Mixed Finite Element Methods for Incompressible Flow: Stationary Stokes Equations

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In this article, we develop and analyze a mixed finite element method for the Stokes equations. Our mixed method is based on the pseudostress-velocity formulation. The pseudostress is approximated by the Raviart-Thomas (RT) element of index $k \geq 0$ and the velocity by piecewise discontinuous polynomials of degree k. It is shown that this pair of finite elements is stable and yields quasi-optimal accuracy. The indefinite system of linear equations resulting from the discretization is decoupled by the penalty method. The penalized pseudostress system is solved by the $H(\text{div})$ type of multigrid method and the velocity is then calculated explicitly. Alternative preconditioning approaches that do not involve penalizing the system are also discussed. Finally, numerical experiments are presented. Numer Methods Partial Differential Eq 000: 000–000, 2009

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I. INTRODUCTION

Let Ω be a bounded, open, connected subset of $\mathcal{R}^d$ ($d = 2$ or 3) with a Lipschitz continuous boundary $\partial\Omega$. Let $\mathbf{f} = (f_1, \ldots, f_d)$ and $\nu > 0$ be the given external body force and kinematic viscosity, respectively. Denote $\mathbf{u} = (u_1, \ldots, u_d)$, $\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_{ij})_{d\times d}$, and p to be the velocity vector, stress tensor, and pressure, respectively. When the density of the fluid is practically constant, the basic equations for incompressible Newtonian flows consist of

$$\begin{cases} \tilde{\boldsymbol{\sigma}} + p\boldsymbol{\delta} - 2\nu\boldsymbol{\epsilon}(\mathbf{u}) & = & \mathbf{0}, & \text{(consitutive law)} \\ \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot\tilde{\boldsymbol{\sigma}} & = & \mathbf{f}, & \text{(balance of linear momentum),} \\ \nabla\cdot\mathbf{u} & = & 0, & \text{(conservation of mass)} \end{cases} \tag{1.1}$$

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where ∇, $\nabla\cdot$, and $\boldsymbol{\delta}$ denote the gradient operator, divergence operator, and identity tensor, respectively; and $\boldsymbol{\epsilon}(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^t)/2$ is the deformation rate tensor. Here $\tilde{\boldsymbol{\sigma}}$, p, and ν are scaled with the density. To close the system, both initial and boundary conditions are needed. The initial condition should be given as $\mathbf{u}|_{t=0} = \mathbf{u}_0$, where $\mathbf{u}_0$ is the initial velocity. There are different kinds of boundary conditions. Many applications in incompressible Newtonian flow are posed under the Dirichlet boundary condition for the velocity

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\mathbf{g} = (g_1, \ldots, g_d)$ is prescribed velocity on the boundary satisfying the compatibility condition

$$\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{g}\, ds = 0. \tag{1.3}$$

In this case, the pressure is only unique up to an additive constant.

System (1.1) is known as the stress-velocity-pressure formulation of incompressible Navier-Stokes equations. Eliminating the stress from (1.1) gives the velocity-pressure formulation of Navier-Stokes equations

$$\begin{cases} \dfrac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot(\nu\boldsymbol{\epsilon}(\mathbf{u})) + \nabla p & = & \mathbf{f}, \\ \nabla\cdot\mathbf{u} & = & 0. \end{cases} \tag{1.4}$$

Without the nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$, Eq. (1.4) becomes the Stokes problem. The Stokes problem is linear and plays a critical role in numerical methods for solving Navier-Stokes equations.

The velocity-pressure formulation in (1.4) has long been the mainstream in computational incompressible Newtonian flows. However, research on the stress-velocity-pressure formulation is gaining consistent attention recently because of the arising interest in non-Newtonian flows. For non-Newtonian flows in which the constitutive law is nonlinear, the stress cannot be eliminated. Therefore, a formulation containing the stress as a fundamental unknown is unavoidable. Notice that the main advantage of the stress-velocity-pressure formulation is that it provides a unified framework for both the Newtonian and the non-Newtonian flows. It has also been pointed out [1] that an accurate and efficient numerical scheme for Newtonian flows under formulation (1.1) is necessary for the successful computation of non-Newtonian flows. Another advantage of the stress-velocity-pressure formulation is that, physical quantity like the stress is computed directly instead of by taking derivatives of the velocity. This avoids degrading of accuracy which is inevitable in the process of numerical differentiation. Accurate calculation of the stress is paramountly important for any flow problems involving obstacle bodies since it is crucial for, e.g., the design of solid structure and the reduction of drag.

However, the stress-velocity-pressure formulation has some obvious disadvantages. The most significant ones are the increase in the number of unknowns and the symmetry requirement for the stress tensor [2]. Both of them pose extra difficulty in the numerical computation. To avoid these disadvantages, this article studies mixed finite element methods based on the pseudostress-velocity formulation [3, 4]. Raviart-Thomas (RT) elements of index $k \geq 0$ [5] are used for approximating each row of the pseudostress, and discontinuous piecewise polynomials of degree $k \geq 0$ for approximating each component of the velocity. It is shown that this pair of mixed finite elements is stable and yields quasi-optimal accuracy $O(h^{k+1})$ for sufficiently smooth solutions. This discretization has two obvious advantages: (i) accurate approximation to physical quantities such as the stress and vorticity and (ii) no essential boundary condition posed in approximation space. Moreover, the method can be easily extended to applications with variable viscosity and/or variable density.

One possible disadvantage on using the pseudostress in incompressible Newtonian flows is that it increases the number of variables. Indeed, at the continuous level, the pseudostress-velocity formulation has d times more independent variables than the velocity-pressure formulation. However, at the discrete level, for lower order elements the number of degrees of freedom for the pseudostress-velocity using Raviart-Thomas elements is comparable with that for the velocity-pressure using Crouzeix-Raviart elements [6–8] (nonconforming velocity and discontinuous pressure) and both approaches have the same accuracy for the H^1 seminorm of the velocity and the L^2 norm of the pressure. More specifically, for the lowest order elements, the pseudostress-velocity has $dN_{\mathrm{f}} + dN_{\mathrm{t}}$ unknowns and the velocity-pressure has $dN_{\mathrm{f}} + N_{\mathrm{t}}$ unknowns, where N_{f} and N_{t} are the number of edges/faces and elements, respectively. The velocity with dN_{t} unknowns in the pseudostress-velocity form is further through either the penalty method for stationary problems or natural time discretization for unsteady-state problems so that we only need to solve numerically the symmetric and positive definite pseudostress system with dN_{f} unknowns. Similarly, for stationary problems one can use the penalty method to eliminate the pressure in the velocity-pressure form to get the Lamé system with dN_{f} unknowns. The large Lamé constant is the reciprocal of the penalty parameter.

To solve the indefinite system of linear equations resulting from the discretization efficiently, we eliminate the velocity by using the penalty method for stationary problems to obtain a smaller system involving only the pseudostress. To avoid accuracy loss in the penalty method, the penalty parameter ε is chosen to be proportional to the discretization accuracy. This means that $\varepsilon = O(h^{k+1})$ for RT elements of index $k \geq 0$. With this choice of ε, the condition number of the pseudostress system is $O(h^{-2}\varepsilon^{-1}) = O(h^{-2-(k+1)})$ and, hence, very ill-conditioned. This is an apparently very difficult problem to solve by any conventional iterative methods whose convergence factor depends also on the penalty parameter. In this article, we numerically solve the reduced pseudostress system by the $H(\mathrm{div})$ type of multigrid method introduced in [9–11]. Preconditioning the pseudostress system by a $V(1,1)$-cycle multigrid method for a weighted $H(\mathrm{div})$ problem, it is shown that the corresponding preconditioned conjugate gradient (PCG) method converges uniformly with respect to the mesh size h, the number of levels, and the penalty parameter ε provided that ε is bounded above by a constant. This is confirmed by our numerical results on uniform rectangular RT elements of the lowest order ($k = 0$). With computed pseudostress, the velocity can then be calculated either explicitly for $k = 0$ or locally for $k \geq 1$. The penalty approach is not the only one possible. We suggest a block-diagonal preconditioner for the (unpenalized) saddle-point problem which utilizes the same tools as the preconditioner for the penalized matrix, namely, one needs an optimal preconditioner for a similar $H(\mathrm{div})$ problem.

The article is organized as follows. The pseudostress-velocity formulation is derived in Section II. Sections III and IV describe and analyze mixed finite element method and the penalty method, respectively. Our preconditioning technique is discussed in Section V. Finally, numerical experiments on the accuracy of mixed finite element method and the condition number of the preconditioned system are presented in Section VI. We end with some concluding remarks in Section VII.

A. Notation

We use the standard notations and definitions for the Sobolev spaces $H^s(\Omega)^d$ and $H^s(\partial\Omega)^d$ for $s \geq 0$. The standard associated inner products are denoted by $(\cdot,\cdot)_{s,\Omega}$ and $(\cdot,\cdot)_{s,\partial\Omega}$, and their respective norms are denoted by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\partial\Omega}$. (We suppress the superscript d because the dependence on dimension will be clear by context. We also omit the subscript Ω from the inner product and norm designation when there is no risk of confusion.) For $s = 0$, $H^s(\Omega)^d$ coincides with $L^2(\Omega)^d$. In this case, the inner product and norm will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Set $H_0^1(\Omega) := \{q \in H^1(\Omega) : q = 0 \text{ on } \partial\Omega\}$. We use $H^{-1}(\Omega)$ to denote the dual of $H_0^1(\Omega)$ with norm defined by

$$\|\phi\|_{-1} = \sup_{0\neq\psi\in H_0^1(\Omega)} \frac{(\phi,\psi)}{\|\psi\|_1}.$$

Denote the product space $H^{-1}(\Omega)^d = \prod_{i=1}^d H^{-1}(\Omega)$ with the standard product norm. Finally, set

$$H(\mathrm{div};\,\Omega) = \{\mathbf{v} \in L^2(\Omega)^d : \nabla\cdot\mathbf{v} \in L^2(\Omega)\},$$

which is a Hilbert space under the norm

$$\|\mathbf{v}\|_{H(\mathrm{div})} = (\|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2)^{\frac{1}{2}}.$$

