EXACT TRAVELING-WAVE SOLUTIONS TO BI-DIRECTIONAL WAVE EQUATIONS

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ABSTRACT. In this paper, we present several systematic ways to find exact traveling-wave solutions of the systems

$$\eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} = 0,$$
$$u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0,$$

where a, b, c and d are real constants. These systems, derived by Bona, Saut and Toland in [3] for describing small-amplitude long waves in a water channel, are formally equivalent to the classical Boussinesq system (cf. [5]) and correct through first order with regard to a small parameter characterizing the typical amplitude to depth ratio. Exact solutions for a large class of systems are presented. The existence of the exact traveling-wave solutions is in general extremely helpful in the theoretical and numerical study of the systems.

1. Introduction.

We consider in this paper the model systems which describe two-way propagation of nonlinear dispersive waves in a water channel. Under the assumptions that the maximum deviation a of the free surface is small and a typical wave length λ is large when compared to the undisturbed water depth h, and that the Stokes number $S = a\lambda^2/h^3$ is of order one, which means the effect of nonlinearity and dispersion is of the same order (S will be taken to be 1 in the rest of the paper for simplicity in notion), a restricted four-parameter family of systems was derived by Bona, Saut and Toland in [3], having the form

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} = 0,\\ &u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \end{aligned} \tag{1.1}$$

where x corresponds to distance along the channel (scaled by h) and t is the elapsed time scaled by $(h/g)^{\frac{1}{2}}$ where g denotes the acceleration of gravity, the variable $\eta(x,t)$ is the non-dimensional deviation of the water surface (scaled by h) from its

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undisturbed position and $u(x,t)$ is the non-dimensional horizontal velocity (scaled by $\sqrt{gh}$) at a height θh with $0 \le \theta \le 1$ above the bottom of the channel, and the real constants a, b, c and d satisfy

$$
\begin{aligned}
a &= \frac{1}{2}(\theta^2 - \frac{1}{3})\lambda, \qquad & b &= \frac{1}{2}(\theta^2 - \frac{1}{3})(1-\lambda), \\
c &= \frac{1}{2}(1-\theta^2)\mu, \qquad & d &= \frac{1}{2}(1-\theta^2)(1-\mu),
\end{aligned} \tag{1.2}
$$

where λ, μ are real numbers. These systems are formally equivalent and have the same formal status as the Kortweg-deVries equation for the uni-directional propagation of waves in a channel in the way that they are correct through first order with regard to the small parameter $\epsilon = a/h$. The reasons for the plethora of different, but formally equivalent Boussinesq system is owed to the fact that the lower-order relation can be used systematically to alter the higher-order terms without disturbing the formal level of approximation and to the considerable number of choices of dependent variables available (see [3] in detail). Some interesting examples included in (1.1) are

Whitham's system ($\theta^2 = 0, \lambda = 1, \mu = 0$) (cf. [17], formula (13.101)):

$$
\begin{aligned}
&\eta_t + u_x + (u\eta)_x - \frac{1}{6}u_{xxx} = 0, \\
&u_t + \eta_x + uu_x - \frac{1}{2}u_{xxt} = 0.
\end{aligned} \tag{1.3}
$$

Regularized Boussinesq system ($\theta^2 = \frac{2}{3}, \lambda = 0, \mu = 0$) (cf. [2]):

$$
\begin{aligned}
&\eta_t + u_x + (u\eta)_x - \frac{1}{6}\eta_{xxt} = 0, \\
&u_t + \eta_x + uu_x - \frac{1}{6}u_{xxt} = 0.
\end{aligned} \tag{1.4}
$$

Coupled K-dV-regularized system ($\theta^2 = \frac{2}{3}, \lambda = 1, \mu = 0$):

$$
\begin{aligned}
&\eta_t + u_x + (u\eta)_x + \frac{1}{6}u_{xxx} = 0, \\
&u_t + \eta_x + uu_x - \frac{1}{6}u_{xxt} = 0.
\end{aligned} \tag{1.5}
$$

Boussinesq's original system ($\theta^2 = \frac{1}{3}, \lambda$ *arbitrary*$, \mu = 0$) (cf. [5]):

$$
\begin{aligned}
&\eta_t + u_x + (u\eta)_x = 0, \\
&u_t + \eta_x + uu_x - \frac{1}{3}u_{xxt} = 0.
\end{aligned} \tag{1.6}
$$

Coupled K-dV system ($\theta^2 = \frac{2}{3}, \lambda = 1, \mu = 1$):

$$
\begin{aligned}
&\eta_t + u_x + (u\eta)_x + \frac{1}{6}u_{xxx} = 0, \\
&u_t + \eta_x + uu_x + \frac{1}{6}\eta_{xxx} = 0.
\end{aligned} \tag{1.7}
$$

Coupled regularized- K-dV system ($\theta^2 = \frac{2}{3}, \lambda = 0, \mu = 1$):

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x - \frac{1}{6}\eta_{xxt} = 0, \\ &u_t + \eta_x + uu_x + \frac{1}{6}\eta_{xxx} = 0. \end{aligned} \tag{1.8}$$

Integrable version of Boussinesq system ($\theta^2 = 1, \lambda = 1, \mu$ arbitrary) (cf. [10]):

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x + \frac{1}{3}u_{xxx} = 0, \\ &u_t + \eta_x + uu_x = 0. \end{aligned} \tag{1.9}$$

Bona-Smith system ($\theta^2 = (\frac{4}{3} - \mu)/(2 - \mu), \lambda = 0, \mu < 0$ arbitrary) (cf. [4]):

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x - b\eta_{xxt} = 0, \\ &u_t + \eta_x + uu_x + c\eta_{xxx} - bu_{xxt} = 0, \end{aligned} \tag{1.10}$$

where in the notation of (1.1) and (1.2)

$$b = \frac{1-\mu}{3(2-\mu)} > 0 \quad \text{and} \quad c = \frac{\mu}{3(2-\mu)} < 0.$$

In this paper, we will concentrate on finding exact traveling-wave solutions of (1.1). The existence of these solutions will be useful in several ways in the study of these model systems. In fact, one of the exact solution we found here for the regularized Boussinesq system (1.4) has been used in [2] to demonstrate the convergence rate of a numerical algorithm.

