tau}) = \|\boldsymbol{\tau}\|^2 - \frac{\lambda}{d\lambda+2\mu}\|\mathrm{tr}\,\boldsymbol{\tau}\|^2 = \|\mathcal{A}_\infty \boldsymbol{\tau}\|^2 + \frac{2\mu}{d(d\lambda+2\mu)}\|\mathrm{tr}\,\boldsymbol{\tau}\|^2. \end{cases} \tag{3.7}$$

Since the constant in (3.6) is independent of λ, taking the limit of (3.6) as $\lambda \to \infty$ and using the first equation in (3.7) we obtain

$$\|\mathrm{tr}\,\boldsymbol{\tau}\| \le C\, \left(\|\mathcal{A}_\infty \boldsymbol{\tau}\| + \|\nabla\cdot\boldsymbol{\tau}\|_{-1,D}\right) \quad \forall\, \boldsymbol{\tau} \in \mathbf{X}_N. \tag{3.8}$$

Let $\|\boldsymbol{\tau}\|_a \equiv \left(\|\mathcal{A}_\infty \boldsymbol{\tau}\|^2 + \|\nabla\cdot\boldsymbol{\tau}\|^2_{-1,\,D}\right)^{\frac{1}{2}}$; then $\|\boldsymbol{\tau}\|_a$ is equivalent to the L^2 norm.

LEMMA 3.1. *There exists a positive constant C such that*

$$\frac{1}{C}\|\boldsymbol{\tau}\|^2 \le \|\boldsymbol{\tau}\|_a^2 \le C\,\|\boldsymbol{\tau}\|^2 \quad \forall\, \boldsymbol{\tau} \in \mathbf{X}_N. \tag{3.9}$$

Proof. From the definition of the inverse norm and the Cauchy–Schwarz inequality, we have that

$$\|\nabla\cdot\boldsymbol{\tau}\|_{-1,\,D} \le \|\boldsymbol{\tau}\|. \tag{3.10}$$

Equation (3.9) follows easily from (3.7), (3.8), and (3.10). ∎

Now we are ready to establish equivalence between functionals G_{-1} and M_{-1} and equivalence between functionals G and M.

THEOREM 3.2. *The homogeneous functionals $G_{-1}(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\,\mathbf{0})$ and $G(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\,\mathbf{0})$ are uniformly equivalent to the functionals $M_{-1}(\boldsymbol{\tau},\,\mathbf{v},\,q)$ and $M(\boldsymbol{\tau},\,\mathbf{v},\,q)$, respectively; i.e., there exist positive constants C_1 and C_2 such that*

$$\frac{1}{C_1}M_{-1}(\boldsymbol{\tau},\,\mathbf{v},\,q) \le G_{-1}(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\,\mathbf{0}) \le C_1 M_{-1}(\boldsymbol{\tau},\,\mathbf{v},\,q) \tag{3.11}$$

and

$$\frac{1}{C_2}M(\boldsymbol{\tau},\,\mathbf{v},\,q) \le G(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\,\mathbf{0}) \le C_2 M(\boldsymbol{\tau},\,\mathbf{v},\,q) \tag{3.12}$$

hold for all $(\boldsymbol{\tau},\,\mathbf{v},\,q) \in \mathcal{V}$.

Proof. The upper bounds in both (3.11) and (3.12) follow easily from the triangle inequality and (3.10).

To show the validity of the lower bound in (3.11), we first note that

$$\begin{aligned} \|\boldsymbol{\tau} - \boldsymbol{\tau}^t\| &= \|\,(\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v})) - (\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v}))^t\,\| \\ &\le 2\,\|\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v})\|. \end{aligned}$$

We used symmetry of I and $\boldsymbol{\epsilon}(\mathbf{v})$ and the triangle inequality above. Now integration by parts and the Cauchy–Schwarz and Korn inequalities lead to

$$\begin{aligned} |(\boldsymbol{\tau},\,\boldsymbol{\epsilon}(\mathbf{v}))| &= \left|\left(\frac{\boldsymbol{\tau}+\boldsymbol{\tau}^t}{2},\,\boldsymbol{\epsilon}(\mathbf{v})\right)\right| = \left|\left(\frac{\boldsymbol{\tau}+\boldsymbol{\tau}^t}{2},\,\nabla\,\mathbf{v}\right)\right| \\ &= \left|(\boldsymbol{\tau},\,\nabla\mathbf{v}) - \left(\frac{\boldsymbol{\tau}-\boldsymbol{\tau}^t}{2},\,\nabla\mathbf{v}\right)\right| = \left|(-\nabla\cdot\boldsymbol{\tau},\,\mathbf{v}) - \left(\frac{\boldsymbol{\tau}-\boldsymbol{\tau}^t}{2},\,\nabla\mathbf{v}\right)\right| \\ &\le \|\nabla\cdot\boldsymbol{\tau}\|_{-1,\,D}\,\|\mathbf{v}\|_1 + \|\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v})\|\,\|\nabla\,\mathbf{v}\| \\ &\le C\,\left(\|\nabla\cdot\boldsymbol{\tau}\|_{-1,\,D} + \|\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v})\|\right)\,\|\boldsymbol{\epsilon}(\mathbf{v})\| \\ &\le C\,G_{-1}^{\frac{1}{2}}(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0})\,\|\boldsymbol{\epsilon}(\mathbf{v})\|, \end{aligned} \tag{3.13}$$

where $G^{\frac{1}{2}}_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0})$ denotes the square root of $G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0})$. To bound the deformation rate tensor $\boldsymbol{\epsilon}(\mathbf{v})$, it follows from the fact that