II. PSEUDOSTRESS-VELOCITY FORMULATION

For a vector function $\mathbf{v} = (v_1,\ldots,v_d)$, define its gradient as a $d\times d$ tensor

$$\nabla\mathbf{v} = \begin{pmatrix} \dfrac{\partial v_1}{\partial x_1} & \cdots & \dfrac{\partial v_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial v_d}{\partial x_1} & \cdots & \dfrac{\partial v_d}{\partial x_d} \end{pmatrix} = \left(\frac{\partial v_i}{\partial x_j}\right)_{d\times d}.$$

For a tensor function $\boldsymbol{\tau} = (\tau_{ij})_{d\times d}$, let $\boldsymbol{\tau}_i = (\tau_{i1},\ldots,\tau_{id})$ denote its ith-row for $i = 1,\ldots,d$ and define its divergence, normal, and trace by

$$\nabla\cdot\boldsymbol{\tau} = (\nabla\cdot\boldsymbol{\tau}_1,\ldots,\nabla\cdot\boldsymbol{\tau}_d), \quad \mathbf{n}\cdot\boldsymbol{\tau} = (\mathbf{n}\cdot\boldsymbol{\tau}_1,\ldots,\mathbf{n}\cdot\boldsymbol{\tau}_d), \quad \text{and} \quad \mathrm{tr}\boldsymbol{\tau} = \sum_{i=1}^d \tau_{ii},$$

respectively. Let $\mathcal{A} : \mathcal{R}^{d\times d} \to \mathcal{R}^{d\times d}$ be a linear map, which is singular, defined by

$$\mathcal{A}\boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{d}(\mathrm{tr}\boldsymbol{\tau})\boldsymbol{\delta},$$

it is easy to see that $\mathcal{A}\boldsymbol{\tau}$ is trace free and that $\boldsymbol{\tau} \in \mathcal{R}^{d\times d}$ has the following orthogonal decomposition

$$\boldsymbol{\tau} = \mathcal{A}\boldsymbol{\tau} + \frac{1}{d}(\mathrm{tr}\boldsymbol{\tau})\boldsymbol{\delta}, \tag{2.1}$$

with respect to the product of tensors

$$\boldsymbol{\sigma} : \boldsymbol{\tau} \equiv \sum_{i,j=1}^d \sigma_{ij}\tau_{ij},$$

for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{R}^{d\times d}$.

Introducing a new independent, nonsymmetric tensor variable, the *pseudostress*, as follows

$$\boldsymbol{\sigma} = \nu \nabla \mathbf{u} - p\boldsymbol{\delta}, \tag{2.2}$$

taking trace of (2.2) and using the divergence free condition in the third equation of (1.1) give

$$p = -\frac{1}{d}\mathrm{tr}\boldsymbol{\sigma}. \tag{2.3}$$

Then (2.2) may be rewritten as

$$\kappa \mathcal{A}\boldsymbol{\sigma} - \nabla \mathbf{u} = \mathbf{0},$$

where $\kappa = 1/\nu$. For incompressible fluids, because the divergence of $(\nabla \mathbf{u})^t$ vanishes, the stress and pseudostress have the same divergence; i.e.,

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot \tilde{\boldsymbol{\sigma}}.$$

Hence, we have the following pseudostress-velocity formulation of the Navier-Stokes equation

$$\begin{cases} \kappa \mathcal{A}\boldsymbol{\sigma} - \nabla \mathbf{u} = \mathbf{0}, \\ \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}. \end{cases} \tag{2.4}$$

The incompressibility condition is implicitly contained in the "constitutive" equation [the first equation in (2.4)]. There are two reasons for eliminating the pressure. An obvious one is to reduce one variable and, hence, many degrees of freedom in the discrete system. A more important reason is that we are able to use economic and accurate stable elements and able to develop fast solvers for the resulting discrete system so that computational cost will be greatly reduced.

In this article, we concentrate on the pseudostress-velocity formulation of the stationary Stokes problem

$$\begin{cases} \kappa \mathcal{A}\boldsymbol{\sigma} - \nabla \mathbf{u} = \mathbf{0} & \text{in} \quad \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} = \mathbf{f} & \text{in} \quad \Omega, \end{cases} \tag{2.5}$$

with boundary condition (1.2) and compatibility condition (1.3). Mixed finite element methods based on the pseudostress-velocity formulation for the stationary Navier-Stokes equation will be presented in [12].

When the viscosity parameter is constant, problem (2.5) is independent of ν (or κ) by scaling the $\boldsymbol{\sigma}$ and $\mathbf{f}$ with the viscosity. Otherwise, assume that there exist positive constants κ_0 and κ_1 such that

$$0 < \kappa_0 \le \kappa(x) \le \kappa_1, \tag{2.6}$$

for almost all $x \in \Omega$. It is well-known that the stationary Stokes equation has a unique solution provided that

$$\int_\Omega p\, dx = 0,$$

which, together with (2.3), implies

$$\int_\Omega \mathrm{tr}\boldsymbol{\sigma}\, dx = 0.$$

Therefore, we introduce a subspace of $H(\mathrm{div};\,\Omega)^d$:

$$\hat{H}(\mathrm{div};\,\Omega)^d = \{\boldsymbol{\tau} \in H(\mathrm{div};\,\Omega)^d : \int_\Omega \mathrm{tr}\boldsymbol{\tau}\, dx = 0\}.$$

To obtain the weak formulation of (2.5), we multiply the first equation in (2.5) by a test tensor function $\boldsymbol{\tau} \in \hat{H}(\mathrm{div};\,\Omega)^d$, integrate it over the domain Ω, and use integration by parts and boundary condition (1.2)

$$(\kappa\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{u}, \nabla\cdot\boldsymbol{\tau}) = \int_{\partial\Omega} \mathbf{g}\cdot(\mathbf{n}\cdot\boldsymbol{\tau})ds \equiv g(\boldsymbol{\tau}).$$

Multiplying a test vector function $\mathbf{v} \in L^2(\Omega)^d$ on both sides of the second equation in (2.5) and integrating it over the domain Ω give that

$$(\nabla\cdot\boldsymbol{\sigma}, \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) \equiv f(\mathbf{v}).$$

Now, the variational problem of the pseudostress-velocity formulation is to find a pair $(\boldsymbol{\sigma}, \mathbf{u}) \in \hat{H}(\mathrm{div};\,\Omega)^d \times L^2(\Omega)^d$ such that

$$\begin{cases} (\kappa\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{u}, \nabla\cdot\boldsymbol{\tau}) = g(\boldsymbol{\tau}) & \forall\,\boldsymbol{\tau} \in \hat{H}(\mathrm{div};\,\Omega)^d, \\ (\nabla\cdot\boldsymbol{\sigma}, \mathbf{v}) = f(\mathbf{v}) & \forall\,\mathbf{v} \in L^2(\Omega)^d. \end{cases} \tag{2.7}$$

It follows from the fact that $\mathcal{A}$ is singular and the orthogonal decomposition in (2.1), that $\|\mathrm{tr}\boldsymbol{\tau}\|$ can not be controlled by $\|\mathcal{A}\boldsymbol{\tau}\|$ alone for any $\boldsymbol{\tau} \in \hat{H}(\mathrm{div};\,\Omega)^d$, but we do have the following inequality (see [13, 14]).

Lemma 2.1. *For any $\boldsymbol{\tau} \in \hat{H}(\mathrm{div};\,\Omega)^d$, we have*

$$\|\mathrm{tr}\,\boldsymbol{\tau}\| \le C(\|\mathcal{A}\boldsymbol{\tau}\| + \|\nabla\cdot\boldsymbol{\tau}\|_{-1}). \tag{2.8}$$

(We use C with or without subscripts in this article to denote a generic positive constant, possibly different at different occurrences, that is independent of the mesh size h and the penalty parameter ε introduced in subsequent sections but may depend on the domain Ω.) It is easy to see that

$$\|\boldsymbol{\tau}\|^2 = \|\mathcal{A}\boldsymbol{\tau}\|^2 + \frac{1}{d}\|\mathrm{tr}\boldsymbol{\tau}\|^2, \tag{2.9}$$

which, together with Lemma 2.1, implies

$$\|\boldsymbol{\tau}\| \le C(\|\mathcal{A}\boldsymbol{\tau}\| + \|\nabla\cdot\boldsymbol{\tau}\|_{-1}) \le C\,(\|\mathcal{A}\boldsymbol{\tau}\| + \|\nabla\cdot\boldsymbol{\tau}\|). \tag{2.10}$$

To prove the existence and uniqueness of problem (2.7), it is convenient to use the following lemma (see, e.g., [15]).

Lemma 2.2. *For any $q \in L^2(\Omega)$, there exists a $\mathbf{v} \in H^1(\Omega)^d$ such that*

$$\nabla \cdot \mathbf{v} = q \quad \textit{in } \Omega \quad \textit{and} \quad \|\mathbf{v}\|_1 \le C\|q\|. \tag{2.11}$$

Theorem 2.3. *The variational problem in* (2.7) *has a unique solution.*

Proof. By the Cauchy-Schwarz inequality, definition of norms, and a trace theorem, it is easy to see that linear forms $f(\mathbf{v})$ and $g(\boldsymbol{\tau})$ are continuous in $L^2(\Omega)^d$ and $\hat{H}(\text{div}; \Omega)^d$, respectively; that is

$$|f(\mathbf{v})| \le \|\mathbf{f}\|\|\mathbf{v}\|, \tag{2.12}$$

and

$$|g(\boldsymbol{\tau})| \le \|\mathbf{g}\|_{1/2,\partial\Omega}\|\mathbf{n}\cdot\boldsymbol{\tau}\|_{-1/2,\partial\Omega} \le \|\mathbf{g}\|_{1/2,\partial\Omega}\|\boldsymbol{\tau}\|_{H(\text{div})}. \tag{2.13}$$

It follows from (2.10) and (2.6) that the bilinear form $(\kappa\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\kappa\mathcal{A}\boldsymbol{\sigma}, \mathcal{A}\boldsymbol{\tau})$ is coercive in the divergence free subspace of $\hat{H}(\text{div}; \Omega)^d$

$$C\kappa_0\|\boldsymbol{\tau}\|^2_{H(\text{div})} \le \kappa_0\|\mathcal{A}\boldsymbol{\tau}\|^2 \le (\kappa\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau}), \tag{2.14}$$

for any $\boldsymbol{\tau}$ in $\hat{H}^0(\text{div};\Omega)^d = \{\boldsymbol{\tau} \in \hat{H}(\text{div}; \Omega)^d | \nabla\cdot\boldsymbol{\tau} = \mathbf{0}\}$.