The structure of the paper is as follows. In section 2, we search for exact solutions $(\eta(x,t), u(x,t))$ where $\eta(x,t)$ and $u(x,t)$ are proportional to each other and they approach zero when x approaches $\pm\infty$. The solutions we found appear to have the form $A\,\mathrm{sech}^2\big(\lambda(x + x_0 - C_s t)\big)$, where A, λ, x_0 and C_s are constants. The explicit requirements on a, b, c, d (or on λ, μ, θ) for such solutions to exist are presented. We then compare these solitary-wave solutions with those of the famous KdV equation which is a model equation describing uni-directional waves. In section 3, we search for exact traveling-wave solutions (η, u) where $u(x,t)$ is of the form $u_\infty + A\,\mathrm{sech}^2\big(\lambda(x + x_0 - C_s t)\big)$ and $\eta(x,t)$ is a function of $u(x,t)$ and approaches a constant η_∞ at $-\infty$. It is shown that such exact solutions can be found by solving a system of nonlinear algebraic equations involving A, λ, C_s, u_∞ and η_∞. Section 4 is similar to section 3, but searching for the traveling-wave solutions where $\eta(x,t)$ is of the form $\eta_\infty + A\,\mathrm{sech}^2\big(\lambda(x + x_0 - C_s t)\big)$. We conclude the paper in section 5.

2. Solitary-wave solutions in the form of $u(x,t) = u(x + x_0 - C_s t)$ and $u(x,t) = B\,\eta(x,t)$.

Denoting $\xi = x + x_0 - C_s t$ with x_0 and C_s being constants, one can write a traveling-wave solution $(\eta(x,t), u(x,t))$ as

$$\eta(x,t) = \eta(\xi) \equiv \eta(x + x_0 - C_s t), \quad u(x,t) = u(\xi) \equiv u(x + x_0 - C_s t). \tag{2.1}$$

Substituting (2.1) into (1.1), one finds

$$\begin{aligned} &-C_s\eta' + u' + (u\eta)' + au''' + bC_s\eta''' = 0, \\ &-C_s u' + \eta' + uu' + c\eta''' + dC_s u''' = 0, \end{aligned} \tag{2.2}$$

where the derivatives are with respect to ξ. Since we are searching for solitary-wave solutions, meaning that the solutions that are asymptotically small at large distance from their crest, so $(\eta(\xi), u(\xi)) \to 0$ as $\xi \to \pm\infty$, system (2.2) can be integrated once to yield

$$\begin{aligned} &(-C_s + u)\eta + bC_s\eta'' = -u - au'', \\ &\eta + c\eta'' = C_s u - \frac{1}{2}u^2 - dC_s u''. \end{aligned} \tag{2.3}$$

Suppose that $\eta(\xi)$ and $u(\xi)$ are proportional to each other, namely $u(\xi) = B\eta(\xi)$ with B being a constant, one obtains

$$\begin{aligned} &(C_s B - B^2)\eta - (bC_s B + aB^2)\eta'' = B^2\eta^2, \\ &(2C_s B - 2)\eta - (2dC_s B + 2c)\eta'' = B^2\eta^2. \end{aligned} \tag{2.4}$$

In order for (2.4) to have nontrivial solitary-wave solutions, it is necessary that the two equations are identical which implies

$$\begin{aligned} &B^2 + C_s B - 2 = 0, \\ &aB^2 + (b - 2d)C_s B - 2c = 0. \end{aligned} \tag{2.5}$$

Above system is linear with respect to unknowns B^2 and $C_s B$ and its solution depends on the values of a, b, c and d as follows,

- **Case I:** if $a - b + 2d \neq 0$, there is an unique solution $B^2 = \frac{2(-b+c+2d)}{a-b+2d}$ and $C_s B = 2 - B^2$;
- **Case II:** if $a - b + 2d = 0$ and $a = c$, there are infinitely many solutions, which reads $C_s B = 2 - B^2$ and B^2 arbitrary;
- **Case III:** if $a - b + 2d = 0$ and $c \neq a$, there is no solution.

In the cases that (2.5) has a solution, taking derivative on one of the equations in (2.4) and using (2.5), one finds $\eta(\xi)$ satisfies

$$2(1 - B^2)\eta' - ((a-b)B^2 + 2b)\eta''' = 2B^2\eta\eta'. \tag{2.6}$$

The solution of (2.6) can be easily obtained with the use of the following lemma which can be found in many textbooks (for example, see [11]).

Lemma 1. *Let α, β be real constants, the equation*

$$\alpha\eta'(\xi) - \beta\eta'''(\xi) = \eta(\xi)\eta'(\xi)$$

has a solitary-wave solution if $\alpha\beta > 0$. Moreover, the solitary-wave solution is

$$\eta(\xi) = 3\,\alpha\, sech^2\left(\frac{1}{2}\sqrt{\frac{\alpha}{\beta}}\,(\xi+\xi_0)\right)$$

where ξ_0 is an arbitrary constant.

In summary, one concludes that the conditions for (1.1) to have solitary-wave solutions which is of the form $u(\xi) = B\,\eta(\xi)$ are the following,

(i) a, b, c and d are in the case I or II;

(ii) the solution B^2 of (2.5) is non-negative and satisfies

$$(B^2-1)((b-a)B^2-2b) > 0. \tag{2.7}$$

Notice that $B = 0$ is not a solution of (2.5), so the solitary-wave solutions can be expressed explicitly,

$$\begin{aligned}
\eta(x,t) &= \frac{3(1-B^2)}{B^2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{2(1-B^2)}{(a-b)B^2+2b}}\,(x+x_0-C_s t)\right),\\
u(x,t) &= B\,\eta(x,t).
\end{aligned}$$

Above solutions can be written in a more familiar form. Let

$$\eta_0 = \frac{3(1-B^2)}{B^2},$$

one sees that

$$\begin{aligned}
\eta(x,t) &= \eta_0\,\mathrm{sech}^2\big(\lambda(x+x_0-C_s t)\big),\\
u(x,t) &= \pm\sqrt{\frac{3}{\eta_0+3}}\,\eta_0\,\mathrm{sech}^2\big(\lambda(x+x_0-C_s t)\big),
\end{aligned} \tag{2.8}$$

where

$$C_s = \frac{3+2\eta_0}{\pm\sqrt{3(3+\eta_0)}}, \qquad \lambda = \frac{1}{2}\sqrt{\frac{2\eta_0}{3(a+b)+2b\eta_0}}. \tag{2.9}$$

After simple calculations, one can prove the following theorem which corresponds to the case I.

Theorem 1. *Suppose $a-b+2d \neq 0$ and let $p = \frac{-b+c+2d}{a-b+2d}$, then the system* (1.1) *has a pair of solitary-wave solution in the form of $u(x,t) = B\eta(x,t)$ if and only if $p > 0$ and $(p-\frac{1}{2})((b-a)p-b) > 0$. Moreover, the exact solitary-wave solution is in the form of* (2.8)–(2.9) *with $\eta_0 = \frac{3(1-2p)}{2p}$.*

Similarly, the situation corresponding to case II can be translated as that for $a-b+2d = 0$ and $a = c$, the solitary-wave solutions exist for any B^2 satisfies $(B^2-1)(dB^2-b) > 0$ and $B^2 > 0$. More specifically, the following theorem holds when one denotes the closed (or open) interval between α and β by $[\alpha, \beta]$ (or (α, β)). For example, if $\alpha = 2$ and $\beta = 1$, we use $[2, 1]$ to denote the closed interval $[1, 2]$.