$$(q\,I,\, \boldsymbol{\epsilon}(\mathbf{v})) = (q,\, \nabla\cdot\mathbf{v}),$$

the Cauchy–Schwarz inequality, and (3.13) that

$$\begin{aligned}
\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 &= (\boldsymbol{\epsilon}(\mathbf{v}) - \boldsymbol{\tau} - q\,I,\, \boldsymbol{\epsilon}(\mathbf{v})) + (\boldsymbol{\tau},\, \boldsymbol{\epsilon}(\mathbf{v})) + (q,\, \nabla\cdot\mathbf{v}) \\
&\le \|\boldsymbol{\epsilon}(\mathbf{v}) - \boldsymbol{\tau} - q\,I\|\, \|\boldsymbol{\epsilon}(\mathbf{v})\| + C\, G^{\frac{1}{2}}_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0})\, \|\boldsymbol{\epsilon}(\mathbf{v})\| + \|q\|\, \|\nabla\cdot\mathbf{v}\| \\
&\le \|\boldsymbol{\epsilon}(\mathbf{v}) - \boldsymbol{\tau} - q\,I\|^2 + C\, G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0}) + \frac{1}{2}\, \|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \|q\|\, \|\nabla\cdot\mathbf{v}\|.
\end{aligned}$$

This implies that

$$\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 \le C\, G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0}) + 2\,\|q\|\, \|\nabla\cdot\mathbf{v}\|. \tag{3.14}$$

Now to bound $\|q\|$ in (3.14), since $\mathrm{tr}\ (\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v})) = \mathrm{tr}\,\boldsymbol{\tau} + d\,q - \nabla\cdot\mathbf{v}$, we have

$$\|\mathrm{tr}\,\boldsymbol{\tau} + d\,q - \nabla\cdot\mathbf{v}\| \le d\,\|\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v})\|.$$

It then follows from the triangle inequality that

$$\begin{aligned}
\|q\| &\le \frac{1}{d}\,(\|\mathrm{tr}\,\boldsymbol{\tau} + d\,q - \nabla\cdot\mathbf{v}\| + \|\mathrm{tr}\,\boldsymbol{\tau}\| + \|\nabla\cdot\mathbf{v}\|) \\
&\le d\,G^{\frac{1}{2}}_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0}) + \|\mathrm{tr}\,\boldsymbol{\tau}\|. \qquad (3.15)
\end{aligned}$$

Next, we bound $\|\mathrm{tr}\,\boldsymbol{\tau}\|$ above by the homogeneous functional and the L^2 norm of the deformation rate tensor. To do so, we first establish a similar upper bound for $\|\mathcal{A}_\infty\boldsymbol{\tau}\|$. Note that $\mathcal{A}_\infty^2 = \mathcal{A}_\infty$ and that $(q\,I,\, \mathcal{A}_\infty\boldsymbol{\tau}) = 0$. These identities and the Cauchy–Schwarz inequality lead to

$$\begin{aligned}
\|\mathcal{A}_\infty\boldsymbol{\tau}\|^2 &= (\boldsymbol{\tau},\, \mathcal{A}_\infty\boldsymbol{\tau}) = (\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v}),\, \mathcal{A}_\infty\boldsymbol{\tau}) + (\boldsymbol{\epsilon}(\mathbf{v}),\, \mathcal{A}_\infty\boldsymbol{\tau}) \\
&\le (\|\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v})\| + \|\boldsymbol{\epsilon}(\mathbf{v})\|)\, \|\mathcal{A}_\infty\boldsymbol{\tau}\|,
\end{aligned}$$

which implies that

$$\|\mathcal{A}_\infty\boldsymbol{\tau}\| \le \|\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v})\| + \|\boldsymbol{\epsilon}(\mathbf{v})\|. \tag{3.16}$$

Together with (3.9), inequality (3.16) yields

$$\begin{aligned}
\|\mathrm{tr}\,\boldsymbol{\tau}\| \le \|\boldsymbol{\tau}\| &\le C\ (\|\mathcal{A}_\infty\boldsymbol{\tau}\| + \|\nabla\cdot\boldsymbol{\tau}\|_{-1,D}) \\
&\le C\ (\|\boldsymbol{\tau} + q\,I - \boldsymbol{\epsilon}(\mathbf{v})\| + \|\boldsymbol{\epsilon}(\mathbf{v})\| + \|\nabla\cdot\boldsymbol{\tau}\|_{-1,D}) \\
&\le C\ \left(G^{\frac{1}{2}}_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0}) + \|\boldsymbol{\epsilon}(\mathbf{v})\|\right). \qquad (3.17)
\end{aligned}$$

Now, combining upper bounds in (3.14), (3.15), and (3.17) and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 &\le C\, G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0}) + \left(d\,G^{\frac{1}{2}}_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0})\ + \|\mathrm{tr}\,\boldsymbol{\tau}\|\right)\, \|\nabla\cdot\mathbf{v}\| \\
&\le C\, G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0}) + C\ \left(G^{\frac{1}{2}}_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0})\ + \|\boldsymbol{\epsilon}(\mathbf{v})\|\right)\, \|\nabla\cdot\mathbf{v}\| \\
&\le C\, G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0}) + C\, \|\boldsymbol{\epsilon}(\mathbf{v})\|\, \|\nabla\cdot\mathbf{v}\|.
\end{aligned}$$

Hence,

$$\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 \le C\, G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0}),$$

which, together with (3.17), (3.15), and (3.9), implies that both $\|\boldsymbol{\tau}\|^2$ and $\|q\|^2$ are also bounded above by the homogeneous functional $G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0})$. This completes the proof of the lower bound in (3.11). Since

$$G_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{0}) \le G(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{0}) \quad \text{and} \quad \|\nabla\cdot\boldsymbol{\tau}\|^2 \le G(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{0}),$$

then the lower bound in (3.12) follows from (3.11). The proof of the theorem is therefore completed. ∎

3.2. Least-squares functionals based on the stress-velocity formulation. In this section, we derive the stress-velocity formulation by using relation (3.2) to eliminate the pressure. We then define least-squares functionals based on this formulation and establish their ellipticity and continuity.