For any $\mathbf{v} \in L^2(\Omega)^d$, Lemma 2.2 implies that there exists $\tilde{\boldsymbol{\tau}} \in H^1(\Omega)^{d\times d}$ such that

$$\nabla\cdot\tilde{\boldsymbol{\tau}} = \mathbf{v} \quad \text{in } \Omega \quad \text{and} \quad \|\tilde{\boldsymbol{\tau}}\|_1 \le C\|\mathbf{v}\|. \tag{2.15}$$

Let $a = \int_\Omega \text{tr}\, \tilde{\boldsymbol{\tau}}\, dx$ and $\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}} - \frac{a}{d|\Omega|}\boldsymbol{\delta}$ where $|\Omega|$ denotes the volume of the domain Ω, then it is easy to check that

$$\boldsymbol{\tau} \in \hat{H}(\text{div}; \Omega)^d, \quad \nabla\cdot\boldsymbol{\tau} = \mathbf{v} \quad \text{in } \Omega, \quad \text{and} \quad \|\boldsymbol{\tau}\|_1 \le C\|\mathbf{v}\|. \tag{2.16}$$

Hence,

$$\sup_{\boldsymbol{\gamma}\in\hat{H}(\text{div};\Omega)^d} \frac{(\nabla\cdot\boldsymbol{\gamma}, \mathbf{v})}{\|\boldsymbol{\gamma}\|_{H(\text{div})}} \ge \frac{\|\mathbf{v}\|^2}{\|\boldsymbol{\tau}\|_{H(\text{div})}} \ge \beta\|\mathbf{v}\|. \tag{2.17}$$

Now, the coercivity condition in (2.14) and the inf-sup condition in (2.17) imply [16] that the variational problem in (2.7) has a unique solution. ∎

It is important to point out that the pseudostress contains more information than the stress

$$\tilde{\boldsymbol{\sigma}} = -p\boldsymbol{\delta} + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^t) = \boldsymbol{\sigma} + \nu(\nabla\mathbf{u})^t.$$

Physical quantities such as the velocity gradient, stress, vorticity, and pressure can be *algebraically* expressed in terms of the pseudostress:

$$\nabla\mathbf{u} = \mathcal{A}\boldsymbol{\sigma}, \quad \tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \nu(\mathcal{A}\boldsymbol{\sigma})^t, \quad \boldsymbol{\omega} = \frac{1}{2}(\mathcal{A}\boldsymbol{\sigma} - (\mathcal{A}\boldsymbol{\sigma})^t), \quad \text{and} \quad p = -\frac{1}{d}\text{tr}\boldsymbol{\sigma}, \tag{2.18}$$

respectively. Therefore, these physical quantities (if needed) can be computed in a postprocessing procedure without degrading accuracy of approximation. In (2.18), we conveniently represent the vorticity $\nabla \times \mathbf{u}$ as the skew symmetric part of the velocity gradient:

$$\boldsymbol{\omega} = \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^t).$$

This defines the vorticity (or the curl operator) in all dimensions by one formula.

III. FINITE ELEMENT APPROXIMATION

Assume that Ω is a polygonal domain, let $\mathcal{T}_h$ be a quasi-regular triangulation of Ω with (triangular/tetrahedral or rectangular) elements of size $O(h)$. Denote spaces of polynomials on an element $K \subset \mathcal{R}^d$

$P_k(K)$ is the space of polynomials of degree $\leq k$;

$$P_{k_1,k_2}(K) = \left\{ p(x_1, x_2) : p(x_1, x_2) = \sum_{i \leq k_1, j \leq k_2} a_{ij} x_1^i x_2^j \right\} \quad d = 2;$$

$$P_{k_1,k_2,k_3}(K) = \left\{ p(x_1, x_2, x_3) : p(x_1, x_2, x_3) = \sum_{i \leq k_1, j \leq k_2, k \leq k_3} a_{ijk} x_1^i x_2^j x_3^k \right\} \quad d = 3;$$

$$Q_k(K) = \begin{cases} P_{k,k}(K) & d = 2, \\ P_{k,k,k}(K) & d = 3. \end{cases}$$

Denote the local Raviart-Thomas (RT) space of index $k \geq 0$ on an element K:

$$RT_k(K) = \begin{cases} P_k(K)^d + (x_1, \ldots, x_d) P_k(K) & K = \text{triangle/tetrahedral}, \\ Q_k(K)^d + (x_1, \ldots, x_d) Q_k(K) & K = \text{rectangle/cube}. \end{cases}$$

In two dimensions, degrees of freedom for $RT_0(K) = (a + bx_1, c + bx_2)$ on triangle or $RT_0(K) = (a + bx_1, c + dx_2)$ on rectangle are normal components of vector field on all edges. For the choice of degrees of freedom of the RT_k space of index $k \geq 1$, see [17]. They are chosen for ensuring continuity of the normal component of vector field at interfaces of elements. Then one can define the $H(\text{div}; \Omega)$ conforming Raviart-Thomas space of index $k \geq 0$ [5] by

$$RT_k = \{\mathbf{v} \in H(\text{div}; \Omega) : \mathbf{v}|_K \in RT_k(K) \quad \forall K \in \mathcal{T}_h\}.$$

Let

$$D_k(K) = \begin{cases} P_k(K) & K = \text{triangle/tetrahedral}, \\ \mathcal{F}(Q_k(K)) & K = \text{rectangle/cube}, \end{cases}$$

where $\mathcal{F}(\hat{v}) \cdot F^{-1}$ and F are affine map from the reference element $\hat{K}$ to the physical element K [17]. Denote the space of piecewise polynomials of degree $\leq k$ by

$$P_k = \{q \in L^2(\Omega) : q|_K \in D_k(K) \quad \forall K \in \mathcal{T}_h\}.$$

Let $\mathcal{P}_h$ be the L^2 projection onto P_k. It is well-known that

$$\|q - \mathcal{P}_h q\| \le Ch^r \|q\|_r \quad \text{for} \quad 0 \le r \le k+1, \tag{3.1}$$

for all $q \in H^r(\Omega)$. Also, it is well-known that there exists an interpolation operator $\Pi_h : H(\text{div};\, \Omega) \cap L^t(\Omega)^d \to RT_k$ for $t > 2$ satisfying the commutativity property

$$\nabla \cdot (\Pi_h \mathbf{v}) = \mathcal{P}_h \nabla \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in H(\text{div};\, \Omega) \cap L^t(\Omega)^d, \tag{3.2}$$

and the following approximation properties

$$\|\mathbf{v} - \Pi_h \mathbf{v}\| \le Ch^r \|\mathbf{v}\|_r \qquad \text{for} \quad 1 \le r \le k+1, \tag{3.3}$$

$$\|\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v})\| \le Ch^r \|\nabla \cdot \mathbf{v}\|_r \quad \text{for} \quad 0 \le r \le k+1. \tag{3.4}$$

Denote the product spaces by $RT_k^d = \prod_{i=1}^d RT_k$ and $P_k^d = \prod_{i=1}^d P_k$ and define

$$R\hat{T}_k^d = \left\{ \boldsymbol{\tau} \in RT_k^d \,\middle|\, \int_\Omega \operatorname{tr} \boldsymbol{\tau}\, dx = 0 \right\}.$$

Then our mixed finite element approximation is to find a pair $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in R\hat{T}_k^d \times P_k^d$ such that

$$\begin{cases} (\kappa \mathcal{A} \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\mathbf{u}_h, \nabla \cdot \boldsymbol{\tau}) = g(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in R\hat{T}_k^d, \\ (\nabla \cdot \boldsymbol{\sigma}_h, \mathbf{v}) = f(\mathbf{v}) & \forall\, \mathbf{v} \in P_k^d. \end{cases} \tag{3.5}$$

To establish well-posedness of (3.5) and error bounds, define an interpolation operator $\boldsymbol{\Pi}_h : \hat{H}(\text{div};\, \Omega)^d \cap L^t(\Omega)^{d\times d} \to R\hat{T}_k^d$ by

$$\boldsymbol{\Pi}_h \boldsymbol{\tau} = (\Pi_h \boldsymbol{\tau}_1, \ldots, \Pi_h \boldsymbol{\tau}_d)^t - b\boldsymbol{\delta} \quad \text{with} \quad b = \frac{1}{d|\Omega|} \int_\Omega \operatorname{tr}(\Pi_h \boldsymbol{\tau}_1, \ldots, \Pi_h \boldsymbol{\tau}_d)^t dx,$$

and the L^2 projection operator onto P_k^d by

$$\mathbf{P}_h \mathbf{v} = (\mathcal{P}_h v_1, \ldots, \mathcal{P}_h v_d).$$

By (3.2), (3.3), (3.4), and (3.1), it is then easy to check the validity of the commutativity property

$$\nabla \cdot (\boldsymbol{\Pi}_h \boldsymbol{\tau}) = \mathbf{P}_h \nabla \cdot \boldsymbol{\tau} \quad \forall\, \boldsymbol{\tau} \in H(\text{div};\, \Omega)^d \cap L^t(\Omega)^{d\times d}, \tag{3.6}$$

and the approximation properties

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\| \le Ch^r \|\boldsymbol{\tau}\|_r \qquad \text{for} \quad 1 \le r \le k+1, \tag{3.7}$$

$$\|\nabla \cdot (\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau})\| \le Ch^r \|\nabla \cdot \boldsymbol{\tau}\|_r \quad \text{for} \quad 0 \le r \le k+1, \tag{3.8}$$

$$\|\mathbf{v} - \mathbf{P}_h \mathbf{v}\| \le Ch^r \|\mathbf{v}\|_r \qquad \text{for} \quad 0 \le r \le k+1. \tag{3.9}$$

Let

$$D = \left\{ \boldsymbol{\tau} \in R\hat{T}_k^d \,|\, (\nabla \cdot \boldsymbol{\tau}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in P_k^d \right\},$$

and denote bilinear forms by

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\kappa \mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\kappa \mathcal{A}\boldsymbol{\sigma}, \mathcal{A}\boldsymbol{\tau}) \quad \text{and} \quad b(\boldsymbol{\tau}, \mathbf{v}) = (\nabla \cdot \boldsymbol{\tau}, \mathbf{v}).$$

Next two lemmas verify the coercivity of the bilinear form $a(\cdot,\cdot)$ in D and the inf-sup condition of the bilinear form $b(\cdot,\cdot)$ in $\hat{RT}_k^d \times P_k^d$.