Theorem 2.

(i) If $a = b = c > 0$, $d = 0$, there are solitary-wave solutions in the form of (2.8)–(2.9) *for any $0 < \eta_0 < +\infty$.*

(ii) If $a = b = c < 0$, $d = 0$, there are solitary-wave solutions in the form of (2.8)–(2.9) *for any $-3 \leq \eta_0 < 0$.*

(iii) If $a-b+2d = 0$, $a = c$, $d > 0$, there are solitary-wave solutions in the form of (2.8)–(2.9) *for any $\eta_0 > -3$ and $\frac{3}{\eta_0+3} \notin [1, \frac{b}{d}]$.*

(iv) If $a-b+2d = 0$, $a = c$, $d < 0$, there are solitary-wave solutions in the form of (2.8)–(2.9) *for any $\eta_0 > -3$ and $\frac{3}{\eta_0+3} \in [1, \frac{b}{d}]$.*

It is worth to note that that the phase velocity C_s for these exact solitary-wave solutions depends on η_0, the amplitude of the wave, but not on the constants a, b, c and d. The spread of the wave which is represented by λ does depend on the individual system.

We now compare the phase velocity C_s of (2.8)–(2.9) with the phase velocity of the full Euler equations

$$c = 1 + \frac{1}{2}\eta_0 - \frac{3}{20}\eta_0^2 + \frac{3}{56}\eta_0^3 + \cdots, \tag{2.10}$$

which is obtained by systematic expansion (cf. [5] and [7]) and is justified in some sense by Craig in [6]. Subtracting c from C_s yields

$$C_s - c = 0.09\,\eta_0^2 + \text{ higher-order terms in } \eta_0.$$

If one also compare the phase velocity of the solitary-wave solutions of the KdV equation, which is $C_k = 1 + \frac{1}{2}\eta_0$, with c, one finds

$$C_K - c = 0.15\,\eta_0^2 + \text{ higher-order terms in } \eta_0.$$

Since the leading order in $C_s - c$ is smaller than that in $C_k - c$, one expects that the exact solitary-wave solutions obtained in Theorem 1 and 2 for systems in (1.1) are better approximations for small-amplitude waves. This fact is also observed numerically for other systems in (1.1) which we were not able to find exact solutions of the form (2.8)-(2.9) (cf. [2]). Comparisons with laboratory data also indicate that the Boussinesq systems capture far more accurately the general drift of amplitude-speed than does the KdV-equation even for somewhat larger amplitude waves.

Theorem 1 and 2 also demonstrate that although all the systems in (1.1)-(1.2) are formally equivalent, they are different. Depending on the values of a, b, c and d, the system admits different number of $sech^2$-solitary-wave solutions. In the cases which satisfy the conditions stated in Theorem 1, there exists only one pair of $sech^2$-solitary-wave solutions (one travels to the left and one travels to the right). In the cases which satisfy the conditions stated in Theorem 2, many interesting situations occur. For instance the $sech^2$-solitary-wave solution can exists for η_0 in a certain range and the waves can be in elevation or depression depending on the sign of η_0. This is different from the result for KdV and BBM equation (cf. [1]) which admit an one-parameter family of solitary-wave solutions with elevation only (cf. [12]). The spread of the solitary-wave solutions are also different depending on the individual system (c.f. (2.9)).

Since the parameters a, b, c, d in (1.1) as model systems for water waves are not free, but form a restricted four-parameter family with the restrictions (1.2), one can prove after tedious calculation that

Corollary. *Let*

$$R_1 = 1 + \frac{\frac{1}{2}(\theta^2 - \frac{1}{3})(2\lambda - 1)}{1 - \theta^2}, \quad R_2 = 2 - \frac{(\theta^2 - \frac{1}{3})(1 - \lambda)}{1 - \theta^2},$$

the system (1.1) *has nontrivial solitary-wave solutions, which are in the form of*

(2.8)–(2.9), *if one of the following is satisfied:*

$$(1)\qquad 0\le\theta^2<\frac{1}{3},\qquad \lambda \textit{ is arbitrary},\qquad \mu<\min\{2\lambda,R_1\};$$

$$(2)\qquad \frac{1}{3}<\theta^2<\frac{7}{9},\qquad \begin{cases}\lambda<\frac12, & 2\lambda<\mu<R_1;\\ \lambda\le\frac12, & \mu>R_2;\\ \lambda>\frac12, & \mu>\max\{2\lambda,R_2\};\end{cases}$$

$$(3)\qquad \theta^2=\frac{7}{9},\qquad \lambda \textit{ is arbitrary},\qquad \mu=2\lambda;$$

$$(4)\qquad \frac{7}{9}<\theta^2<1,\qquad \begin{cases}\lambda<\frac12, & R_1<\mu<2\lambda;\\ \lambda\le\frac12, & \mu<R_2;\\ \lambda>\frac12, & \mu<\min\{2\lambda,R_2\}.\end{cases}$$

We now applying Theorems 1 and 2 to some of the systems included in (1.1).

Example 1. Applying Theorem 1 to system (Whitham) ($a=-\frac16$, $b=c=0$, $d=\frac12$), one recovers the exact solitary-wave solution

$$\begin{aligned}\eta(x,t)&=-\frac{7}{4}\text{sech}^2\left(\frac12\sqrt7(x+x_0\pm\frac{1}{\sqrt{15}}t)\right),\\ u(x,t)&=\mp\frac72\sqrt{\frac35}\text{sech}^2\left(\frac12\sqrt7(x+x_0\pm\frac{1}{\sqrt{15}}t)\right),\end{aligned}\tag{2.11}$$

obtained in [15] by using a homogeneous balance method.

Example 2. Let $\theta^2=\frac45,\lambda=-1,\mu=-4$, the resulting system is

$$\begin{aligned}&\eta_t+u_x+(u\eta)_x-\frac{7}{30}u_{xxx}-\frac{7}{15}\eta_{xxt}=0,\\ &u_t+\eta_x+uu_x-\frac25\eta_{xxx}-\frac12u_{xxt}=0.\end{aligned}$$

According to Theorem 1, this system has an exact solitary-wave solution

$$\begin{aligned}\eta(x,t)&=\frac38\text{sech}^2\left(\frac12\sqrt{\frac57}(x+x_0\pm\frac{5\sqrt2}{6}t)\right),\\ u(x,t)&=\pm\frac{1}{2\sqrt2}\text{sech}^2\left(\frac12\sqrt{\frac57}(x+x_0\pm\frac{5\sqrt2}{6}t)\right).\end{aligned}$$

Example 3. For Bona-Smith system (1.10), $p=\frac{b+c}{b}$. Applying Theorem 1, one finds the solitary-wave solutions in the form of (2.8)–(2.9) exist in the cases

$$(i)\quad b>0 \text{ and } c>1;\qquad (ii)\quad b<0, c>0 \text{ and } b+2c<0.$$

Since

$$\eta_0=\frac{-3(b+2c)}{2(b+c)},$$

it is easy to check that $\eta_0<0$ in both cases.