Assume that the first equation in (2.3) holds. Then it is easy to see that (3.2) is equivalent to the incompressible condition, the second equation in (2.3). Relation (3.2) says that the pressure is the negative of the arithmetic average of the normal stress. Since the stress is a variable in our first-order system, using (3.2) we eliminate the pressure in the first equation of (2.3) to obtain the following constitutive equation:

$$\mathcal{A}_\infty\, \boldsymbol{\sigma} = \boldsymbol{\sigma} - \frac{1}{d}(\operatorname{tr}\boldsymbol{\sigma})\, I = \boldsymbol{\epsilon}(\mathbf{u}) \quad \text{in} \quad \Omega. \tag{3.18}$$

Note that taking the trace of this equation yields the incompressible condition. This and the momentum equation define the stress-velocity formulation for incompressible Newtonian fluid flow problems. In particular, for linear stationary problems, we have

$$\begin{cases} \mathcal{A}_\infty\, \boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u}) &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \nabla\cdot\boldsymbol{\sigma} + \mathbf{f} &= \mathbf{0} \quad \text{in} \quad \Omega, \end{cases} \tag{3.19}$$

with boundary conditions (3.1). Let

$$\tilde{\mathcal{V}} = \mathbf{X}_N \times H^1_D(\Omega)^d.$$

For $\mathbf{f} \in L^2(\Omega)^d$, we define the following least-squares functionals:

$$\tilde{G}_{-1}(\boldsymbol{\sigma},\, \mathbf{u}\,;\, \mathbf{f}) = \|\mathcal{A}_\infty\, \boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u})\|^2 + \|\nabla\cdot\boldsymbol{\sigma} + \mathbf{f}\|^2_{-1,D} \tag{3.20}$$

and

$$\tilde{G}(\boldsymbol{\sigma},\, \mathbf{u}\,;\, \mathbf{f}) = \|\mathcal{A}_\infty\, \boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u})\|^2 + \|\nabla\cdot\boldsymbol{\sigma} + \mathbf{f}\|^2 \tag{3.21}$$

for $(\boldsymbol{\sigma},\, \mathbf{u}) \in \tilde{\mathcal{V}}$. We also define the norm functionals

$$\tilde{M}_{-1}(\boldsymbol{\tau},\, \mathbf{v}) = \|\mathbf{v}\|_1^2 + \|\boldsymbol{\tau}\|^2$$

and

$$\tilde{M}(\boldsymbol{\tau},\, \mathbf{v}) = \|\mathbf{v}\|_1^2 + \|\boldsymbol{\tau}\|^2 + \|\nabla\cdot\boldsymbol{\tau}\|^2.$$

THEOREM 3.3. *The homogeneous functionals $\tilde{G}_{-1}(\boldsymbol{\tau},\, \mathbf{v};\, \mathbf{0})$ and $\tilde{G}(\boldsymbol{\tau},\, \mathbf{v};\, \mathbf{0})$ are uniformly equivalent to the functionals $\tilde{M}_{-1}(\boldsymbol{\tau},\, \mathbf{v})$ and $\tilde{M}(\boldsymbol{\tau},\, \mathbf{v})$, respectively; i.e., there exist positive constants C_1 and C_2 such that*

$$\frac{1}{C_1}\, \tilde{M}_{-1}(\boldsymbol{\tau},\, \mathbf{v}) \le \tilde{G}_{-1}(\boldsymbol{\tau},\, \mathbf{v}\,;\, \mathbf{0}) \le C_1\, \tilde{M}_{-1}(\boldsymbol{\tau},\, \mathbf{v}) \tag{3.22}$$

and

$$\frac{1}{C_2}\, \tilde{M}(\boldsymbol{\tau},\, \mathbf{v}) \le \tilde{G}(\boldsymbol{\tau},\, \mathbf{v}\,;\, \mathbf{0}) \le C_2\, \tilde{M}(\boldsymbol{\tau},\, \mathbf{v}) \tag{3.23}$$

hold for all $(\boldsymbol{\tau},\, \mathbf{v}) \in \tilde{\mathcal{V}}$.

Proof. Since $\|\mathrm{tr}\, \boldsymbol{\tau}\| \le d\, \|\boldsymbol{\tau}\|$, Theorem 3.2 with the choice of $q = -\mathrm{tr}\, \boldsymbol{\tau}/d$ yields the upper bounds in both (3.22) and (3.23) and the following lower bounds:

$$\tilde{M}_{-1}(\boldsymbol{\tau},\, \mathbf{v}) \le C\, \left(\tilde{G}_{-1}(\boldsymbol{\tau},\, \mathbf{v}\,;\, \mathbf{0}) + \|\nabla\cdot \mathbf{v}\|^2\right)$$

and

$$\tilde{M}(\boldsymbol{\tau},\, \mathbf{v}) \le C\, \left(\tilde{G}(\boldsymbol{\tau},\, \mathbf{v}\,;\, \mathbf{0}) + \|\nabla\cdot \mathbf{v}\|^2\right).$$

Now the lower bounds in both (3.22) and (3.23) are a direct consequence of the bound

$$\|\nabla\cdot \mathbf{v}\| = \|\mathrm{tr}\,(\boldsymbol{\epsilon}(\mathbf{v}) - \mathcal{A}_\infty\, \boldsymbol{\tau})\| \le d\, \|\boldsymbol{\epsilon}(\mathbf{v}) - \mathcal{A}_\infty\, \boldsymbol{\tau}\|. \qquad \square$$

3.3. Least-squares finite element methods. In this section, we restrict our attention to the least-squares method based on the L^2 norm least-squares functional $\tilde{G}$ for the stress-velocity formulation, although the method developed in this section can be developed in the same manner for the stress-velocity-pressure formulation, and discrete inverse norm methods can be developed for the inverse norm functionals (see [6]). In fact, it seems that the least-squares method based on the stress-velocity formulation may be preferable since it does not involve the pressure and, more importantly, since it has mathematical structure similar to that of linear elasticity. Consequently, we develop a unified numerical approach for both linear elasticity and linear, stationary incompressible Newtonian flows. The pressure, if desired, can be recovered using (3.2).