Lemma 3.1. *There exists a positive constant $\hat{\alpha}$ independent of the mesh size h such that*

$$C\kappa_0 \|\boldsymbol{\tau}\|^2_{H(\mathrm{div})} \le a(\boldsymbol{\tau}, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in D. \tag{3.10}$$

Proof. The commutativity property (3.6) gives that $\nabla \cdot \hat{RT}_k^d \subset P_k^d$, which, in turn, implies that D is the divergence free subspace of $\hat{RT}_k^d$. Hence, coercivity (3.10) follows from (2.14). ■

Lemma 3.2. *There exists a positive constant $\hat{\beta}$ independent of the mesh size h such that*

$$\sup_{\boldsymbol{\tau} \in \hat{RT}_k^d} \frac{(\nabla \cdot \boldsymbol{\tau}, \mathbf{v})}{\|\boldsymbol{\tau}\|_{H(\mathrm{div})}} \ge \hat{\beta}\|\mathbf{v}\| \quad \forall \mathbf{v} \in P_k^d. \tag{3.11}$$

Proof. By the triangle inequality and (3.7) with $r = 1$ we have the stability of the interpolation operator

$$\|\boldsymbol{\Pi}_h \boldsymbol{\tau}\| \le C\|\boldsymbol{\tau}\|_1 \quad \forall \boldsymbol{\tau} \in H^1(\Omega)^{d\times d}. \tag{3.12}$$

For any $\mathbf{v} \in P_k^d \subset L^2(\Omega)^d$, there exists a $\boldsymbol{\tau} \in \hat{H}(\mathrm{div};\, \Omega)^d$ satisfying (2.16):

$$\nabla \cdot \boldsymbol{\tau} = \mathbf{v} \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\tau}\|_1 \le C\|\mathbf{v}\|.$$

Taking $\boldsymbol{\gamma} = \boldsymbol{\Pi}_h \boldsymbol{\tau} \in \hat{RT}_k^d$ and using commutativity property (3.6), we have

$$\nabla \cdot \boldsymbol{\gamma} = \nabla \cdot (\boldsymbol{\Pi}_h \boldsymbol{\tau}) = \mathbf{P}_h \nabla \cdot \boldsymbol{\tau} = \mathbf{P}_h \mathbf{v} = \mathbf{v}.$$

Hence, by (3.12) and (2.16)

$$\begin{aligned} \|\boldsymbol{\gamma}\|_{H(\mathrm{div})} &= \|\boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{H(\mathrm{div})} = (\|\boldsymbol{\Pi}_h \boldsymbol{\tau}\|^2 + \|\nabla \cdot (\boldsymbol{\Pi}_h \boldsymbol{\tau})\|^2)^{\frac{1}{2}} \\ &\le (C\|\boldsymbol{\tau}\|_1^2 + \|\mathbf{v}\|^2)^{\frac{1}{2}} \le C\|\mathbf{v}\|. \end{aligned}$$

Now, for any $\mathbf{v} \in P_k^d \subset L^2(\Omega)^d$

$$\sup_{\boldsymbol{\tau} \in \hat{RT}_k^d} \frac{(\nabla \cdot \boldsymbol{\tau}, \mathbf{v})}{\|\boldsymbol{\tau}\|_{H(\mathrm{div})}} \ge \frac{(\nabla \cdot \boldsymbol{\gamma}, \mathbf{v})}{\|\boldsymbol{\gamma}\|_{H(\mathrm{div})}} \ge \hat{\beta}\|\mathbf{v}\|,$$

where $\hat{\beta}$ is independent of the mesh size h. This proves the lemma. ■

Now, we are ready to establish the well-posedness and error bounds of mixed finite element approximation.

Theorem 3.3. *The discrete problem in* (3.5) *has a unique solution* $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ *in* $\hat{RT}_k^d \times P_k^d$. *Let* $(\boldsymbol{\sigma}, \mathbf{u})$ *be the solution of* (2.7)*, we then have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div})} \leq C \inf_{\boldsymbol{\tau} \in \hat{RT}_k^d} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{H(\mathrm{div})} \tag{3.13}$$

and

$$\|\mathbf{u} - \mathbf{u}_h\| \leq C \left(\inf_{\mathbf{v} \in P_k^d} \|\mathbf{u} - \mathbf{v}\| + \inf_{\boldsymbol{\tau} \in \hat{RT}_k^d} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{H(\mathrm{div})} \right). \tag{3.14}$$

Moreover, for $1 \leq r \leq k+1$, *assume that* $\mathbf{f} \in H^r(\Omega)^d$ *and* $(\boldsymbol{\sigma}, \mathbf{u}) \in H^r(\Omega)^{d\times d} \times H^r(\Omega)^d$. *Then we have the following error bounds:*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div})} \leq Ch^r(\|\boldsymbol{\sigma}\|_r + \|\mathbf{f}\|_r) \tag{3.15}$$

and

$$\|\mathbf{u} - \mathbf{u}_h\| \leq Ch^r(\|\mathbf{u}\|_r + \|\boldsymbol{\sigma}\|_r + \|\mathbf{f}\|_r). \tag{3.16}$$

Proof. Existence and uniqueness of problem (3.5) and error bounds in (3.13) and (3.14) follow from the abstract theory for the saddle-point problem (see, e.g., [16, 17]) and Lemmas 3.1 and 3.2. Error bounds in (3.15) and (3.16) follow from (3.13), (3.14), and the approximation properties in (3.7), (3.8), and (3.9). ■

We end this section by establishing an a priori estimate for a slightly more general system that contains both (3.5) and its perturbation. This estimate will be used for bounding the penalty error in next section.

Lemma 3.4. *For a constant parameter* $0 \leq \varepsilon < 1$, *let a pair* $(\boldsymbol{\gamma}_h, \mathbf{w}_h) \in \hat{RT}_k^d \times P_k^d$ *be the unique solution of*

$$\begin{cases} (\kappa \mathcal{A}\boldsymbol{\gamma}_h, \boldsymbol{\tau}) + (\mathbf{w}_h, \nabla \cdot \boldsymbol{\tau}) &= g'(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \hat{RT}_k^d, \\ (\nabla \cdot \boldsymbol{\gamma}_h, \mathbf{v}) - \varepsilon(\mathbf{w}_h, \mathbf{v}) &= f'(\mathbf{v}) \qquad \forall\, \mathbf{v} \in P_k^d. \end{cases} \tag{3.17}$$

Assume that g' *and* f' *are continuous linear functionals defined on* $H(\mathrm{div};\, \Omega)^d$ *and* $L^2(\Omega)^d$ *with norms* $\|g'\|$ *and* $\|f'\|$*, respectively. Then the following a priori estimate holds*

$$\|\boldsymbol{\gamma}_h\|_{H(\mathrm{div})} + \|\mathbf{w}_h\| \leq C(\|f'\| + \|g'\|), \tag{3.18}$$

where C *is a positive constant independent of the mesh size* h *and the parameter* ε.

Proof. To bound $(\boldsymbol{\gamma}_h, \mathbf{w}_h)$ in $H(\mathrm{div}) \times L^2$ norm, we first bound $\|\mathbf{w}_h\|$ above in terms of $\|\mathcal{A}\boldsymbol{\gamma}_h\|$ and $\|g'\|$ by using (3.11), the first equation of (3.17), the Cauchy-Schwarz inequality, and (2.6)

$$\hat{\beta}\|\mathbf{w}_h\| \leq \sup_{\boldsymbol{\tau} \in \hat{RT}_k^d} \frac{(\nabla \cdot \boldsymbol{\tau}, \mathbf{w}_h)}{\|\boldsymbol{\tau}\|_{H(\mathrm{div})}} = \sup_{\boldsymbol{\tau} \in \hat{RT}_k^d} \frac{g'(\boldsymbol{\tau}) - (\kappa \mathcal{A}\boldsymbol{\gamma}_h, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{H(\mathrm{div})}} \leq \|g'\| + \kappa_1 \|\mathcal{A}\boldsymbol{\gamma}_h\|. \tag{3.19}$$

Next, choosing $\mathbf{v} = \nabla\cdot\boldsymbol{\gamma}_h \in P_k^d$ in the second equation of (3.17) and using the Cauchy-Schwarz inequality give

$$\|\nabla \cdot \boldsymbol{\gamma}_h\|^2 = \varepsilon(\mathbf{w}_h, \nabla \cdot \boldsymbol{\gamma}_h) + f'(\nabla \cdot \boldsymbol{\gamma}_h) \le (\varepsilon\|\mathbf{w}_h\| + \|f'\|)\|\nabla \cdot \boldsymbol{\gamma}_h\|.$$

Dividing $\|\nabla \cdot \boldsymbol{\gamma}_h\|$ on both sides and using (3.19) yield, for $0 \le \varepsilon < 1$,

$$\|\nabla \cdot \boldsymbol{\gamma}_h\| \le \varepsilon\|\mathbf{w}_h\| + \|f'\| \le C(\varepsilon\|g'\| + \|f'\| + \varepsilon\kappa_1\|\mathcal{A}\boldsymbol{\gamma}_h\|),$$

which, together with (2.10), implies

$$\|\boldsymbol{\gamma}_h\|_{H(\mathrm{div})} \le C(\|\mathcal{A}\boldsymbol{\gamma}_h\| + \|\nabla \cdot \boldsymbol{\gamma}_h\|) \le C(\|g'\| + \|f'\| + \|\mathcal{A}\boldsymbol{\gamma}_h\|). \tag{3.20}$$