Combining the results in [3], one can find systems which admit exact solitary-wave solutions and have also other properties such as the linearized system is L_2-well-posed. Recalling from [3], one has

Proposition. *The linearized system of* (1.1) *is L_2-well-posed if and only if one of the following condition hold.*

- $a < 0$, $c < 0$, $b > 0$, $d > 0$;
- $a = c$, $b < 0$, $d > 0$;
- $b = d$, $a < 0$, $c > 0$;
- $a = c$, $b = d$;
- $a = 0$, $b = 0$, $c < 0$, $d > 0$;
- $c = 0$, $d - 0$, $a < 0$, $b > 0$;
- $a = -b$, $c < 0$, $d > 0$;
- $a = -b$, $d = -c$;
- $d = -c$, $b > 0$, $a < 0$.

Applying the proposition to the systems in Example 1 and 2, one finds that the linearized system from Example 1 is not L_2-well-posed while the linearized system from Example 2 is L_2-well-posed. One might ask at this point that which system in (1.1) should be chosen in a concrete modeling situation. This issue is addressed both in paper [3] and [2]. The primary criteria for the choice include but limited to that the system is mathematically well-posed, preserve energy or other physical quantities which was conserved by the full Euler equations, has stable solitary-wave solutions and suitable for numerical simulations on a well-posed initial- and boundary-value problem.

Due to the restrictions posed on the form of the exact solutions we are searching, it is clear from Theorem 1 and 2 that there are many systems which do not admit solutions of the form (2.8)–(2.9), including the systems listed in (1.4)–(1.9). We therefore continuous our search in the next section.

3. Traveling-wave solutions with $u(x,t) = u_\infty + A\,\mathbf{sech}^2\big(\lambda(x + x_0 - C_s t)\big)$.

In this section, we use a more general approach to search for more exact solutions. Denoting again that $\xi = x + x_0 - C_s t$, the traveling-wave solution $\big(u(\xi), \eta(\xi)\big)$ we search in this section are the ones that $u(\xi)$ is of the form $u_\infty + A\,\mathrm{sech}^2\,(\lambda\xi)$ where u_∞, A, λ are constants and $\eta(\xi)$ tends to η_∞ as ξ tends to $-\infty$. Notice that $\big(u(\xi), \eta(\xi)\big) = (\alpha, \beta)$ are solutions of (1.1) for any constants α and β, one can restrict $\lambda > 0$ and search only for non-constant solutions. The exact solutions found in section 2 are in this form and can be recovered by using the method presented in this section. Solutions with non-zero η_∞ or u_∞ are not physically relevant, but will be useful in the theoretical and numerical study of the corresponding systems. One of the exact solutions found in this section for the regularized Boussinesq system

(1.4) has been used to verify the convergence rate of a proposed numerical schemes (cf [2]).

Starting from (2.2) and introducing functions $h(\xi) = \eta(\xi) - \eta_\infty$ and $v(\xi) = u(\xi) - u_\infty$, one obtains

$$
\begin{aligned}
&- C_s h' + v' + (vh)' + \eta_\infty v' + u_\infty h' + a v''' + bC_s h''' = 0, \\
&- C_s v' + h' + vv' + u_\infty v' + ch''' + dC_s v''' = 0.
\end{aligned} \tag{3.1}
$$

Since $h(\xi)$ and $v(\xi)$ tends to zero as ξ tends to $-\infty$, the system can be integrated once to obtain

$$
\begin{aligned}
&h(-C_s + v + u_\infty) + bC_s h'' = -v - \eta_\infty v - a v'', \\
&h + ch'' = C_s v - \frac{1}{2} v^2 - u_\infty v - dC_s v''.
\end{aligned} \tag{3.2}
$$

Assuming c is not zero and solving h and h'' yields

$$
h = g_1(v)/f(v) \quad \text{and} \quad h'' = g_2(v)/f(v), \tag{3.3}
$$

where

$$
f(v) = c(-C_s + v + u_\infty) - bC_s, \tag{3.4}
$$

$$
\begin{aligned}
g_1(v) &= c(-v - a v'' - \eta_\infty v) - bC_s(C_s v - \frac{1}{2} v^2 - dC_s v'' - u_\infty v), \\
g_2(v) &= v + a v'' + \eta_\infty v + (-C_s + v + u_\infty)(C_s v - \frac{1}{2} v^2 - dC_s v'' - u_\infty v).
\end{aligned} \tag{3.5}
$$

Differentiating the first equation in (3.3) twice with respect to ξ and using the second equation, one obtains a 4th-order ordinary differential equation with dependent variable $v(\xi)$

$$
f^2 g_2 = g_1'' f^2 - g_1 f'' f - 2 g_1' f f' + 2 g_1 (f')^2,
$$

which is of the form after substituting f, g_1 and g_2 into the equation

$$
\begin{aligned}
&(a_1 v^2 + a_2 v + a_3) v'''' + (a_4 v + a_5) v' v''' + (a_6 v + a_7)(v'')^2 \\
&\quad + a_8 (v')^2 v'' + (a_9 v^3 + a_{10} v^2 + a_{11} v + a_{12}) v'' + a_{13} (v')^2 \\
&\quad + a_{14} v^5 + a_{15} v^4 + a_{16} v^3 + a_{17} v^2 + a_{18} v = 0,
\end{aligned} \tag{3.6}
$$

where $a_i, i = 1, \cdots, 18$ depend on a, b, c, d, C_s, η_∞ and u_∞. With the help of

