Find a such that $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + a\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ are perpendicular.

A. 3 B. 2 C. 1 D. -1 E. -2

$\boxed{\text{if } \vec{u} \perp \vec{w}, \text{ then } \vec{u} \cdot \vec{w} = 0}$

$\vec{u} \cdot \vec{w} = \langle 2, -1, a \rangle \cdot \langle 1, 4, 2 \rangle = 0$

$(2)(1) + (-1)(4) + (a)(2) = 0$

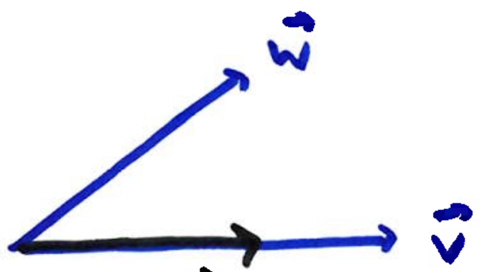
$2 - 4 + 2a = 0$

$2a = 2$

$a = 1$

If $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{k}$, find $|\text{proj}_{\mathbf{v}}(\mathbf{w})|$.

A. $1/\sqrt{3}$ B. $\sqrt{3}$ C. $\sqrt{3}/5$ D. $2\sqrt{3}$ E. $\sqrt{3}/2$



$\text{proj}_{\vec{v}}\vec{w} = \text{scal}_{\vec{v}}\vec{w}\,\dfrac{\vec{v}}{|\vec{v}|}$

$|\text{proj}_{\vec{v}}\vec{w}| = \text{scal}_{\vec{v}}\vec{w}$

$\cos\theta = \dfrac{\text{scal}_{\vec{v}}\vec{w}}{|\vec{w}|}$

$\text{scal}_{\vec{v}}\vec{w} = |\vec{w}|\cos\theta$

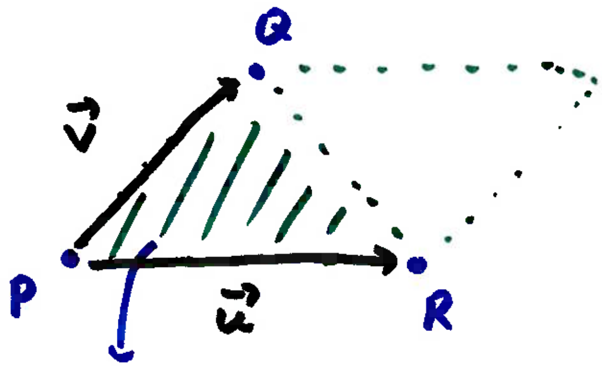
$\boxed{\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta}$

$\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$

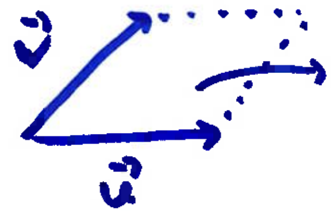
$\text{scal}_{\vec{v}}\vec{w} = \cancel{|\vec{w}|}\dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cancel{|\vec{w}|}} = \dfrac{\langle 1,1,1\rangle\cdot\langle 2,0,-1\rangle}{\sqrt{1^2+1^2+1^2}} = \dfrac{1}{\sqrt{3}}$

Find the area of the triangle with vertices $P = (0,0,0)$, $Q = (1,2,1)$, and $R = (2,1,-1)$.