Finally, we establish an upper bound for $\|\mathcal{A}\boldsymbol{\gamma}_h\|$. To this end, in (3.17), we take $\boldsymbol{\tau} = \boldsymbol{\gamma}_h$ and $\mathbf{v} = \mathbf{w}_h$ and subtract the second equation from the first equation to obtain

$$\|\sqrt{\kappa}\mathcal{A}\boldsymbol{\gamma}_h\|^2 + \varepsilon\|\mathbf{w}_h\|^2 = g'(\boldsymbol{\gamma}_h) - f'(\mathbf{w}_h) \le \|g'\|\|\boldsymbol{\gamma}_h\|_{H(\mathrm{div})} + \|f'\|\|\mathbf{w}_h\|.$$

It then follows from (2.6), (3.20), (3.19), and the δ-inequality ($2ab \le \delta a^2 + b^2/\delta$ for all positive δ) that

$$\begin{aligned}
\kappa_0\|\mathcal{A}\boldsymbol{\gamma}_h\|^2 &\le \|\sqrt{\kappa}\mathcal{A}\boldsymbol{\gamma}_h\|^2 \le \|g'\|\|\boldsymbol{\gamma}_h\|_{H(\mathrm{div})} + \|f'\|\|\mathbf{w}_h\| \\
&\le C\|g'\|(\|g'\| + \|f'\| + \|\mathcal{A}\boldsymbol{\gamma}_h\|) + C\|f'\|(\|g'\| + \|\mathcal{A}\boldsymbol{\gamma}_h\|) \\
&\le C(\|g'\| + \|f'\|)\|g'\| + C(\|g'\| + \|f'\|)\|\mathcal{A}\boldsymbol{\gamma}_h\| \\
&\le C(\|g'\|^2 + \|f'\|^2) + \frac{1}{2}\|\mathcal{A}\boldsymbol{\gamma}_h\|^2.
\end{aligned}$$

Hence,

$$\|\mathcal{A}\boldsymbol{\gamma}_h\| \le C(\|g'\| + \|f'\|). \tag{3.21}$$

Now, (3.18) is a direct consequence of (3.19), (3.20), and (3.21). This completes the proof of the lemma. ∎

Corollary 3.5. *Let $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \hat{RT}_k^d \times P_k^d$ be the solution of* (3.5)*, then the following a priori estimate holds*

$$\|\boldsymbol{\sigma}_h\|_{H(\mathrm{div})} + \|\mathbf{u}_h\| \le C(\|\mathbf{f}\| + \|\mathbf{g}\|_{1/2,\partial\Omega}). \tag{3.22}$$

Proof. The a priori estimate in (3.22) follows from Lemma 3.4 with $\varepsilon = 0$, $\|g'\| = \|\mathbf{g}\|_{1/2,\partial\Omega}$, and $\|f'\| = \|\mathbf{f}\|$. ∎

IV. PENALTY METHOD

To solve the saddle-point problem in (3.5) efficiently, we eliminate the velocity by using the penalty method [17–19] to obtain a smaller system involving only the pseudostress which will be solved by a fast multigrid method. The velocity can then be calculated for piecewise polynomials of degree k either explicitly for $k = 0$ or locally for $k \geq 1$. To this end, let $0 \leq \varepsilon < 1$ be a small parameter. We perturb (3.5) by finding $(\boldsymbol{\sigma}_h^\varepsilon, \mathbf{u}_h^\varepsilon) \in \hat{RT}_k^d \times P_k^d$ such that

$$\begin{cases} \left(\kappa \mathcal{A}\boldsymbol{\sigma}_h^\varepsilon, \boldsymbol{\tau}\right) + \left(\mathbf{u}_h^\varepsilon, \nabla \cdot \boldsymbol{\tau}\right) & = & g(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \hat{RT}_k^d, \\ \left(\nabla \cdot \boldsymbol{\sigma}_h^\varepsilon, \mathbf{v}\right) - \varepsilon\left(\mathbf{u}_h^\varepsilon, \mathbf{v}\right) & = & f(\mathbf{v}) & \forall\, \mathbf{v} \in P_k^d. \end{cases} \tag{4.1}$$

It is easy to check that the perturbed problem in (4.1) has a unique solution $(\boldsymbol{\sigma}_h^\varepsilon, \mathbf{u}_h^\varepsilon)$ in $\hat{RT}_k^d \times P_k^d$. By using Lemma 3.4, next lemma shows that the perturbed solution of (4.1) is close to the original solution of (3.5).

Lemma 4.1. *Let* $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ *and* $(\boldsymbol{\sigma}_h^\varepsilon, \mathbf{u}_h^\varepsilon)$ *be the solutions of* (3.5) *and* (4.1), *respectively. Then, for all* $0 \leq \varepsilon < 1$, *there exists a positive constant C independent of both h and* ε *such that*

$$\left\|\mathbf{u}_h - \mathbf{u}_h^\varepsilon\right\| + \left\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^\varepsilon\right\|_{H(\mathrm{div})} \leq C\varepsilon\|\mathbf{u}_h\| \leq C\varepsilon(\|\mathbf{f}\| + \|\mathbf{g}\|_{1/2,\partial\Omega}). \tag{4.2}$$

Proof. Let $\boldsymbol{\gamma}_h = \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^\varepsilon$ and $\mathbf{w}_h = \mathbf{u}_h - \mathbf{u}_h^\varepsilon$. Then difference of (3.5) and (4.1) gives the following well-posed system

$$\begin{cases} (\kappa \mathcal{A}\boldsymbol{\gamma}_h, \boldsymbol{\tau}) + (\mathbf{w}_h, \nabla \cdot \boldsymbol{\tau}) & = & 0 & \forall\, \boldsymbol{\tau} \in \hat{RT}_k^d, \\ (\nabla \cdot \boldsymbol{\gamma}_h, \mathbf{v}) - \varepsilon(\mathbf{w}_h, \mathbf{v}) & = & -\varepsilon(\mathbf{u}_h, \mathbf{v}) & \forall\, \mathbf{v} \in P_k^d. \end{cases} \tag{4.3}$$

Now, the first inequality in (4.2) follows from the *a priori* estimate in Lemma 3.4 with $\|g'\| = 0$ and $\|f'\| = \varepsilon\|\mathbf{u}_h\|$, and the second inequality is a direct consequence of a priori estimate (3.22) for the discrete solution. This completes the proof of the lemma. ■

Theorem 4.2. *Let* $(\boldsymbol{\sigma}, \mathbf{u})$ *and* $(\boldsymbol{\sigma}_h^\varepsilon, \mathbf{u}_h^\varepsilon)$ *be the solutions of* (2.7) *and* (4.1), *respectively. Then, for all* $0 \leq \varepsilon < 1$, *there exists a positive constant C independent of both h and* ε *such that*

$$\begin{aligned} &\left\|\mathbf{u} - \mathbf{u}_h^\varepsilon\right\| + \left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^\varepsilon\right\|_{H(\mathrm{div})} \\ &\leq C\Big(\inf_{\mathbf{v}_h \in P_k^d} \|\mathbf{u} - \mathbf{v}_h\| + \inf_{\boldsymbol{\tau}_h \in \hat{RT}_k^d} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{H(\mathrm{div})} + \varepsilon(\|\mathbf{f}\| + \|\mathbf{g}\|_{1/2,\partial\Omega})\Big). \end{aligned} \tag{4.4}$$

Moreover, choosing $\varepsilon = O(h^r)$, *we then have the following error estimate:*

$$\left\|\mathbf{u} - \mathbf{u}_h^\varepsilon\right\| + \left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^\varepsilon\right\|_{H(\mathrm{div})} \leq Ch^r(\|\mathbf{u}\|_r + \|\boldsymbol{\sigma}\|_r + \|\mathbf{f}\|_r + \|\mathbf{g}\|_{1/2,\partial\Omega}) \tag{4.5}$$

for all $1 \leq r \leq k + 1$.

Proof. Inequality (4.4) is an immediate consequence of the triangle inequality, Theorem 3.3, and Lemma 4.1. Error bound in (4.5) follows from (4.4) and the approximation properties in (3.7), (3.8), and (3.9). This proves the theorem. ■

Remark 4.3. Theorem 4.2 *indicates that the penalty method does not deteriorate the accuracy of approximation provided that* $\varepsilon = O(h^r)$.

Corollary 4.4. *Let* $(\boldsymbol{\sigma}_h^\varepsilon, \mathbf{u}_h^\varepsilon)$ *be the solution of* (4.1). *Let* $\tilde{\boldsymbol{\sigma}}$, $\boldsymbol{\omega}$, *and p be the respective stress, vorticity, and pressure defined in* (2.18) *and define their approximations as follows*

$$\tilde{\boldsymbol{\sigma}}_h^\varepsilon = \boldsymbol{\sigma}_h^\varepsilon + \nu\left(\mathcal{A}\boldsymbol{\sigma}_h^\varepsilon\right)^t, \quad \boldsymbol{\omega}_h^\varepsilon = \frac{1}{2}\left(\mathcal{A}\boldsymbol{\sigma}_h^\varepsilon - (\mathcal{A}\boldsymbol{\sigma}_h^\varepsilon)^t\right), \quad \textit{and} \quad p_h^\varepsilon = -\frac{1}{d}\mathrm{tr}\boldsymbol{\sigma}_h^\varepsilon. \tag{4.6}$$

Then, for $\varepsilon = O(h^r)$*, we have the following error estimate:*

$$\begin{aligned}&\|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h^\varepsilon\| + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h^\varepsilon\| + \|p - p_h^\varepsilon\|\\ &\le C\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^\varepsilon\| \le Ch^r(\|\mathbf{u}\|_r + \|\boldsymbol{\sigma}\|_r + \|\mathbf{f}\|_r + \|\mathbf{g}\|_{1/2,\partial\Omega})\end{aligned} \tag{4.7}$$

for all $1 \le r \le k+1$.