Mathematica, one sees

$$
\begin{aligned}
&a_1 = bc^2dC_s^2 - ac^3, \qquad a_2 = 2c(-ac + bdC_s^2)\big(-(b+c)C_s + cu_\infty\big),\\
&a_3 = (-ac + bdC_s^2)((b+c)C_s - cu_\infty)^2, \qquad a_4 = -2bc^2dC_s^2 + 2ac^3,\\
&a_5 = 2c(-ac + bdC_s^2)((b+c)C_s - cu_\infty), \qquad a_6 = a_4/2,\\
&a_7 = a_5/2, \qquad a_8 = -a_4, \qquad a_9 = c^2(d + b/2)C_s,\\
&a_{10} = -c(3cd + 2bd + 3bc/2 + 3b^2/2)C_s^2 - ac^2 + 3c^2C_su_\infty(b/2 + d),\\
&a_{11} = (-(b+c)C_s + cu_\infty)(-2ac - c^2(1+\eta_\infty) -\\
&\qquad\qquad (b^2 + 2bc + bd + 3cd)C_s^2 + c(2b + 3d)C_su_\infty),\\
&a_{12} = (-(b+c)C_s + cu_\infty)^2(-a - c(1+\eta_\infty) - (b+d)C_s(-C_s + u_\infty)),\\
&a_{13} = (-(b+c)C_s + cu_\infty)(2c^2(1+\eta_\infty) - (b-c)bC_s^2 - bcC_su_\infty),\\
&a_{14} = c^2/2, \qquad a_{15} = -bcC_s - 5c^2/2C_s + 5c^2u_\infty/2,\\
&a_{16} = -c^2(1+\eta_\infty) + (b^2/2 + 4bc + 9c^2/2)C_s^2\\
&\qquad\qquad - c(4b + 9c)C_su_\infty + 9c^2u_\infty^2/2,\\
&a_{17} = ((b+c)C_s - cu_\infty)(4c(1+\eta_\infty) - (3b + 7c)C_s^2\\
&\qquad\qquad + (3b + 14c)C_su_\infty - 7cu_\infty^2))/2,\\
&a_{18} = (-(b+c)C_s + cu_\infty)^2(-1 - \eta_\infty + (C_s - u_\infty)^2).
\end{aligned}
\tag{3.7}
$$

We therefore established the fact that in order to find a traveling-wave solution of (1.1), it is suffice to find a solution of the ordinary differential equation (3.6).

Theorem 3. *For given $a, b, c \neq 0, d, C_s$, u_∞ and η_∞, any solution $v(\xi)$ of (3.6) will provide a traveling-wave solution $u(x,t) = u_\infty + v(\xi)$, $\eta(x,t) = \eta_\infty + \frac{g_1(v(\xi))}{f(v(\xi))}$, where $\xi = x + x_0 - C_st$, f and g_1 are defined in (3.4) and (3.5). On the other hand, any traveling-wave solution $(\eta(x,t), u(x,t)) \equiv (\eta(\xi), u(\xi))$ of system (1.1) with $c \neq 0$ which approaches constants (η_∞, u_∞) as ξ approaches $-\infty$ has the property that $u(\xi) - u_\infty$ satisfies (3.6).*

Instead of solving $v(\xi)$ from (3.6) directly, which is very difficult if it is not impossible, the technique used by Kichennassamy and Olver in [9] for a single 5th-order equation is adopted here. In the present case, the ordinary differential equation has coefficients depending on C_s, u_∞ and η_∞ which are part of the unknowns. Assuming that $v(\xi)$ can be reconstructed as the solution of a simple first-order ordinary differential equation

$$w(v) = (v')^2. \tag{3.8}$$

Once the function $w(v)$ is known, $v(\xi)$ can be solved by a simple quadrature:

$$\int_{\alpha}^{v} \frac{ds}{\sqrt{w(s)}} = \xi + C.$$

Examples of solutions that have this form are the soliton and cnoidal wave solutions of the KDV equation [17] where the function $w(v)$ is a cubic polynomial. Solutions corresponding to $w(v)$ in other forms can be found in [18]. Using (3.8), one finds that for $v' \neq 0$

$$\begin{aligned} (v')^2 &= w, & v'' &= \frac{1}{2}w', \\ v'v''' &= \frac{1}{2}ww'', & v'''' &= \frac{1}{2}ww''' + \frac{1}{4}w'w'', \end{aligned} \tag{3.10}$$

where the primes on w indicate derivatives with respect to v. Substituting above relationships into (3.6), one finds w must satisfy a third-order ordinary differential equation

$$\begin{aligned} &(a_1 v^2 + a_2 v + a_3)(\frac{1}{2}ww''' + \frac{1}{4}w'w'') + \frac{1}{2}(a_4 v + a_5)ww'' \\ &\quad + \frac{1}{4}(a_6 v + a_7)(w')^2 + \frac{1}{2}a_8 ww' + \frac{1}{2}(a_9 v^3 + a_{10}v^2 + a_{11}v + a_{12})w' \\ &\quad + a_{13}w + a_{14}v^5 + a_{15}v^4 + a_{16}v^3 + a_{17}v^2 + a_{18}v = 0. \end{aligned} \tag{3.11}$$

For a solitary-wave solution of the form

$$v(\xi) = A\,\mathrm{sech}^2\,(\lambda\,\xi)\,, \quad \lambda > 0, \tag{3.12}$$

the corresponding function $w(v)$ must be a cubic polynomial:

$$w(v) = 4\lambda^2(v^2 - \frac{1}{A}v^3) \equiv \rho v^2 + \sigma v^3,$$

where $\rho = 4\lambda^2 > 0$ and $\sigma = -\frac{4\lambda^2}{A}$. Substituting $w(v)$ into (3.11), the left-hand side become a degree-five homogeneous polynomial in v. In order for v to be a nontrivial solution, all the coefficients have to be zero which yields that ρ, σ, C_s, u_∞ and η_∞ have to satisfy the following algebraic equations:

$$\begin{aligned} &a_{18} + a_{12}\rho + a_3\rho^2 = 0, \\ &a_{17} + (a_{11} + a_{13})\rho + (a_2 + a_5 + a_7)\rho^2 + 3a_{12}\sigma/2 + 15a_3\rho\sigma/2 = 0, \\ &a_{16} + a_{10}\rho + (a_1 + a_4 + a_6 + a_8)\rho^2 + (3a_{11}/2 + a_{13})\sigma \\ &\qquad + (15a_2/2 + 4a_5 + 3a_7)\rho\sigma + 15a_3\sigma^2/2 = 0, \\ &a_{15} + a_9\rho + 3a_{10}\sigma/2 + (15a_1/2 + 4a_4 \\ &\qquad 3a_6 + 5a_8/2) + (15a_2/2 + 3a_5 + 9a_7/4)\sigma^2 = 0, \\ &4a_{14} + 6a_9\sigma + (30a_1 + 12a_4 + 9a_6 + 6a_8)\sigma^2 = 0, \end{aligned} \tag{3.13}$$

which is the ultimate equations one has to solve to find solutions of the form (3.12).

In the cases that $c = 0$, one sees from the second equation of (3.2) that

$$h = C_s v - \frac{1}{2}v^2 - u_\infty v - dC_s v''.$$

Substituting h into the first equation of (3.2), one again obtains an ordinary differential equation on v. The same technique used on (3.6) can then be used again. We therefore can prove the following theorem.