The variational problem corresponding to the L^2 norm least-squares functional for the stress-velocity formulation is to minimize functional (3.21) over $\tilde{\mathcal{V}}$, that is, to find $(\boldsymbol{\sigma},\, \mathbf{u}) \in \tilde{\mathcal{V}}$ such that

$$\tilde{G}(\boldsymbol{\sigma},\, \mathbf{u}\,;\, \mathbf{f}) = \inf_{(\boldsymbol{\tau},\, \mathbf{v}) \in \tilde{\mathcal{V}}} \tilde{G}(\boldsymbol{\tau},\, \mathbf{v}\,;\, \mathbf{f}). \tag{3.24}$$

By Theorem 3.3, we can conclude that (3.24) has a unique solution.

Now (3.24) very much resembles the variational problem for the least-squares formulation of linear elasticity developed in [14]. In particular, the elasticity least-squares problem for limiting case $\lambda \to \infty$ is precisely (3.24). In [14], optimal accuracy for the least-squares finite element approximations and optimal multigrid convergence rates for solving the resulting algebraic equations are established to be uniform in λ. This indicates that using the finite elements in [14] to discretize the least-squares problem in (3.24) will give optimal accuracy, and multigrid methods with optimal complexity can be used to solve the resulting algebraic equations. For completeness, we describe these finite elements and their approximation properties and comment on multigrid methods for solving the resulting algebraic systems. For simplicity, we take the two-dimensional case ($d = 2$).

Assuming that the domain Ω is polygonal, let $\mathcal{T}_h$ be a regular triangulation of Ω (see [16]) with triangular elements of size $\mathcal{O}(h)$. Let $P_k(K)$ be the space of polynomials of degree k on triangle K, and denote the local Raviart–Thomas space of order k on K:

$$RT_k(K) = P_k(K)^2 + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} P_k(K).$$

Then the standard $H(\mathrm{div};\,\Omega)$ conforming Raviart–Thomas space of order k [22] and the standard (conforming) continuous piecewise polynomials of degree $k+1$ are defined, respectively, by

$$\Sigma_h^k = \{\boldsymbol{\tau} \in \mathbf{X}_N \,:\, \boldsymbol{\tau}|_K \in RT_k(K)^2 \;\forall\, K \in \mathcal{T}_h\} \subset \mathbf{X}_N, \tag{3.25}$$

$$V_h^{k+1} = \{\mathbf{v} \in C^0(\Omega)^2 \,:\, \mathbf{v}|_K \in P_{k+1}(K)^2 \;\forall\, K \in \mathcal{T}_h,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\} \subset H_D^1(\Omega)^2. \tag{3.26}$$

Space Σ_h^k is used to approximate the stress, and space V_h^{k+1} is used to approximate the velocity. These spaces have the following approximation properties: let $k \ge 0$ be an integer, and let $l \in (0,\, k+1]$:

$$\inf_{\boldsymbol{\tau} \in \Sigma_h^k} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{H(\mathrm{div};\,\Omega)} \le C\, h^l \left(\|\boldsymbol{\sigma}\|_l + \|\nabla \cdot \boldsymbol{\sigma}\|_l\right) \tag{3.27}$$

for $\boldsymbol{\sigma} \in H^l(\Omega)^{2\times 2} \cap \mathbf{X}_N$ with $\nabla \cdot \boldsymbol{\sigma} \in H^l(\Omega)^2$ and

$$\inf_{\mathbf{u} \in V_h^{k+1}} \|\mathbf{u} - \mathbf{v}\|_1 \le C\, h^l\, \|\mathbf{u}\|_{l+1} \tag{3.28}$$

for $\mathbf{u} \in H^{l+1}(\Omega)^2 \cap H_D^1(\Omega)^2$. Based on the smoothness of $\boldsymbol{\sigma}$ and $\mathbf{u}$, we will choose $k+1$ to be the smallest integer greater than or equal to l.

The finite element discretization of our stress-velocity least-squares variational problem is as follows: find $(\boldsymbol{\sigma}^h,\, \mathbf{u}^h) \in \Sigma_h^k \times V_h^{k+1}$ such that

$$\tilde{G}(\boldsymbol{\sigma}^h,\, \mathbf{u}^h;\, \mathbf{f}) = \min_{(\boldsymbol{\tau},\, \mathbf{v}) \in \Sigma_h^k \times V_h^{k+1}} \tilde{G}(\boldsymbol{\tau},\, \mathbf{v};\, \mathbf{f}). \tag{3.29}$$

By Theorem 3.3 and the fact that $\Sigma_h^k \times V_h^{k+1}$ is a subspace of $\tilde{\mathcal{V}}$, (3.29) has a unique solution. As proved in [14], we have the following error estimations.