Proof. Let $\boldsymbol{\gamma} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^\varepsilon$, then it is an immediate consequence of (2.18) and (4.6) that

$$\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h^\varepsilon = \boldsymbol{\gamma} + \nu(\mathcal{A}\boldsymbol{\gamma})^t, \quad \boldsymbol{\omega} - \boldsymbol{\omega}_h^\varepsilon = \frac{1}{2}(\mathcal{A}\boldsymbol{\gamma} - (\mathcal{A}\boldsymbol{\gamma})^t), \quad \text{and} \quad p - p_h^\varepsilon = -\frac{1}{d}\mathrm{tr}\boldsymbol{\gamma}.$$

Now, the first inequality in (4.7) follows from the triangle inequality and the second inequality from (4.5). ■

The penalty system in (4.1) can be efficiently solved by decoupling the velocity and pseudostress as follows. Choosing $\mathbf{v} = \nabla\cdot\boldsymbol{\tau} \in P_k^d$ in the second equation of (4.1) gives

$$\left(\mathbf{u}_h^\varepsilon, \nabla\cdot\boldsymbol{\tau}\right) = \frac{1}{\varepsilon}\left(\nabla\cdot\boldsymbol{\sigma}_h^\varepsilon, \nabla\cdot\boldsymbol{\tau}\right) - \frac{1}{\varepsilon}f(\nabla\cdot\boldsymbol{\tau}) \qquad \forall\boldsymbol{\tau} \in \hat{RT}_k^d. \tag{4.8}$$

Substituting (4.8) into the first equation of (4.1) yields the penalized system for only the pseudostress

$$\left(\kappa\mathcal{A}\boldsymbol{\sigma}_h^\varepsilon, \boldsymbol{\tau}\right) + \frac{1}{\varepsilon}\left(\nabla\cdot\boldsymbol{\sigma}_h^\varepsilon, \nabla\cdot\boldsymbol{\tau}\right) = g(\boldsymbol{\tau}) + \frac{1}{\varepsilon}f(\nabla\cdot\boldsymbol{\tau}) \qquad \forall\boldsymbol{\tau} \in \hat{RT}_k^d. \tag{4.9}$$

This system will be numerically solved by effective multigrid methods discussed in the next section. As $\nabla\cdot\hat{RT}_k^d = P_k^d$, with known pseudostress $\boldsymbol{\sigma}_h^\varepsilon$, the velocity $\mathbf{u}_h^\varepsilon$ can then be calculated by

$$\mathbf{u}_h^\varepsilon = \frac{1}{\varepsilon}\left(\nabla\cdot\boldsymbol{\sigma}_h^\varepsilon + \mathbf{P}_h\mathbf{f}\right), \tag{4.10}$$

where $\mathbf{P}_h$ is the L^2 projection operator into P_k^d defined in the previous section. For $k = 0$, the calculation of $\mathbf{P}_h\mathbf{f}$ is explicit because for every $K \in \mathcal{T}_h$

$$\mathbf{P}_h\mathbf{f}|_K = \frac{1}{|K|}\int_K \mathbf{f}dx.$$

For $k \ge 1$, the calculation of $\mathbf{P}_h\mathbf{f}$ requires numerical solutions of local problems on each element $K \in \mathcal{T}_h$

$$(\mathbf{P}_h\mathbf{f}|_K, \mathbf{v})_K = (\mathbf{f}, \mathbf{v})_K \qquad \forall\mathbf{v} \in P_k(K).$$

We end this section with description of matrix forms of (4.9) and (4.10). To do so, let $\{\boldsymbol{\Phi}_i^h, i = 1, 2, \ldots, N\}$ and $\{\boldsymbol{\psi}_i^h, i = 1, 2, \ldots, M\}$ be basis functions for $\hat{RT}_k^d$ and P_k^d, respectively. The solutions $\boldsymbol{\sigma}_h^\varepsilon$ and $\mathbf{u}_h^\varepsilon$ of (4.9) and (4.10) may be represented in terms of these basis functions

$$\boldsymbol{\sigma}_h^\varepsilon = \sum_{j=1}^N \Sigma_j^h\boldsymbol{\Phi}_j^h \quad \text{and} \quad \mathbf{u}_h^\varepsilon = \sum_{j=1}^M U_j^h\boldsymbol{\psi}_j^h,$$

respectively. Denote the unknown vectors by

$$\mathbf{\Sigma}_h = \left(\Sigma_1^h, \Sigma_2^h, \ldots, \Sigma_N^h\right)^t \quad \text{and} \quad \mathbf{U}_h = \left(U_1^h, U_2^h, \ldots, U_M^h\right)^t,$$

the coefficient matrices by

$$\mathbf{A}_h = \mathbf{A}_h^0 + \frac{1}{\varepsilon}\mathbf{A}_h^1 \quad \text{with} \quad \mathbf{A}_h^0 = \left(\left(\kappa\mathcal{A}\mathbf{\Phi}_j^h, \mathbf{\Phi}_i^h\right)\right)_{N\times N} \quad \text{and} \quad \mathbf{A}_h^1 = \left(\left(\nabla\cdot\mathbf{\Phi}_j^h, \nabla\cdot\mathbf{\Phi}_i^h\right)\right)_{N\times N},$$
$$\mathbf{D}_h = \left(D_{ij}^h\right)_{M\times M} \quad \text{with} \quad D_{ij}^h = \left(\boldsymbol{\psi}_j^h, \boldsymbol{\psi}_i^h\right),$$

and the right-hand side vectors by

$$\mathbf{G}_h = \left(G_i^h\right)_{N\times 1} \quad \text{with} \quad G_i^h = g\left(\mathbf{\Phi}_i^h\right) + \frac{1}{\varepsilon}f\left(\nabla\cdot\mathbf{\Phi}_i^h\right),$$
$$\mathbf{F}_h = \left(F_i^h\right)_{M\times 1} \quad \text{with} \quad F_i^h = f\left(\boldsymbol{\psi}_i^h\right),$$

then the matrix forms of (4.9) and (4.10) are

$$\mathbf{A}_h\mathbf{\Sigma}_h = \mathbf{G}_h \tag{4.11}$$

and

$$\mathbf{D}_h\mathbf{U}_h = \frac{1}{\varepsilon}(\mathbf{B}_h\mathbf{\Sigma}_h + \mathbf{F}_h), \tag{4.12}$$

respectively, where $\mathbf{B}_h = (B_{ij}^h)_{M\times N}$ with $B_{ij}^h = (\nabla\cdot\mathbf{\Phi}_j^h, \boldsymbol{\psi}_i^h)$. Note that support of the basis function $\boldsymbol{\psi}_i^h$ for the velocity is one element. Hence, the coefficient matrix $\mathbf{D}^h$ is a *block-diagonal mass* matrix with each block of size $(k+1)\times(k+1)$, where k is the degree of piecewise discontinuous polynomials approximating the velocity. This indicates that computational cost of solving (4.12) is negligible and, hence, the main cost of the new method for solving the Stokes equation is the solution of (4.11).

V. MULTIGRID PRECONDITIONERS

In this section, we study efficient multigrid preconditioners for both the penalty system (4.9) and the global (unpenalized) saddle-point problem (3.5). Consider first the penalized problem (4.9). Denote the corresponding bilinear form of (4.9) by

$$A_\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\kappa\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) + \frac{1}{\varepsilon}(\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{\tau})$$

and introduce a weighted $H(\text{div})$ inner product by

$$B_\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau}) + \frac{1}{\varepsilon}(\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{\tau}).$$

Theorem 5.1. *Assume that the penalty parameter ε is bounded above by a constant. Then bilinear forms $A_\varepsilon(\cdot,\cdot)$ and $B_\varepsilon(\cdot,\cdot)$ are spectrally equivalent and uniform in ε; i.e., there exist two positive constants C_1 and C_2 independent of ε such that*

$$C_1 B_\varepsilon(\boldsymbol{\tau}, \boldsymbol{\tau}) \le A_\varepsilon(\boldsymbol{\tau}, \boldsymbol{\tau}) \le C_2 B_\varepsilon(\boldsymbol{\tau}, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \hat{H}(\text{div};\, \Omega)^d. \tag{5.1}$$

Proof. Equality (2.9) gives

$$(\kappa\mathcal{A}\boldsymbol{\tau},\boldsymbol{\tau}) = \|\sqrt{\kappa}\mathcal{A}\boldsymbol{\tau}\|^2 \le C\|\boldsymbol{\tau}\|^2,$$

which, in turn, implies the upper bound in (5.1). The lower bound in (5.1) is a direct consequence of (2.10) and the assumption that $\varepsilon \le C$. ■

The above theorem shows that the form $B_\varepsilon(\cdot,\cdot)$ can be used to precondition the form $A_\varepsilon(\cdot,\cdot)$ effectively. It is well-known (see, e.g., [20]) that the form $B_\varepsilon(\cdot,\cdot)$ can be efficiently preconditioned by the multigrid (V-cycle) preconditioner with appropriate additive or multiplicative Schwarz smoothers. This, in turn, implies that the multigrid V-cycle for the form $B_\varepsilon(\cdot,\cdot)$ is an efficient preconditioner for the form $A_\varepsilon(\cdot,\cdot)$.

As an alternative, we discuss a spectrally equivalent preconditioner for the saddle-point problem in (3.5) without using the penalty method. This discrete problem takes a saddle-point two-by-two block matrix form:

$$\mathcal{M}_h = \begin{pmatrix} \mathbf{A}_h^0 & \mathbf{B}_h^T \\ \mathbf{B}_h & \mathbf{0} \end{pmatrix}. \tag{5.2}$$

Denote by $\mathbf{H}_h = (B_1(\boldsymbol{\Phi}_j^h, \boldsymbol{\Phi}_i^h))_{N\times N}$ the matrix representation of the $H(\text{div})$ bilinear form $B_1(\boldsymbol{\sigma},\boldsymbol{\tau})$, and denote by $\langle\cdot,\cdot\rangle$ the Euclidean inner product. In what follows will study the spectral relations between the symmetric and indefinite matrix $\mathcal{M}_h$ and the symmetric, positive definite, and block-diagonal matrix

$$\mathcal{D}_h = \begin{pmatrix} \mathbf{H}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_h \end{pmatrix}.$$

Lemma 5.2. *There exist positive constants $\tilde{\alpha}$ and $\tilde{\beta}$ independent of h such that*

$$\langle \mathcal{D}_h\mathcal{M}_h^{-1}\mathcal{F}, \mathcal{M}_h^{-1}\mathcal{F}\rangle \le \tilde{\beta}\langle \mathcal{D}_h^{-1}\mathcal{F},\mathcal{F}\rangle \tag{5.3}$$

for any $\mathcal{F} = (\mathbf{g},\mathbf{f})^t$ and that

$$|\langle \mathcal{M}_h\mathcal{X},\mathcal{X}\rangle| \le \tilde{\alpha}\langle \mathcal{D}_h\mathcal{X},\mathcal{X}\rangle \tag{5.4}$$

for any $\mathcal{X} = (\mathbf{x},\mathbf{y})^t$.