A. $\sqrt{27}$ B. $\dfrac{\sqrt{27}}{2}$ C. $\dfrac{\sqrt{11}}{2}$ D. $\sqrt{19}$ E. $\dfrac{\sqrt{3}}{2}$



area $= \frac{1}{2}|\vec{u} \times \vec{v}|$



area $= |\vec{v} \times \vec{u}|$

$\vec{u} = \vec{PR} = \langle 2-0, 1-0, -1-0 \rangle = \langle 2, 1, -1 \rangle$

$\vec{v} = \vec{PQ} = \langle 1, 2, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$= \vec{i}(1 - -2) - \vec{j}(2 - -1) + \vec{k}(4 - 1)$

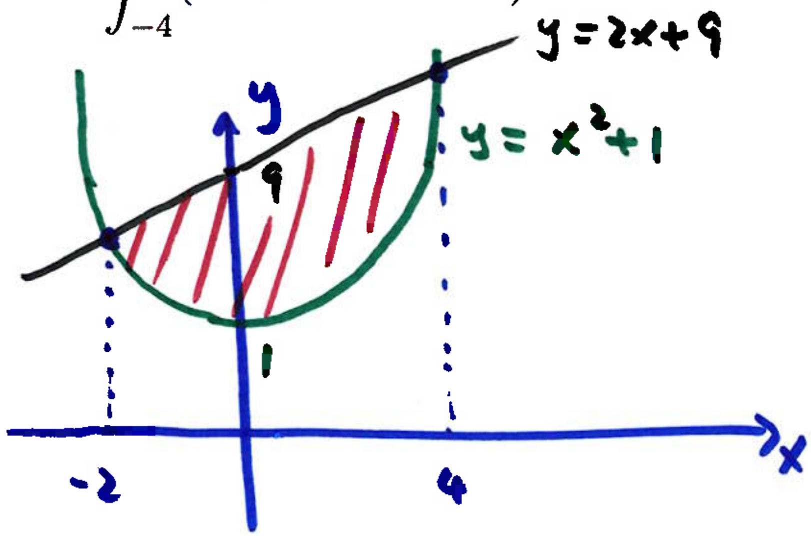
$= \langle 3, -3, 3 \rangle$

$|\vec{u} \times \vec{v}| = \sqrt{9+9+9} = \sqrt{27}$ = area of parallelogram

triangle: $\dfrac{\sqrt{27}}{2}$

The area of the region enclosed by the curves $y = x^2 + 1$ and $y = 2x + 9$ is given by

A. $\int_{-2}^{4} (x^2 + 1 - 2x - 9)\,dx$ B. $\int_{-2}^{4} (2x + 9 - x^2 - 1)\,dx$ C. $\int_{-2}^{2} (2x + 9 - x^2 - 1)\,dx$ D. $\int_{-4}^{2} (2x + 9 - x^2 - 1)\,dx$ E. $\int_{-4}^{2} (x^2 + 1 - 2x - 9)\,dx$



intersection: $x^2+1 = 2x+9$

$x^2-2x-8=0$

$(x-4)(x+2)=0$

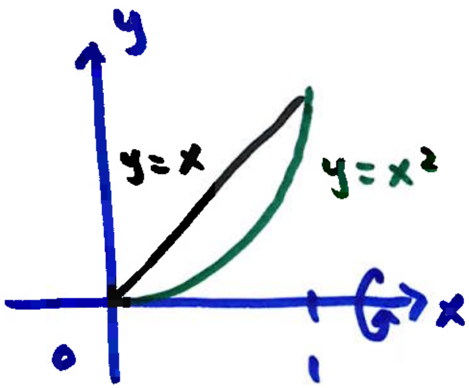
$x=-2,\ x=4$

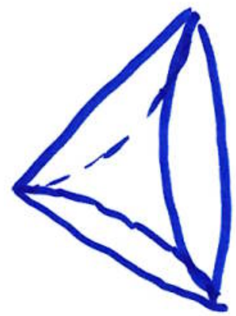
area $= \int_{\text{left}}^{\text{right}} (\text{top} - \text{bottom})\,dx = \int_{-2}^{4} \left[(2x+9)-(x^2+1)\right]dx$

$= \int_{-2}^{4} (2x+9-x^2-1)\,dx$

Let R be the region between the graphs of $y = x^2$ and $y = x$. Find the volume of the solid generated by revolving R about the x–axis.