Theorem 4. *For a specified system, that is for a given a, b, c and d, let $a_i, i = 1, \cdots, 18$ be as in* (3.7)*,*

(i) if $c \neq 0$ and $(\rho \geq 0, \sigma, C_s, u_\infty, \eta_\infty)$ is a solution of (3.13)*,*

or

(ii) if $c = 0$ and $(\rho \geq 0, \sigma, C_s, u_\infty, \eta_\infty)$ is a solution of

$$(3.14) \qquad \begin{aligned} &(a_{18} + a_{12}\rho + a_3\rho^2)/b^2 = 0, \\ &(2a_{17} + 2(a_{11} + a_{13})\rho + 3a_{12}\sigma + 15a_3\rho\sigma)/b^2 = 0, \\ &(2a_{16} + (3a_{11} + 2a_{13})\sigma + 15a_3\sigma^2)/b^2 = 0, \end{aligned}$$

then system (1.1) *has a traveling-wave solution which reads*

$$(3.15) \qquad \begin{aligned} &u(x,t) \equiv u(\xi) = u_\infty + v(\xi), \\ &\eta(x,t) \equiv \eta(\xi) = \begin{cases} \eta_\infty + g_1(v(\xi))/f(v(\xi)), & \text{if } c \neq 0 \\ \eta_\infty + C_s v - \frac{1}{2}v^2 - dC_s v'' - u_\infty v, & \text{if } c = 0 \end{cases} \end{aligned}$$

where $v(\xi) = -\frac{\rho}{\sigma} sech^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right)$, $\xi = x + x_0 - C_s t$, f and g_1 are defined in (3.4) *and* (3.5)*.*

We now present the exact solutions found for the examples listed in the Introduction, except for the integrable version of Boussinesq system (1.9) which does not possess solutions in the form of (3.15). Since (3.13) and (3.14) are systems of nonlinear algebraic equations on ρ, σ, C_s, u_∞ and η_∞, one can solve them with the help of *Mathematica*. The solutions we found may only be a part of the whole set of solutions. The equalities

$$\operatorname{sech}^2(x)(\cosh(2x) + 1) = 2, \quad \operatorname{sech}^4(x)(\cosh(4x) + 4\cosh(2x) + 3) = 8$$

are used to simplifying the solutions. We will also give examples of the exact solutions where $\eta_\infty = 0$ and (or) $u_\infty = 0$. The notation $\xi = x + x_0 - C_s t$ is used where x_0 and C_s are real constants. When it is not specified, x_0 and C_s are arbitrary constants and ρ is a non-negative constant.

Example 4. More exact solutions of Whitham's system ($a = -1/6$, $b = 0$, $c = 0$, $d = 1/2$).

Substituting a, b, c and d into (3.14) and solving the nonlinear algebraic system, one finds

$$u_\infty = \frac{1}{6C_s} + C_s - \frac{1}{2}C_s\rho, \quad \eta_\infty = -1 + \frac{1}{36C_s^2} + \frac{\rho}{12}, \quad \sigma = \frac{-2}{3C_s},$$

which leads to the exact solutions

$$\begin{aligned} u(x,t) &= u_\infty + \frac{3}{2}C_s\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right), \\ \eta(x,t) &= \eta_\infty - \frac{1}{4}\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right). \end{aligned}$$

Setting $\eta_\infty = 0$ and $u_\infty = 0$ yields $C_s^2 = \frac{1}{15}$ and $\rho = 7$, which recovers the exact solution (2.11) found in Example 1.

Example 5. Exact solutions of the regularized Boussinesq system ($a = 0, b = 1/6, c = 0, d = 1/6$).

Using Theorem 4, one finds two sets of exact traveling-wave solutions

$$\begin{aligned} u(x,t) &= C_s(1 - \frac{1}{6}\rho) + \frac{1}{2}C_s\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right), \\ \eta(x,t) &= -1, \end{aligned}$$

and

$$\begin{aligned} u(x,t) &= C_s(1 - \frac{5}{18}\rho) + \frac{5C_s\rho}{6}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right), \\ \eta(x,t) &= -1 + \frac{1}{81}C_s^2\rho^2 + \frac{5}{108}C_s^2\rho^2\left(2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right) - 3\mathrm{sech}^4\left(\frac{1}{2}\sqrt{\rho}\,\xi\right)\right). \end{aligned}$$

Setting $\rho = 18/5$ and $C_s^2 = \frac{25}{4}$, one finds $\eta_\infty = u_\infty = 0$, which leads to the exact traveling-wave solution

$$\begin{aligned} u(x,t) &= \pm\frac{15}{2}\mathrm{sech}^2\left(\frac{3}{\sqrt{10}}(x + x_0 \mp \frac{5}{2}t)\right), \\ \eta(x,t) &= \frac{15}{4}\left(2\mathrm{sech}^2\left(\frac{3}{\sqrt{10}}(x + x_0 \mp \frac{5}{2}t)\right) - 3\mathrm{sech}^4\left(\frac{3}{\sqrt{10}}(x + x_0 \mp \frac{5}{2}t)\right)\right), \end{aligned}$$

which we used in [2] to demonstrate numerically the rate of the convergence of a numerical method.

Example 6. Exact solutions of Coupled K-dV-regularized system ($b = c = 0$, $a = d = 1/6$).

Theorem 4 yields

$$u(x,t) = -\frac{1}{2C_s} + C_s - \frac{1}{6}C_s\rho + \frac{1}{2}C_s\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right),$$
$$\eta(x,t) = -1 + \frac{1}{4C_s^2} - \frac{1}{12}\rho + \frac{1}{4}\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right).$$

where setting $\rho = 6$ and $C_s^2 = \frac{1}{6}$ gives a pair of exact traveling-wave solutions with $\eta_\infty = 0$

$$u(x,t) = \mp\sqrt{\frac{3}{2}} \pm \sqrt{\frac{3}{2}}\mathrm{sech}^2\left(\sqrt{\frac{3}{2}}(x + x_0 \mp \frac{1}{\sqrt{6}}t)\right),$$
$$\eta(x,t) = \frac{3}{2}\mathrm{sech}^2\left(\sqrt{\frac{3}{2}}(x + x_0 \mp \frac{1}{\sqrt{6}}t)\right).$$

Example 7. Exact solutions of Boussinesq's original system ($a = 0,\, b = 0,\, c = 0,\, d = \frac{1}{6}$).

One can easily verify using Theorem 4 that

$$u(x,t) = (1 - \frac{1}{6}\rho)C_s + \frac{1}{2}C_s\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right),$$
$$\eta(x,t) = -1,$$

are exact solutions.

Example 8. Exact solutions of Coupled K-dV system ($a = 1/6,\, b = 0,\, c = 1/6,\, d = 0$).