Proof. For any given $\mathcal{F} = (\mathbf{g},\mathbf{f})^t$, let $\mathcal{X} = (\mathbf{x},\mathbf{y})^t$ be the unique solution of the following saddle-point problem

$$\mathcal{M}_h\mathcal{X} = \mathcal{F}. \tag{5.5}$$

Denote by g' and f' the corresponding linear functionals of $\mathbf{g}$ and $\mathbf{f}$, respectively, and by $(\boldsymbol{\tau},\mathbf{v})$ the corresponding function representation of $\mathcal{X}$ in $\hat{RT}_k^d \times P_k^d$. The a priori estimate in Corollary 3.5 implies

$$\|\boldsymbol{\tau}\|_{H(\text{div})}^2 + \|\mathbf{v}\|^2 \le \bar{\beta}(\|g'\| + \|f'\|), \tag{5.6}$$

which translates to

$$\langle \mathcal{D}_h \mathcal{X}, \mathcal{X} \rangle \leq \tilde{\beta}\left(\langle \mathbf{H}_h^{-1}\mathbf{g}, \mathbf{g}\rangle + \langle \mathbf{D}_h^{-1}\mathbf{f}, \mathbf{f}\rangle\right) = \tilde{\beta}\langle \mathcal{D}_h^{-1}\mathcal{F}, \mathcal{F}\rangle$$

in terms of matrices and coefficient vectors. Now, (5.3) follows from the fact that $\mathcal{X} = \mathcal{M}_h^{-1}\mathcal{F}$ due to (5.5).

For any $\mathcal{X} = (\mathbf{x}, \mathbf{y})^t$, let $(\boldsymbol{\tau}, \mathbf{v})$ be the corresponding function representation of $\mathcal{X}$ in $\hat{RT}_k^d \times P_k^d$. Then (5.4) may be rewritten as

$$|(\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\tau}) + 2(\mathbf{v}, \nabla \cdot \boldsymbol{\tau})| \leq \tilde{\alpha}\left(\|\boldsymbol{\tau}\|_{H(\mathrm{div})}^2 + \|\mathbf{v}\|^2\right),$$

which is an immediate consequence of the definition of $\mathcal{A}$ and the Cauchy-Schwarz inequality. ∎

We note that the same result as in Lemma 5.2 holds if $\mathcal{D}_h$ is replaced with any spectrally equivalent matrix. As $\mathbf{D}_h$ is a simple mass-matrix coming from discontinuous elements, hence easily invertible, we only need a spectrally equivalent preconditioner for $\mathbf{H}_h$ which was already discussed previously in the case of the penalty matrix.

We comment at the end that, in other words, Lemma 5.2 shows that the absolute value of eigenvalues of the symmetric matrix $\mathcal{D}_h^{-\frac{1}{2}} \mathcal{M}_h D_h^{-\frac{1}{2}}$ are bounded above and away from the origin and that these bounds are independent of h. As it is well known (see, e.g., the original reference [21]), these facts are sufficient to prove mesh-independent convergence bounds for the preconditioned minimum residual method applied to the system

$$\mathcal{M}_h \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix}, \tag{5.7}$$

using $\mathcal{D}_h$ as a preconditioner. A mesh-independent convergence bound is also valid, if one simply uses the preconditioned conjugate gradient method applied to the (weighted) normal system

$$\mathcal{M}_h \mathcal{D}_h^{-1} \mathcal{M}_h \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathcal{M}_h \mathcal{D}_h^{-1} \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix}, \tag{5.8}$$

using $\mathcal{D}_h$ as a preconditioner.

In conclusion, the saddle-point matrix $\mathcal{M}_h$ can be optimally preconditioned by appropriate block-diagonal matrix $\mathcal{D}_h$ in a preconditioned minimum residual algorithm, or in the preconditioned conjugate gradient method applied to the weighted normal form (5.8). Such techniques were explored previously, as early as in [22].

VI. NUMERICAL RESULTS

In this section, we present numerical results on accuracy of mixed finite element approximation and on the condition number of the preconditioned pseudostress system in (4.9). Test problems are defined on the unit square $\Omega = (0, 1)^2$ with the viscosity parameter being one ($\nu = 1$).

To measure the discretization error, we consider a model problem with a known nonzero solution. Let

$$\mathbf{f} = 2\begin{pmatrix} (2\pi)^2 \sin(2\pi x)\cos(2\pi y) + x \\ -(2\pi)^2 \cos(2\pi x)\sin(2\pi y) + y \end{pmatrix} \quad \text{and} \quad \mathbf{g} = \begin{cases} \begin{pmatrix} \sin(2\pi x) \\ 0 \end{pmatrix}, & \text{if } y = 0, 1, \\ \begin{pmatrix} 0 \\ -\sin(2\pi y) \end{pmatrix}, & \text{if } x = 0, 1 \end{cases}$$

TABLE I. L^2 errors for the pseudostress $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^\varepsilon\|$.

h	d.o.f.	$\varepsilon = h^2$	$\varepsilon = h/4$	$\varepsilon = h/2$	$\varepsilon = h$	$\varepsilon = 2h$	$\varepsilon = 10h$
$\frac{1}{4}$	80	3.0111	3.0111	3.0120	3.0136	3.0170	3.0448
$\frac{1}{8}$	288	1.4638	1.4641	1.4646	1.4656	1.4677	1.4858
$\frac{1}{16}$	1088	$7.1866e-1$	$7.1878e-1$	$7.1895e-1$	$7.1928e-1$	$7.1997e-1$	$7.2668e-1$
$\frac{1}{32}$	4224	$3.5721e-1$	$3.5724e-1$	$3.5729e-1$	$3.5738e-1$	$3.5758e-1$	$3.5983e-1$
$\frac{1}{64}$	16,640	$1.7832e-1$	$1.7833e-1$	$1.7834e-1$	$1.7837e-1$	$1.7842e-1$	$1.7922e-1$
$\frac{1}{128}$	66,048	$8.9383e-2$	$8.9126e-2$	$8.9129e-2$	$8.9136e-2$	$8.9154e-2$	$8.9469e-2$

be the right-hand side function and the prescribed velocity on the boundary, respectively. Then

$$\mathbf{u} = \begin{pmatrix} \sin(2\pi x)\cos(2\pi y) \\ -\cos(2\pi x)\sin(2\pi y) \end{pmatrix} \quad \text{and} \quad p = x^2 + y^2$$

are the exact solution of the stationary Stokes equation. By the definition of the pseudostress in (2.2), we have

$$\begin{aligned} \boldsymbol{\sigma} &= -p\boldsymbol{\delta} + \nabla \mathbf{u} \\ &= 2\pi \begin{pmatrix} \cos(2\pi x)\cos(2\pi y) - \frac{1}{2\pi}(x^2+y^2) & \sin(2\pi x)\sin(2\pi y) \\ -\sin(2\pi x)\sin(2\pi y) & -\cos(2\pi x)\cos(2\pi y) - \frac{1}{2\pi}(x^2+y^2) \end{pmatrix}. \end{aligned}$$

Obviously, $(\boldsymbol{\sigma}, \mathbf{u})$ is then the exact solution of variational problem (2.7) in the pseudostress-velocity formulation.

Partition the domain $\Omega = (0,1)^2$ by uniform rectangular elements $K_{ij} = (ih, jh)$ for $i, j = 0, 1, \ldots, N$ with $h = 1/N$. Finite element approximation $\boldsymbol{\sigma}_h^\varepsilon \in RT_0^d$ to the pseudostress is computed through solving system (4.9) with the lowest order RT element ($k = 0$) by a direct method. Finite element approximation $\mathbf{u}_h^\varepsilon \in P_0^d$ to the velocity is calculated explicitly by using (4.10). Discretization errors for $\varepsilon = ch$ and h^m with different values of constant c and exponent m are reported in Tables I–III. The pseudostress and velocity are $O(h)$ accurate in the L^2 norm for $m \geq 1$ as predicted theoretically in Section IV and their dependence on the constant c is weak. The second equations in (4.1) and (2.5) imply

$$\varepsilon \mathbf{u}_h^\varepsilon = \nabla \cdot \boldsymbol{\sigma}_h^\varepsilon + \mathbf{P}_h \mathbf{f} = \nabla \cdot \boldsymbol{\sigma}_h^\varepsilon - \mathbf{P}_h \nabla \cdot \boldsymbol{\sigma}.$$

As $\|\mathbf{u}_h^\varepsilon\|$ is bounded (see Lemma 3.4), $\|\mathbf{P}_h \nabla \cdot \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\sigma}_h^\varepsilon\| = O(\varepsilon)$ which is confirmed numerically in Table II.