THEOREM 3.4. *Assume that the solution* $(\boldsymbol{\sigma},\, \mathbf{u})$ *of* (3.24) *is in* $H^l(\Omega)^{2\times 2} \times H^{l+1}(\Omega)^2$ *and that the divergence of the stress* $\nabla \cdot \boldsymbol{\sigma}$ *is in* $H^l(\Omega)^2$. *Let* $k+1$ *be the smallest integer greater than or equal to* l. *Then with* $(\boldsymbol{\sigma}^h,\, \mathbf{u}^h) \in \Sigma_h^k \times V_h^{k+1}$ *denoting the solution to* (3.29), *the following error estimate holds:*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\|_{H(\mathrm{div};\,\Omega)} + \|\mathbf{u} - \mathbf{u}^h\|_1 \le C\, h^l \left(\|\boldsymbol{\sigma}\|_l + \|\nabla \cdot \boldsymbol{\sigma}\|_l + \|\mathbf{u}\|_{l+1}\right). \tag{3.30}$$

As for the pressure, it can be recovered algebraically using (3.2):

$$p^h = -\frac{1}{d}\, \mathrm{tr}\, \boldsymbol{\sigma}^h. \tag{3.31}$$

It follows from (3.2), (3.31), the triangle inequality, and Theorem 3.4 that

$$\|p - p^h\| = \frac{1}{d}\|\mathrm{tr}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}^h)\| \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\| \le C\,h^l. \tag{3.32}$$

Remark. Theorem 3.3 states that the homogeneous functional $\tilde{G}(\boldsymbol{\tau},\,\mathbf{v};\,\mathbf{0})$ is equivalent to the $H(\mathrm{div};\,\Omega)$ norm for the tensor variable and the H^1 norm for the vector variable. It is then well known that multigrid methods applied to discrete linear system (3.29) have optimal convergence properties (see, e.g., [19, 2, 12, 20, 24]).

4. Pure Dirichlet boundary conditions. Many applications in incompressible Newtonian fluid flow are not posed under traction boundary conditions. It is then not necessary to use the stress as the independent variable. In fact, the stress and the deformation rate tensor may not be the variables of choice, especially if the vorticity is needed. This is because the vorticity is the skew-symmetric part of the velocity gradient, and thus the stress and deformation rate tensor do not contain information on the vorticity. For this reason, in this section we develop a least-squares method involving variables that can recover the velocity gradient and vorticity without differentiation. This least-squares method will use the finite element spaces described in section 3.3.

For simplicity, we assume the homogeneous Dirichlet boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{4.1}$$

4.1. First-order systems. Whereas the vorticity is the skew-symmetric part of the velocity gradient, the deformation rate tensor $\boldsymbol{\epsilon}(\mathbf{u})$ is the symmetric part of the velocity gradient. From the second equation of first-order system (2.4), it is then not possible to algebraically obtain the vorticity from the stress tensor. To accomplish this, a new variable must be introduced in place of the stress. This new variable should be chosen such that the resulting least-squares functionals have properties similar to G_{-1} and G ($\tilde{G}_{-1}$ and $\tilde{G}$) and such that both the stress and vorticity can be algebraically obtained from this variable. Insight into designing this new variable can be obtained by noting that for incompressible fluids the divergence of $(\nabla\mathbf{u})^t$ vanishes:

$$\nabla\cdot(\nabla\mathbf{u})^t = \nabla\cdot\begin{pmatrix} \partial_1 u_1 & \cdots & \partial_1 u_d \\ \vdots & \vdots & \vdots \\ \partial_d u_1 & \cdots & \partial_d u_d \end{pmatrix} = \nabla\,(\nabla\cdot\mathbf{u}) = \mathbf{0}. \tag{4.2}$$

Specifically, defining the new independent tensor variable, the pseudostress, to be

$$\tilde{\boldsymbol{\sigma}} = \frac{1}{2}\,\nabla\,\mathbf{u} - p\,I, \tag{4.3}$$

then

$$\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} + \frac{1}{2}\,(\nabla\,\mathbf{u})^t, \tag{4.4}$$

and so by (4.2) we have

$$\nabla\cdot\tilde{\boldsymbol{\sigma}} = \nabla\cdot\boldsymbol{\sigma}. \tag{4.5}$$

Moreover, using the incompressibility of $\mathbf{u}$, we have

$$\mathrm{tr}\,\tilde{\boldsymbol{\sigma}} = \mathrm{tr}\,\boldsymbol{\sigma} = -d\,p. \tag{4.6}$$

The pseudostress is not symmetric and probably not a primitive physical quantity. However, the resulting first-order system is

$$\left\{\begin{array}{rcll} -\nabla\cdot\tilde{\boldsymbol{\sigma}} &=& \mathbf{f} & \text{in} \quad \Omega, \\ \tilde{\boldsymbol{\sigma}} + p\,I - \frac{1}{2}\nabla\mathbf{u} &=& \mathbf{0} & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in} \quad \Omega, \end{array}\right. \tag{4.7}$$

which is essentially equivalent to (2.4). Differentiating and eliminating $\tilde{\boldsymbol{\sigma}}$ in (4.7) leads to the incompressible Stokes equations:

$$\left\{\begin{array}{rcll} -\frac{1}{2}\Delta\mathbf{u} + \nabla p &=& \mathbf{f} & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in} \quad \Omega. \end{array}\right. \tag{4.8}$$

4.2. Least-squares functionals. For $\mathbf{f} \in L^2(\Omega)^d$, we define the following least-squares functionals based on first-order system (4.7):

$$F_{-1}(\tilde{\boldsymbol{\sigma}},\, \mathbf{u},\, p\,;\, \mathbf{f}) = \|\nabla\cdot\tilde{\boldsymbol{\sigma}} + \mathbf{f}\|_{-1,0}^2 + \left\|\tilde{\boldsymbol{\sigma}} + p\,I - \frac{1}{2}\nabla\mathbf{u}\right\|^2 + \|\nabla\cdot\mathbf{u}\|^2 \tag{4.9}$$

and

$$F(\tilde{\boldsymbol{\sigma}},\, \mathbf{u},\, p\,;\, \mathbf{f}) = \|\nabla\cdot\tilde{\boldsymbol{\sigma}} + \mathbf{f}\|^2 + \left\|\tilde{\boldsymbol{\sigma}} + p\,I - \frac{1}{2}\nabla\mathbf{u}\right\|^2 + \|\nabla\cdot\mathbf{u}\|^2 \tag{4.10}$$

for $(\tilde{\boldsymbol{\sigma}},\, \mathbf{u},\, p) \in \mathcal{V}_0 \equiv \mathbf{X}_0 \times H_0^1(\Omega)^d \times L_0^2(\Omega)$.