Thanks to Theorem 4 again, one finds the exact solutions

$$u(x,t) = \mp\frac{\sqrt{2}}{2}(1 + \frac{1}{6}\rho) + C_s \pm \frac{1}{2\sqrt{2}}\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right),$$
$$\eta(x,t) = -\frac{1}{2}(1 + \frac{1}{6}\rho) + \frac{1}{4}\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right).$$

Setting $\rho = 6$ and $C_s = \pm\sqrt{2}$, one finds

$$u(x,t) = \pm\frac{3}{\sqrt{2}}\mathrm{sech}^2(\sqrt{\frac{3}{2}}(x + x_0 \mp \sqrt{2}t),$$
$$\eta(x,t) = -1 + \frac{3}{2}\mathrm{sech}^2(\sqrt{\frac{3}{2}}(x + x_0 \mp \sqrt{2}t).$$

Example 9. Exact solutions of Coupled regularized-K-dV system ($a = 0,\, b = 1/6,\, c = 1/6,\, d = 0$).

Using Theorem 4 yields the exact solutions

$$u(x,t) = C_s(\frac{1}{2} - \frac{1}{12}\rho) + \frac{1}{4}C_s\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right),$$
$$\eta(x,t) = -1 + C_s^2(\frac{1}{4} - \frac{1}{24}\rho) + \frac{1}{8}C_s^2\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right).$$

Setting $\rho = \frac{6(C_s^2-4)}{C_s^2}$ for $|C_s| \geq 2$, one obtains

$$u(x,t) = \frac{1}{2C_s} + \frac{3}{2C_s}(C_s^2 - 4)\mathrm{sech}^2\left(\sqrt{\frac{3}{2} - \frac{6}{C_s^2}}\,\xi\right),$$
$$\eta(x,t) = \frac{3}{4}(C_s^2 - 4)\mathrm{sech}^2\left(\sqrt{\frac{3}{2} - \frac{6}{C_s^2}}\,\xi\right).$$

Example 10. Exact solutions to one of the Bona-Smith systems ($a = 0$, $b = d = \frac{1}{3}$, $c = -\frac{1}{3}$).

Substituting a, b, c and d into (3.13) and using Theorem 4, one finds two sets of the exact solutions, which are

$$\begin{aligned} u(x,t) &= (1 - \frac{1}{3}\rho)C_s + C_s\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right), \\ \eta(x,t) &= -1, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} u(x,t) &= \frac{1}{2}C_s(1 - \frac{1}{3}\rho) + \frac{1}{2}C_s\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right), \\ \eta(x,t) &= -1 + \frac{1}{4}C_s^2(1 - \frac{1}{3}\rho) + \frac{1}{4}C_s^2\rho\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right). \end{aligned} \tag{3.17}$$

Setting $\rho = \frac{3(C_s^2-4)}{C_s^2}$ for $|C_s| \geq 2$ in (3.17), one obtains

$$u(x,t) = \frac{1}{2C_s} + \frac{3}{2C_s}(C_s^2 - 4)\mathrm{sech}^2\left(\frac{\sqrt{3}}{2}\frac{\sqrt{C_s^2-4}}{C_s}\,\xi\right),$$
$$\eta(x,t) = \frac{3(C_s^2-4)}{4}\mathrm{sech}^2\left(\frac{\sqrt{3}}{2}\frac{\sqrt{C_s^2-4}}{C_s}\,\xi)\right),$$

which recovers the exact solution found by Bona (cf. [14]).

For any systems with $a = 0$ and $d > 0$, which includes the Boussinesq's original system and the Bona-Smith system in Example 7 and 10, the following theorem provides a one-parameter family of exact traveling-wave solutions.

Theorem 5. *If $a = 0$, then*

$$u(x,t) = (1 - d\rho)C_s + 3dC_s\rho\, sech^2\left(\frac{1}{2}\sqrt{\rho}(x + x_0 - C_s t)\right),$$
$$\eta(x,t) = -1,$$

are exact solutions, where x_0 and C_s are arbitrary constants and $\rho \geq 0$.

Proof. Substituting $a = 0$ and $\eta(x,t) = -1$, (i.e. $h = 0$ and $\eta_\infty = -1$), into (3.2), the first equation is true for any u (i.e. for any v and u_∞) and the second equation becomes after being integrated once

$$(C_s - u_\infty)v - \frac{1}{2}v^2 - dC_s v'' = 0.$$

The same technique used on (3.6) can then be used (or applying the Lemma 1 directly) to find $u_\infty = (1 - d\rho)C_s$ and $A = 3dC_s\rho$. The theorem therefore holds.

A similar procedure can be performed to find solutions (u, η) where η is of the form $\eta_\infty + A\text{sech}^2(\lambda\xi)$. We will show in next section that such technique will enable us to find exact solutions for the integrable version of Boussinesq system (1.9).

4. Traveling-wave solutions with $\eta(x,t) = \eta_\infty + A\text{sech}^2(\lambda(x + x_0 - C_s t))$.

Instead of searching exact traveling-wave solutions $(\eta(x,t), u(x,t))$ where $u(x,t)$ has the form $u_\infty + A\text{sech}^2(\lambda(x + x_0 - C_s t))$, we now search for the exact solutions where $\eta(x,t)$ has the form $\eta_\infty + A\text{sech}^2(\lambda(x + x_0 - C_s t))$.

Similar to that in section 3, one needs first to find an ordinary differential equation for $\eta(\xi)$. In the cases that $a \neq 0$, multiplying the first equation in (3.2) by dC_s and subtracting a times the second equation, one obtains a quadratic equation of v,

$$-\frac{1}{2}av^2 + y(h)v + z(h) = 0,$$

where

$$y(h) = dC_s(1 + h + \eta_\infty) + aC_s - au_\infty,$$
$$z(h) = -dC_s^2 h + dC_s u_\infty h + bdC_s^2 h'' - ah - ach''.$$

Denoting $g(h) = y(h)^2 + 2az(h)$ and solving for v, one finds

$$v = \frac{y(h)}{a} \mp \frac{1}{a}g(h)^{\frac{1}{2}}, \tag{4.1}$$

which is then being substituted into (3.2) to yield

$$(1 + h + \eta_\infty)\frac{y(h)}{a} - C_s h + u_\infty h + bC_s h'' + dC_s h''$$
$$= \mp\left\{-\frac{1}{a}(1 + h + \eta_\infty)g(h)^{\frac{1}{2}} + \frac{1}{4}g(h)^{-3/2}g'(h)^2 - \frac{1}{2}g(h)^{-1/2}g''(h)\right\}.$$

Squaring both sides of the equation and multiplying $g(h)^3$, one finds a 4th-order ordinary differential equation for $h(\xi)$, which reads

$$
\begin{aligned}
&g(h)^3\{(1+h+\eta_\infty)\frac{y(h)}{a}+(u_\infty-C_s)h+(b+d)C_sh''\}^2 \\
&=\{-\frac{1}{a}(1+h+\eta_\infty)g(h)^2+\frac{1}{4}g'(h)^2-\frac{1}{2}g(h)g''(h)\}^2
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
&g'(h)=2dC_sy(h)h'+2a\{d(-C_s+u_\infty)C_sh'+bdC_s^2h'''-ah'-ach'''\}, \\
&g''(h)=2dC_sy(h)h''+2(dC_sh')^2+2a\{d(-C_s+u_\infty)C_sh''+ \\
&\qquad bdC_s^2h''''-ah''-ach''''\}.
\end{aligned}
$$

Hence one can obtain a traveling-wave solution by solving the ordinary differential equation (4.2).