Next we study the multigrid convergence rates using different values of ε. Random right hand sides are used with the zero energy mode eliminated. We apply a classical V(1,1)-cycle multigrid algorithm with multiplicative Schwarz smoothers where the overlapped blocks are formed

TABLE II. Discrete L^2 errors for the divergence of pseudostress $\|\mathbf{P}_h\nabla\cdot\boldsymbol{\sigma} - \nabla\cdot\boldsymbol{\sigma}_h^{\varepsilon}\|$.

h	d.o.f.	$\varepsilon = h^4$	$\varepsilon = h^3$	$\varepsilon = h^2$	$\varepsilon = h/4$	$\varepsilon = h$	$\varepsilon = 10h$
$\frac{1}{4}$	80	$1.9438e-3$	$7.7746e-3$	$3.1089e-2$	$3.1089e-2$	$1.2421e-1$	1.2251
$\frac{1}{8}$	288	$1.5858e-4$	$1.2686e-3$	$1.0148e-2$	$2.0294e-2$	$8.1126e-2$	$8.0552e-1$
$\frac{1}{16}$	1088	$1.0565e-5$	$1.6905e-4$	$2.7047e-3$	$1.0818e-2$	$4.3259e-2$	$4.3105e-1$
$\frac{1}{32}$	4224	$6.7084e-7$	$2.1467e-5$	$6.8693e-4$	$5.4952e-3$	$2.1977e-2$	$2.1938e-1$
$\frac{1}{64}$	16,640	$4.2092e-8$	$2.6939e-6$	$1.7241e-4$	$2.7585e-3$	$1.1033e-2$	$1.1023e-1$
$\frac{1}{128}$	66,048	$2.6333e-9$	$3.3706e-7$	$4.3144e-5$	$1.3806e-3$	$5.5222e-3$	$5.5197e-2$

by collecting the edge variables incident on each node (that is, each block is 8×8, because there are 4 edges and each edge has degree of freedom 2); and the coarsest problem is solved by the conjugate gradient method. The prolongation (or coarse-to-fine) operators, which are widely used for nested rectangular meshes, are defined by (from the coarser level $l+1$ to the finer level l):

$$P_{l+1}^{l}(e_j)_{l+1} = \begin{cases} (e_k)_l & \text{if } (e_j)_{l+1} \in (e_k)_l \\ \dfrac{1}{2}(e_k)_l & \text{if } (e_j)_{l+1}, (e_k)_l \in (E_n)_{l+1}, (e_j)_{l+1} \notin (e_k)_l, (e_j)_{l+1} || (e_k)_l \end{cases}$$

where $(e_j)_l$ denotes the j-th edge on level l, $(E_j)_l$ denotes the j-th element on level l, and $\|$ denotes that the two edges are parallel. The second clause of the above formula basically states that the fine edges that are not part of the coarse mesh are interpolated by their neighboring edges that are parallel to them. The restriction (or fine-to-coarse) operators are defined as the transpose of the corresponding prolongation operators. Finally, we form the Galerkin coarse operators for all coarse levels via $A_{l+1} = (P_{l+1}^{l})^T A_l P_{l+1}^{l}$.

TABLE III. L^2 errors for the velocity $\|\mathbf{u} - \mathbf{u}_h^{\varepsilon}\|$.

h	d.o.f.	$\varepsilon = h^2$	$\varepsilon = h/4$	$\varepsilon = h/2$	$\varepsilon = h$	$\varepsilon = 2h$	$\varepsilon = 10h$
$\frac{1}{4}$	80	$4.2115e-1$	$4.2115e-1$	$4.2118e-1$	$4.2125e-1$	$4.2140e-1$	$4.2256e-1$
$\frac{1}{8}$	288	$2.2277e-1$	$2.2278e-1$	$2.2279e-1$	$2.2282e-1$	$2.2288e-1$	$2.2338e-1$
$\frac{1}{16}$	1088	$1.1287e-1$	$1.1288e-1$	$1.1288e-1$	$1.1289e-1$	$1.1291e-1$	$1.1307e-1$
$\frac{1}{32}$	4224	$5.6620e-2$	$5.6621e-2$	$5.6622e-2$	$5.6624e-2$	$5.6629e-2$	$5.6681e-2$
$\frac{1}{64}$	16,640	$2.8333e-2$	$2.8333e-2$	$2.8334e-2$	$2.8334e-2$	$2.8335e-2$	$2.8353e-2$
$\frac{1}{128}$	66,048	$1.4169e-2$	$1.4169e-2$	$1.4169e-2$	$1.4170e-2$	$1.4170e-2$	$1.4177e-2$

TABLE IV. MG convergence rate for different ε's.

		ρ(iter)					
h	N	$\varepsilon = h^2$	$\varepsilon = 0.1h$	$\varepsilon = 0.5h$	$\varepsilon = h$	$\varepsilon = 5h$	$\varepsilon = 10h$
$\frac{1}{8}$	288	0.20(12)	0.20(12)	0.20(12)	0.20(12)	0.20(12)	0.20(12)
$\frac{1}{16}$	1088	0.21(12)	0.21(12)	0.21(12)	0.21(12)	0.21(12)	0.21(12)
$\frac{1}{32}$	4224	0.21(12)	0.21(12)	0.21(12)	0.21(12)	0.21(12)	0.21(12)
$\frac{1}{64}$	16,640	0.22(12)	0.21(12)	0.21(12)	0.21(12)	0.21(12)	0.21(12)
$\frac{1}{128}$	66,048	NC	0.21(12)	0.21(12)	0.21(12)	0.21(12)	0.21(12)
$\frac{1}{256}$	263,168	NC	NC	0.24(13)	0.21(12)	0.21(12)	0.21(12)

N, total degree of freedom; ρ, average convergence rates; (iter), number of MG iterations (V(1,1) using Schwarz smoother); NC, does not converge.

The results for a few different values of ε are given in Table IV. We observe here that multigrid is not robust when ε is small. A remedy is to use the generalized minimal residual (GMRES) (or conjugate gradient (CG)) method with one V(1,1)-cycle multigrid as the preconditioner, the results of which are given in Table V.

In the next set of numerical experiments we use the bilinear form $B_\varepsilon(\cdot,\cdot)$ as a preconditioner of the bilinear form $A_\varepsilon(\cdot,\cdot)$. We again apply GMRES with one V(1,1)-cycle multigrid as the preconditioner. Again the iteration counts for a few different values of ε are given in Table VI.

In all our experiments we observe essential spectral equivalent convergence rates for small ε, i.e., $\varepsilon = O(h)$. When the method converges, its convergence rate is independent of ε. Finally, Tables V and VI show that the preconditioned GMRES using the bilinear form $A_\varepsilon(\cdot,\cdot)$ is twice faster than that using the bilinear form $B_\varepsilon(\cdot,\cdot)$.

TABLE V. MG convergence rate for different ε's.

		ρ(iter)					
h	N	$\varepsilon = h^2$	$\varepsilon = 0.1h$	$\varepsilon = 0.5h$	$\varepsilon = h$	$\varepsilon = 5h$	$\varepsilon = 10h$
$\frac{1}{8}$	288	0.124(9)	0.124(9)	0.124(9)	0.124(9)	0.130(9)	0.130(9)
$\frac{1}{16}$	1088	0.146(10)	0.146(10)	0.146(12)	0.146(10)	0.146(10)	0.146(10)
$\frac{1}{32}$	4224	0.154(10)	0.154(10)	0.154(10)	0.154(10)	0.154(10)	0.154(10)
$\frac{1}{64}$	16,640	0.155(10)	0.155(10)	0.155(10)	0.155(10)	0.155(10)	0.155(10)
$\frac{1}{128}$	66,048	0.157(10)	0.157(12)	0.157(10)	0.157(10)	0.157(10)	0.157(10)
$\frac{1}{256}$	263,168	0.158(10)	0.158(10)	0.158(10)	0.158(10)	0.158(10)	0.158(10)

N, total degree of freedom; ρ, average convergence rates; (iter), number of GMRES iterations using MG V(1,1) as preconditioner.

TABLE VI. GMRES-MG convergence rates for different ε's.

h	N	ρ(iter)					
		$\varepsilon = h^2$	$\varepsilon = 0.1h$	$\varepsilon = 0.5h$	$\varepsilon = h$	$\varepsilon = 5h$	$\varepsilon = 10h$
$\frac{1}{8}$	288	0.38(19)	0.38(19)	0.38(19)	0.38(19)	0.38(19)	0.38(19)
$\frac{1}{16}$	1088	0.40(20)	0.40(20)	0.40(20)	0.40(20)	0.40(21)	0.40(21)
$\frac{1}{32}$	4224	0.42(22)	0.42(22)	0.42(22)	0.42(22)	0.42(22)	0.42(22)
$\frac{1}{64}$	16,640	0.42(22)	0.42(22)	0.42(22)	0.42(22)	0.42(22)	0.42(22)
$\frac{1}{128}$	66,048	0.41(21)	0.41(21)	0.41(21)	0.41(21)	0.41(21)	0.41(21)
$\frac{1}{256}$	263,168	0.41(21)	0.41(21)	0.41(21)	0.41(21)	0.41(21)	0.41(21)

N, total degree of freedom; ρ, average convergence rates; (iter), number of GMRES iterations using MG V(1,1) as preconditioner.

VII. CONCLUSION REMARKS

In this article, we studied a new numerical method for solving the stationary Stokes equation, which may be easily extended to Navier-Stokes equations in principle. The method is more accurate than existing methods for applications in which the shear stress are important. The main cost of the method is the computation of the solution of the pseudostress system in (4.9). Even though the pseudostress has more variables than the velocity and pressure, the numbers of degrees of freedom for the pseudostress using Raviart-Thomas elements of index $k = 0, 1$ and BDM elements of index $k = 1, 2$ [17] are comparable to those for the velocity-pressure using Crouzeix-Raviart (nonconforming) elements of order $k = 1, 2$ and $k = 2, 3$, respectively. Calculations of the other physical quantities such as the velocity, pressure, stress, and vorticity are straightforward and have negligible cost. Our numerical results have shown that the positive, definite pseudostress system can be solved by a highly efficient PCG with a spectrally equivalent multigrid preconditioner. Uniform convergence analysis, with respect to the mesh size, the number of levels, and the large penalty parameter, on multigrid method for the pseudostress system will be presented in [23]. If one wants to avoid the penalty formulation, then one has to work with the indefinite saddle-point system in the way described in the second part of Section V. The latter approach is somewhat more expensive because one has to work with bigger size matrices and vectors, but nevertheless the discussed preconditioned methods (the minimum residual and conjugate gradient applied to the weighted normal system) exhibit proven convergence rates bounded independently of the mesh size.

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