# 4.7 l'Hospital's Rule

revisit an old problem: $\lim_{x \to 2} \frac{x^2+x-6}{x^2-4} \to \frac{4+2-6}{4-4} \to \frac{0}{0}$ indeterminate form of $\frac{0}{0}$

old way: factoring and canceling

$$\lim_{x \to 2} \frac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \lim_{x \to 2} \frac{x+3}{x+2} = \boxed{\frac{5}{4}}$$

today, we'll learn l'Hospital's Rule to handle

$\frac{0}{0}$ or $\frac{\infty}{\infty}$ when computing limits

$\frac{\text{small \#}}{\text{small \#}}$ $\qquad$ $\frac{\text{big \#}}{\text{big \#}}$

## l'Hospital's Rule

if $\lim_{x \to a} \frac{f(x)}{g(x)} \to \frac{0}{0}$ or $\frac{\infty}{\infty}$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

named after Marquis de l'Hospital

(old spelling l'Hôpital)

1661-1704

published the first calculus textbook

in 1696

**example** $\lim_{x \to 2} \frac{x^2+x-6}{x^2-4} \to \frac{0}{0}$ so we can use l'Hospital's Rule

$\overset{L}{=} \lim_{x \to 2} \frac{2x+1}{2x}$ now let $x=2$

this L means I used l'Hospital's Rule this step

$= \frac{2(2)+1}{2(2)} = \boxed{\frac{5}{4}}$

**example** $\lim_{x\to 0} \dfrac{x+\sin 2x}{x-\sin 2x} \xrightarrow{\text{as } x\to 0} \dfrac{0}{0}$

MUST see $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to use l'Hospital's Rule

$\stackrel{L}{=} \lim_{x\to 0} \dfrac{1+\cos(2x)\cdot 2}{1-\cos(2x)\cdot 2}$ now try letting $x=0$

$= \dfrac{1+\cos(0)\cdot 2}{1-\cos(0)\cdot 2} = \dfrac{1+2}{1-2} = \dfrac{3}{-1} = \boxed{-3}$

**example** $\lim_{x\to 0} \dfrac{x+\sin 2x}{x-\cos 2x} \xrightarrow{\text{as } x\to 0} \dfrac{0}{0-1} = \dfrac{0}{-1}$

NOT $\frac{0}{0}$ or $\frac{\infty}{\infty}$

DO NOT use l'Hospital's Rule

$= \dfrac{0}{-1} = \boxed{0}$

<u>example</u> which of the following grows faster as $x \to \infty$

$f(x) = x^2$ or $g(x) = 2^x$ ?

if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x^2}{2^x} = 0$ then that means $2^x$ eventually becomes much larger than $x^2$ so that also means $2^x$ grows faster.

if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x^2}{2^x} = \infty$ then $x^2$ eventually is much bigger so must be growing faster

if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x^2}{2^x} \to \frac{\infty}{\infty} = ?$ we can use l'Hospital's Rule to see what is going on

$$\lim_{x\to\infty} \frac{x^2}{2^x} \xrightarrow{\text{as } x\to\infty} \frac{\infty}{\infty} = ?$$

Since it is $\frac{\infty}{\infty}$ or $\frac{0}{0}$ we can use l'Hospital's Rule

$$\overset{L}{=} \lim_{x\to\infty} \frac{2x}{2^x \cdot \ln 2} \xrightarrow{\text{as } x\to\infty} \frac{\infty}{\infty}$$

$$\frac{d}{dx}(a^x) = a^x \cdot \ln a$$

after applying l'Hospital's Rule, the limit remains $\frac{\infty}{\infty}$ (or $\frac{0}{0}$)

when this happens, use the Rule again (until the limit is no longer $\frac{\infty}{\infty}$ or $\frac{0}{0}$)

$$\overset{L}{=} \lim_{x\to\infty} \frac{2}{\underbrace{(2^x \cdot \ln 2)}_{\text{deriv. of } 2^x} \cdot \ln 2} \xrightarrow{\text{as } x\to\infty} \frac{2}{\infty}$$