THEOREM 4.1. *The homogeneous functionals $F_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{0})$ and $F(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{0})$ are uniformly equivalent to the functionals $M_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q)$ and $M(\boldsymbol{\tau},\, \mathbf{v},\, q)$, respectively; i.e., there exist positive constants C_1 and C_2 such that*

$$\frac{1}{C_1} M_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q) \le F_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{0}) \le C_1 M_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q) \tag{4.11}$$

and

$$\frac{1}{C_2} M(\boldsymbol{\tau},\, \mathbf{v},\, q) \le F(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{0}) \le C_2 M(\boldsymbol{\tau},\, \mathbf{v},\, q) \tag{4.12}$$

hold for all $(\boldsymbol{\tau},\, \mathbf{v},\, q) \in \mathcal{V}_0$.

Proof. The theorem can be proved in a similar manner as in Theorem 3.2. Actually, the key inequality

$$\|\nabla\mathbf{v}\|^2 \le C\, F_{-1}(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\mathbf{0}) + C\,\|q\|\,\|\nabla\cdot\mathbf{v}\|, \tag{4.13}$$

which is similar to inequality (3.14), can be established easily: integration by parts and the Cauchy–Schwarz inequality lead to

$$\begin{aligned} \frac{1}{2}\,\|\nabla\mathbf{v}\|^2 &= \left(\frac{1}{2}\nabla\mathbf{v} - \boldsymbol{\tau} - q\,I,\, \nabla\mathbf{v}\right) - (\nabla\cdot\boldsymbol{\tau},\, \mathbf{v}) + (q,\, \nabla\cdot\mathbf{v}) \\ &\le \left\|\frac{1}{2}\nabla\mathbf{v} - \boldsymbol{\tau} - q\,I\right\|\, \|\nabla\mathbf{v}\| + \|\nabla\cdot\boldsymbol{\tau}\|_{-1,0}\,\|\mathbf{v}\|_1 + \|q\|\,\|\nabla\cdot\mathbf{v}\|. \end{aligned}$$

Now (4.13) follows from the Poincaré and ϵ inequalities. □

As in section 3.2, we can derive the following first-order system without the pressure:

$$\begin{cases} \mathcal{A}_\infty \tilde{\boldsymbol{\sigma}} - \frac{1}{2}\nabla \mathbf{u} = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \tilde{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0} & \text{in } \Omega. \end{cases} \tag{4.14}$$

The corresponding least-squares functionals are

$$\tilde{F}_{-1}(\tilde{\boldsymbol{\sigma}}, \mathbf{u}\,; \mathbf{f}) = \left\| \mathcal{A}_\infty \tilde{\boldsymbol{\sigma}} - \frac{1}{2}\nabla \mathbf{u} \right\|^2 + \|\nabla \cdot \tilde{\boldsymbol{\sigma}} + \mathbf{f}\|^2_{-1,0} \tag{4.15}$$

and

$$\tilde{F}(\tilde{\boldsymbol{\sigma}}, \mathbf{u}\,; \mathbf{f}) = \left\| \mathcal{A}_\infty \tilde{\boldsymbol{\sigma}} - \frac{1}{2}\nabla \mathbf{u} \right\|^2 + \|\nabla \cdot \tilde{\boldsymbol{\sigma}} + \mathbf{f}\|^2 \tag{4.16}$$

for $(\tilde{\boldsymbol{\sigma}}, \mathbf{u}) \in \tilde{\mathcal{V}}_0 \equiv \mathbf{X}_0 \times H^1_0(\Omega)^d$.

THEOREM 4.2. *The homogeneous functionals $\tilde{F}_{-1}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})$ and $\tilde{F}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{0})$ are uniformly equivalent to the functionals $\tilde{M}_{-1}(\boldsymbol{\tau}, \mathbf{v})$ and $\tilde{M}(\boldsymbol{\tau}, \mathbf{v})$, respectively; i.e., there exist positive constants C_1 and C_2 such that*

$$\frac{1}{C_1}\tilde{M}_{-1}(\boldsymbol{\tau}, \mathbf{v}) \le \tilde{F}_{-1}(\boldsymbol{\tau}, \mathbf{v}\,; \mathbf{0}) \le C_1 \tilde{M}_{-1}(\boldsymbol{\tau}, \mathbf{v}) \tag{4.17}$$

and

$$\frac{1}{C_2}\tilde{M}(\boldsymbol{\tau}, \mathbf{v}) \le \tilde{F}(\boldsymbol{\tau}, \mathbf{v}\,; \mathbf{0}) \le C_2 \tilde{M}(\boldsymbol{\tau}, \mathbf{v}) \tag{4.18}$$

hold for all $(\boldsymbol{\tau}, \mathbf{v}) \in \tilde{\mathcal{V}}_0$.

Proof. The theorem can be shown in a similar fashion as in Theorem 3.3. ∎

Remark. The mixed variational problem based on (4.14) is to find $(\tilde{\boldsymbol{\sigma}}, \mathbf{u}) \in \mathbf{X}_0 \times L^2_0(\Omega)^d$ such that

$$\begin{cases} (\mathcal{A}_\infty \tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + \frac{1}{2}(\mathbf{u}, \nabla \cdot \boldsymbol{\tau}) = \mathbf{0} & \forall\, \boldsymbol{\tau} \in \mathbf{X}_0, \\ (\nabla \cdot \tilde{\boldsymbol{\sigma}}, \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) & \forall\, \mathbf{v} \in L^2(\Omega)^d. \end{cases} \tag{4.19}$$

It is easy to see that (4.19) is essentially a vector version of the mixed formulation for the second-order elliptic problems. Therefore, any stable pair of finite elements for the second-order elliptic problems (see [8]) is also a stable approximation for (4.19). This will be studied in a forthcoming paper.