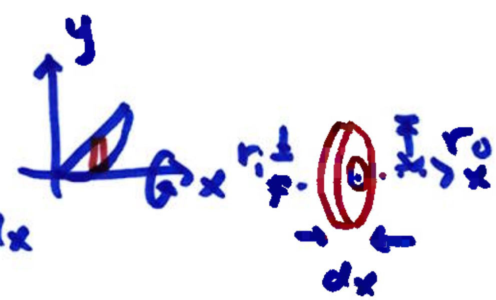
A. $\frac{\pi}{6}$ B. $\frac{\pi}{12}$ C. $\frac{\pi}{4}$ D. $\frac{\pi}{15}$ E. $\frac{2\pi}{15}$





two methods : disk/washer
shell

disk : rectangle perpendicular to axis of revolution



disk volume = $\cancel{\pi r^2 dx}$ $\left[\pi (r_{out})^2 - \pi (r_{in})^2\right] dx$

$= \left[\pi (x)^2 - \pi (x^2)^2\right] dx$

$= \pi (x^2 - x^4)\, dx$

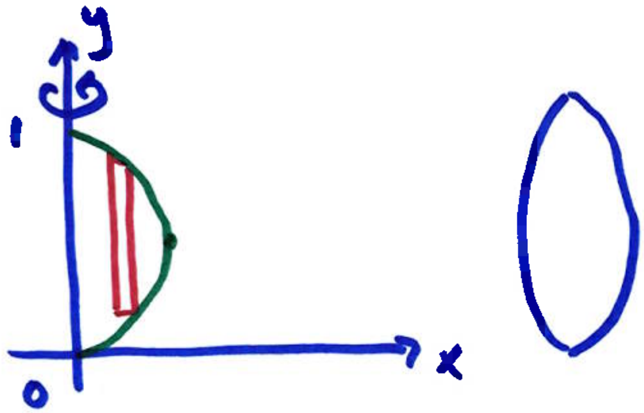
total volume $= \int_0^1 \pi (x^2 - x^4)\, dx = \pi \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{2}{15}\pi$

y-axis

If R is the region bounded by the curves $x = 0$ and $x = y - y^2$, and if R is revolved around the y–axis, then the volume of the solid is

A. $\frac{\pi}{15}$ B. $\frac{\pi}{30}$ C. $\frac{\pi}{12}$ D. $\frac{\pi}{3}$ E. $\frac{\pi}{6}$



$x = y - y^2$ parabola opening left/right

intercepts: $0 = y - y^2$

$= y(1-y)$

$y = 0, \; y = 1$

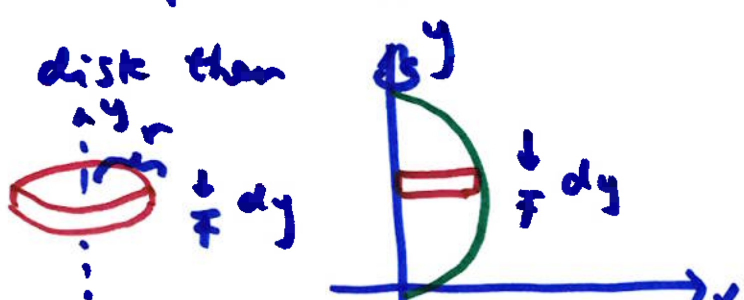
at $y = \frac{1}{2}$ $\quad x = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