Do <u>NOT</u> use l'Hospital's Rule again since it is not $\frac{\infty}{\infty}$ or $\frac{0}{0}$

$$= \boxed{0}$$

so $2^x$ grows faster than $x^2$

other indeterminate forms : $\underbrace{\infty - \infty}_{\substack{\text{not necessarily } 0 \\ \text{means big \# - big \#}}}$ , $\underbrace{1^{\infty}}_{(\text{\# close to } 1)^{\text{big \#}}}$ , $\underbrace{\infty \cdot 0}_{\substack{\text{means} \\ (\text{big \#})(\text{small \#})}}$

<u>example</u> $\lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) \to \infty - \infty = ?$

transform into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then use l'Hospital

$$= \lim_{x \to 0^+} \left( \frac{(e^x - 1)}{x \cdot (e^x - 1)} - \frac{x}{x \cdot (e^x - 1)} \right)$$

$$= \lim_{x \to 0^+} \frac{e^x - 1 - x}{x(e^x - 1)} \xrightarrow{\text{as } x \to 0} \frac{0}{0}$$ now we can use l'Hospital

$$\overset{L}{=} \lim_{x \to 0^+} \frac{e^x - 1}{xe^x + (e^x - 1)} \xrightarrow{\text{as } x \to 0} \frac{0}{0}$$ l'Hospital again

$$\overset{L}{=} \lim_{x \to 0^+} \frac{e^x}{xe^x + e^x + e^x} \xrightarrow{\text{as } x \to 0} \boxed{\frac{1}{2}}$$ this is the limit

example $\lim_{x\to 0^+} (1+x)^{\cot x} \xrightarrow{\text{as } x\to 0} 1^\infty = ?$

transform into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then use l'Hospital

$$\lim_{x\to 0^+} \underbrace{(1+x)^{\cot x}}_{y}$$

so, we want to know $\lim_{x\to 0^+} y$

$$\ln y = \ln (1+x)^{\cot x}$$

$$= \cot x \cdot \ln(1+x) = \frac{\ln(1+x)}{\tan x}$$

now notice as $x\to 0^+$ $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+x)}{\tan x} \to \frac{0}{0}$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+x)}{\tan x} \overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{\sec^2 x} \xrightarrow{\text{as } x\to 0} \frac{1}{1} = 1$$

not done yet!

we want $\lim_{x\to 0^+} y$ but found $\lim_{x\to 0^+} \ln y = 1$

we know $y = e^{\ln y}$

so, if $\ln y \to 1$, then $y \to e^1 \to e$

so, the limit is $\boxed{e}$

<u>example</u> $\lim_{x\to\infty} x^2\left(\frac{1}{x} - \sin\frac{1}{x}\right) \xrightarrow{\text{as } x\to\infty} \infty\cdot 0 = ?$

transform into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then use l'Hospital

$$= \lim_{x\to\infty} x^2\left(\frac{1}{x} - \sin\frac{1}{x}\right)$$

note $x^2 = \dfrac{1}{\frac{1}{x^2}}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} - \sin\frac{1}{x}}{\frac{1}{x^2}} \xrightarrow{\text{as } x\to\infty} \frac{0}{0}$$

now use l'Hospital

(some steps of deriv. skipped)

$$\stackrel{L}{=} \lim_{x\to\infty} \frac{1-\cos\left(\frac{1}{x}\right)}{\frac{2}{x}} \xrightarrow{\text{as } x\to\infty} \frac{0}{0}$$

l'Hospital's again

(deriv. steps skipped)

$$\stackrel{L}{=} \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{2} \xrightarrow{\text{as } x\to\infty} \frac{0}{2}$$

no more l'Hospital's

$$= \boxed{0}$$

that's the limit