4.3. Least-squares finite element methods. The variational problem for the L^2 norm least-squares formulation of (4.14) is to minimize least-squares functional (4.16) over $\tilde{\mathcal{V}}_0$, that is, to find $(\tilde{\boldsymbol{\sigma}}, \mathbf{u}) \in \tilde{\mathcal{V}}_0$ such that

$$\tilde{F}(\tilde{\boldsymbol{\sigma}}, \mathbf{u}\,; \mathbf{f}) = \inf_{(\boldsymbol{\tau}, \mathbf{v}) \in \tilde{\mathcal{V}}_0} \tilde{F}(\boldsymbol{\tau}, \mathbf{v}\,; \mathbf{f}). \tag{4.20}$$

By Theorem 4.2, (4.20) has a unique solution. The discrete finite element problem is to find $(\tilde{\boldsymbol{\sigma}}^h, \mathbf{u}^h) \in \Sigma^k_h \times V^{k+1}_h$ such that

$$\tilde{F}(\tilde{\boldsymbol{\sigma}}^h, \mathbf{u}^h; \mathbf{f}) = \min_{(\boldsymbol{\tau}, \mathbf{v}) \in \Sigma^k_h \times V^{k+1}_h} \tilde{F}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{f}). \tag{4.21}$$

Since $\Sigma_h^k \times V_h^{k+1}$ is a subspace of $\tilde{\mathcal{V}}_0$ ($\Gamma_D = \partial\Omega$ and $\Gamma_N = \emptyset$), by Theorem 4.2, (4.21) has a unique solution. We have the following error estimate for the finite element approximation.

THEOREM 4.3. *Assume that the solution* $(\tilde{\boldsymbol{\sigma}}, \mathbf{u})$ *of* (4.20) *is in* $H^l(\Omega)^{2\times 2} \times H^{l+1}(\Omega)^2$ *and that* $\nabla \cdot \tilde{\boldsymbol{\sigma}}$ *is in* $H^l(\Omega)^2$. *Let* $k+1$ *be the smallest integer greater than or equal to* l. *Then with* $(\tilde{\boldsymbol{\sigma}}^h, \mathbf{u}^h) \in \Sigma_h^k \times V_h^{k+1}$ *denoting the solution to* (4.21), *the following error estimate holds:*

$$\|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^h\|_{H(\mathrm{div};\,\Omega)} + \|\mathbf{u} - \mathbf{u}^h\|_1 \le C\,h^l\,(\|\tilde{\boldsymbol{\sigma}}\|_l + \|\nabla\cdot\tilde{\boldsymbol{\sigma}}\|_l + \|\mathbf{u}\|_{l+1})\,. \tag{4.22}$$

4.4. Computation of pressure, stress, and vorticity. Physical quantities such as pressure, stress, and vorticity can be approximated in terms of $\tilde{\boldsymbol{\sigma}}^h$. For the pressure, (4.6) gives

$$p = -\frac{1}{d}\,\mathrm{tr}\,\tilde{\boldsymbol{\sigma}}. \tag{4.23}$$

For the stress, note that the first equation in (4.14) gives

$$\nabla\,\mathbf{u} = 2\,\mathcal{A}_\infty\,\tilde{\boldsymbol{\sigma}}. \tag{4.24}$$

This, together with (4.4), implies

$$\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} + \frac{1}{2}\,(\nabla\,\mathbf{u})^t = \tilde{\boldsymbol{\sigma}} + (\mathcal{A}_\infty\,\tilde{\boldsymbol{\sigma}})^t = \mathcal{A}_\infty\,\tilde{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}}^t. \tag{4.25}$$

For the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, it can be expressed in terms of the entries of the skew-symmetric part of the velocity gradient and, hence, the pseudostress $\tilde{\boldsymbol{\sigma}}$. More precisely, letting $\mathbf{s} = \frac{1}{2}\,(\nabla\,\mathbf{u} - (\nabla\,\mathbf{u})^t)$, the definition of the curl operator gives

$$\boldsymbol{\omega} = \begin{cases} 2s_{21}(\mathbf{u}) & \text{if } d = 2,\\ 2(s_{32}(\mathbf{u}),\, s_{13}(\mathbf{u}),\, s_{21}(\mathbf{u}))^t & \text{if } d = 3. \end{cases}$$

Then by (4.24) we have

$$\mathbf{s}(\mathbf{u}) = \mathcal{A}_\infty\,\tilde{\boldsymbol{\sigma}} - (\mathcal{A}_\infty\,\tilde{\boldsymbol{\sigma}})^t = \tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^t$$

and, hence,

$$\boldsymbol{\omega} = 2\begin{cases} \tilde{\sigma}_{21} - \tilde{\sigma}_{12} & \text{if } d = 2,\\ (\tilde{\sigma}_{32} - \tilde{\sigma}_{23},\, \tilde{\sigma}_{13} - \tilde{\sigma}_{31},\, \tilde{\sigma}_{21} - \tilde{\sigma}_{12})^t & \text{if } d = 3. \end{cases} \tag{4.26}$$

Equations (4.23), (4.25), and (4.26) suggest that we can approximate the pressure, stress, and vorticity as

$$p^h = -\frac{1}{d}\,\mathrm{tr}\,\tilde{\boldsymbol{\sigma}}^h, \qquad \boldsymbol{\sigma}^h = \mathcal{A}_\infty\,\tilde{\boldsymbol{\sigma}}^h + (\tilde{\boldsymbol{\sigma}}^h)^t,$$

$$\boldsymbol{\omega}_h = 2\begin{cases} \tilde{\sigma}^h_{21} - \tilde{\sigma}^h_{12} & \text{if } d = 2,\\ (\tilde{\sigma}^h_{32} - \tilde{\sigma}^h_{23},\, \tilde{\sigma}^h_{13} - \tilde{\sigma}^h_{31},\, \tilde{\sigma}^h_{21} - \tilde{\sigma}^h_{12})^t & \text{if } d = 3. \end{cases}$$

From (4.23), (4.25), (4.26), the triangle inequality, and Theorem 4.3, we have the following error estimates:

$$\|p - p^h\| = \frac{1}{d}\,\|\mathrm{tr}(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^h)\| \le C\,h^l,$$

$$\|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^h\| = \|\mathcal{A}_\infty(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^h) + (\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^h)^t\| \le C\,h^l,$$

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}^h\| \le C\,\|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^h\| \le C\,h^l.$$

4.5. Weakly imposed boundary conditions. In the previous sections, boundary conditions were imposed on the solution spaces. This leads to least-squares finite element approximations that are more accurate on the boundary than in the interior of the domain. In the context of least-squares methods, it is natural to treat boundary conditions weakly through boundary functionals. This is also convenient for many applications.