In the cases that $a=0$, one finds from the first equation of (3.2) that

$$v(h)=-\frac{\tilde{z}(h)}{\tilde{y}(h)}$$

where

$$\tilde{y}(h)=1+h+\eta_\infty,\quad \tilde{z}(h)=-C_sh+u_\infty h+bC_sh''.$$

Substituting v into the second equation of (3.2) and multiplying $\tilde{y}(h)^3$, one obtains again a 4th-order ordinary differential equation on $h(\xi)$

$$
\begin{aligned}
&(C_s-u_\infty)\tilde{z}(h)\tilde{y}(h)^2+(h+ch'')\tilde{y}(h)^3+\frac{1}{2}\tilde{z}(h)^2\tilde{y}(h) \\
&-dC_s^2(h''+bh'''')\tilde{y}(h)^2+2dC_s^2h'\tilde{y}(h)(-h+bh''')-2dC_s\tilde{z}(h)h''=0.
\end{aligned}
\tag{4.3}
$$

The following theorem provides the relationship between the solutions of ordinary differential equations (4.2) or (4.3) and the traveling-wave solution of (1.1).

Theorem 6. *For given $(a,b,c,d,C_s,\ u_\infty,\ \eta_\infty)$, any solution $h(\xi)$ of* (4.2) *(or* (4.3) *if $a=0$) will provide a traveling-wave solution $\eta(x,t)=\eta_\infty+h(\xi)$, $u(x,t)=u_\infty+v(\xi)$, where $v(\xi)$ is defined by* (4.1) *(or $v(\xi)=-\frac{\tilde{z}(h(\xi))}{\tilde{y}(h(\xi))}$ if $a=0$). On the other hand, any traveling-wave solution $(\eta(x,t),u(x,t))\equiv(\eta(\xi),u(\xi))$ of* (1.1) *which approaches constants (u_∞,η_∞) as ξ approaches $-\infty$ has the property that $\eta(\xi)-\eta_\infty$ satisfies* (4.2) *(or* (4.3) *if $a=0$).*

Similar technique can be employed to find the solution h of (4.2) (or (4.3)) in the form of $A\mathrm{sech}^2(\lambda\xi)$. Setting $\rho=4\lambda^2,\sigma=-4\lambda^2/A$, and using the formulas similar to (3.10), one finds that

$$
\begin{aligned}
&(h')^2=\rho h^2+\sigma h^3,\quad h''=\rho h+\frac{3}{2}\sigma h^2, \\
&h'h'''=\rho^2h^2+4\rho\sigma h^3+3\sigma^2h^4,\quad h''''=\rho^2h+\frac{15}{2}\sigma\rho h^2+\frac{15}{2}\sigma^2h^3.
\end{aligned}
\tag{4.4}
$$

Substituting these into ordinary differential equation (4.2) (or (4.3)), one obtains a rather complicated homogeneous polynomial equation of degree ten (or degree five if $a = 0$). For the equation to have a nontrivial solution, all the coefficients have to be zero which leads to a system of nonlinear algebraic equations involving $a, b, c, d, u_\infty, \eta_\infty, \rho, \sigma$ and C_s. A solution of the algebraic system leads to a solution of the ordinary differential equation. Theorem 6 can then be used to find a traveling-wave solution.

Example 11. Exact solutions of Integrable version of Boussinesq system ($b = c = d = 0$).

With the procedure described above, one finds the exact solutions

$$\begin{aligned} u(\xi) &= C_s \pm \sqrt{-a\rho}\,\mathrm{sech}\left(\frac{1}{2}\sqrt{\rho}\,\xi\right), \\ \eta(\xi) &= -1 + \frac{1}{4}a\rho + \frac{1}{2}a\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right). \end{aligned} \qquad \text{if } a < 0,$$

and

$$\begin{aligned} u(\xi) &= C_s \pm \sqrt{a\rho}\tanh\left(\frac{1}{2}\sqrt{\rho}\,\xi\right), \\ \eta(\xi) &= -1 + \frac{1}{2}a\rho\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right). \end{aligned} \qquad \text{if } a > 0$$

As an example, one solution for $a = 1$ (setting $\rho = C_s^2$) is

$$\begin{aligned} \eta(x,t) &= -1 + \frac{1}{2}C_s^2\mathrm{sech}^2\left(\frac{C_s}{2}(x + x_0 - C_s t)\right), \\ u(x,t) &= C_s\left(1 \pm \tanh\left(\frac{C_s}{2}(x + x_0 - C_s t)\right)\right). \end{aligned}$$

Remark. The system with $a = 1$ has been studied using different methods. For example, a homogeneous balance method was used in [15] and [16]. In addition, a rational solution was found by Sach using Hirota form (cf. [13]). For the system with $a = 1/3$, a solution of the form $u(\xi) = A\mathrm{sech}^2\xi/(1 + B\mathrm{sech}^2\xi)$ was found using Jacobin Cosine elliptic function (cf. [10]).

5. Conclusion.

In this paper we have showed that in order to find the exact traveling-wave solutions of a given system of partial differential equations of the form (1.1), it is sufficient to find a solution of a nonlinear fourth-order ordinary differential equation. In consequence, any method for finding exact solutions of ordinary differential equation can be used to generate exact traveling-wave solutions of (1.1). For instance, one can use Weierstrass elliptic function to construct periodic wave solutions

[8], inverse scattering theory to find N-soliton solutions if they exist, and use certain ansatz equations, for instance the ones in [18], to find traveling wave solutions in some prescribed form.

We have found in this paper all the solutions $(\eta(x,t), u(x,t))$ where either $u(x,t)$ is of the form $u_\infty + A\,\mathrm{sech}^2(\lambda(x+x_0-C_st))$ or $\eta(x,t)$ is of the form $\eta_\infty + A\,\mathrm{sech}^2(\lambda(x+x_0-C_st))$ for each system listed in the Introduction. The method presented here can also be used for the systems

$$\tilde{\eta}_t + (u\tilde{\eta})_x + au_{xxx} - b\tilde{\eta}_{xxt} = 0,$$

$$u_t + \tilde{\eta}_x + uu_x + c\tilde{\eta}_{xxx} - du_{xxt} = 0,$$

by observing that under the transformation $\tilde{\eta}(x,t) = \eta(x,t) + 1$ these systems become (1.1).

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