As an example of least-squares boundary functionals, we describe a least-squares functional with boundary terms for first-order system (4.14):

$$\tilde{F}_b(\tilde{\boldsymbol{\sigma}},\, \mathbf{u}\,;\, \mathbf{f}) = \left\| \mathcal{A}_\infty\, \tilde{\boldsymbol{\sigma}} - \frac{1}{2}\, \nabla\, \mathbf{u} \right\|^2 + \|\nabla \cdot \tilde{\boldsymbol{\sigma}} + \mathbf{f}\|^2 + \|\mathbf{u}\|^2_{\frac{1}{2},\partial\Omega}. \tag{4.27}$$

The least-squares variational problem is to minimize this functional over a solution space free of imposed boundary conditions: find $(\tilde{\boldsymbol{\sigma}},\, \mathbf{u}) \in \tilde{\mathcal{V}}_b \equiv \mathbf{X}_0 \times H^1(\Omega)^d$ such that

$$\tilde{F}_b(\tilde{\boldsymbol{\sigma}},\, \mathbf{u}\,;\, \mathbf{f}) = \inf_{(\boldsymbol{\tau},\, \mathbf{v}) \in \tilde{\mathcal{V}}_b} \tilde{F}_b(\boldsymbol{\tau},\, \mathbf{v}\,;\, \mathbf{f}). \tag{4.28}$$

Using techniques in this paper and in the proof of Theorem 5.1 in [14], we can show that there exists a positive constant C such that

$$\frac{1}{C}\, \tilde{M}(\boldsymbol{\tau},\, \mathbf{v}) \le \tilde{F}_b(\boldsymbol{\tau},\, \mathbf{v}\,;\, \mathbf{0}) \le C\, \tilde{M}(\boldsymbol{\tau},\, \mathbf{v}) \tag{4.29}$$

for all $(\boldsymbol{\tau},\, \mathbf{v}) \in \tilde{\mathcal{V}}_b$. To develop computable finite element methods and the corresponding iterative solvers based on this functional, see [23].

4.6. Relation to existing least-squares methods. There are many existing least-squares methods for the Stokes equations. Since the pseudostress $\tilde{\boldsymbol{\sigma}}$ involves the velocity gradient and the pressure, our approach has some similarities with the methods in [11, 15]. In [11], the velocity gradient is introduced as an independent variable; two additional (consistent) constraints (vanishing trace and curl of the velocity gradient) are added to the original system; the variables of the least-squares method are the velocity, velocity gradient, and pressure; and the homogeneous L^2 norm least-squares functional is elliptic and continuous in $(H(\text{div};\, \Omega)^d \cap H(\mathbf{curl}\,;\, \Omega)^d) \times H^1(\Omega)^d \times H^1(\Omega)$, where $H(\mathbf{curl}\,;\, \Omega)$ is the Hilbert space consisting of square-integrable vectors whose curls are also square-integrable. In [15], a constrained velocity gradient (the velocity gradient satisfying the incompressibility condition) is introduced as an independent variable; the least-squares method is based on the div-curl system of the constraint velocity gradient and the pressure; and the homogeneous functional is elliptic and continuous in $(H(\text{div};\, \Omega)^d \cap H(\mathbf{curl}\,;\, \Omega)^d) \times H^1(\Omega)$. Both methods require sufficient smoothness for the original problem, and, hence, their applicability is very limited.

As a side remark, we comment that the div-curl least-squares method can be developed for our formulations. To see this, applying the curl operator to the first equation of (4.14) leads to the following div-curl system:

$$\left\{ \begin{array}{rcll} \nabla \times (\mathcal{A}_\infty\, \tilde{\boldsymbol{\sigma}}) & = & \mathbf{0} & \text{in} \quad \Omega, \\ \nabla \cdot \tilde{\boldsymbol{\sigma}} + \mathbf{f} & = & \mathbf{0} & \text{in} \quad \Omega, \end{array} \right. \tag{4.30}$$

with boundary conditions $\mathbf{n} \times (\mathcal{A}_\infty\, \tilde{\boldsymbol{\sigma}}) = \mathbf{n} \times \nabla\, \mathbf{u} = \mathbf{0}$ on $\partial\Omega$. The corresponding least-squares functionals are defined as

$$\bar{F}_{-1}(\tilde{\boldsymbol{\sigma}};\, \mathbf{f}) = \|\nabla \times (\mathcal{A}_\infty\, \tilde{\boldsymbol{\sigma}})\,\|^2_{-1,0} + \|\nabla \cdot \tilde{\boldsymbol{\sigma}} + \mathbf{f}\|^2_{-1} \tag{4.31}$$

and

$$\bar{F}(\tilde{\boldsymbol{\sigma}}; \mathbf{f}) = \|\nabla \times (\mathcal{A}_\infty \tilde{\boldsymbol{\sigma}})\|^2 + \|\nabla \cdot \tilde{\boldsymbol{\sigma}} + \mathbf{f}\|^2. \tag{4.32}$$

These div-curl approaches will be studied in a forthcoming paper.

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