this time, lets use shell : rectangle // axis

but the same curve is at the ends of the rectangle, not convenient

difficult to express height

so, back to disk then



disk volume $= \pi (r)^2 dy$

$= \pi (y - y^2)^2 dy$

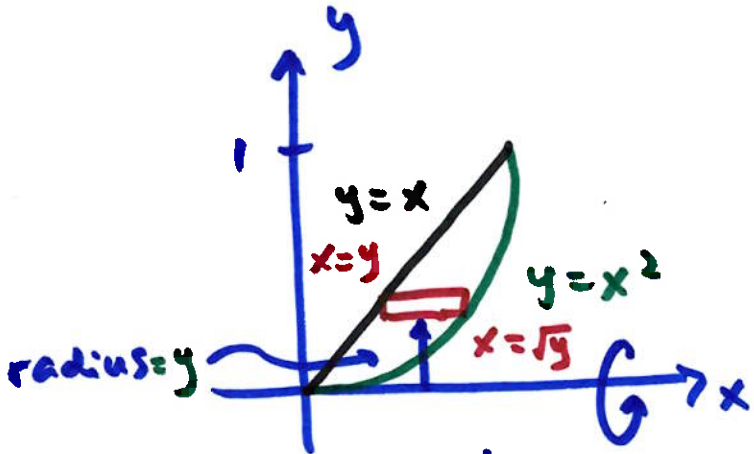
integrate buttom $(y=0)$ to top $(y=1)$

$$\int_0^1 \pi (y-y^2)^2\, dy = \pi \int_0^1 y^2 - 2y^3 + y^4\, dy$$

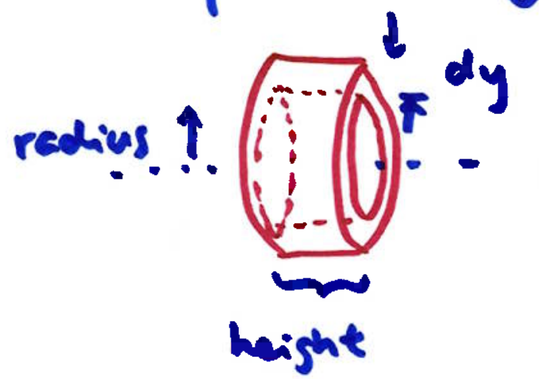
$$= \pi \left(\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \pi \left(\frac{10 - 15 + 6}{30} \right) = \pi \left(\frac{1}{30} \right)$$

R bounded by $y = x^2$, $y = x$, around x-axis but shell this time

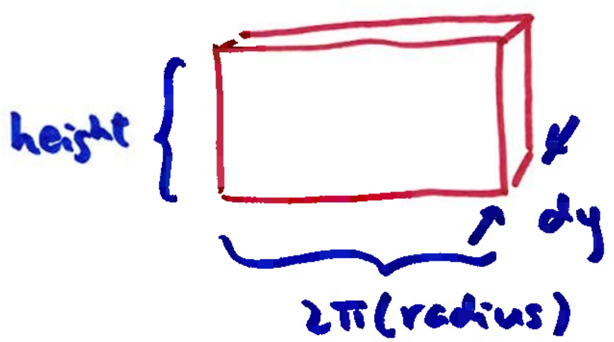


rectangle // axis of rev



$$\text{volume} = 2\pi\,(\text{radius})\,(\text{height})\,dy$$

$$= 2\pi\,(y)\,(\sqrt{y} - y)\,dy = 2\pi\,(y^{3/2} - y^2)\,dy$$



volume of whole thing

$$\int_{0\ \text{bottom}}^{1\ \text{top}} 2\pi\,(y^{3/2} - y^2)\,dy = 2\pi\left(\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right)\Big|_0^1$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = 2\pi\left(\frac{1}{15}\right)$$

$$= \frac{2\pi}{15}$$

A force of 4 lb. is required to stretch a spring $1/2$ ft. beyond its natural length. How much work is required to stretch the spring from its natural length to 2 ft.

$\equiv$ beyond natural

A. 8 ft–lbs. B. 12 ft–lbs. C. 16 ft–lbs. D. 24 ft–lbs. E. 32 ft–lbs.

spring: $F = kx$ ← change from natural length

force ↗ ↑ spring constant

here, $4 = k \cdot \frac{1}{2}$ ← stretch $\frac{1}{2}$ ft beyond natural ($-\frac{1}{2}$ if compressed)

so, $k = 8$

work: $\int \text{force} = \int_{\text{start-natural}}^{\text{end-natural}} kx\,dx = \int_0^2 8x\,dx = 4x^2\Big|_0^2 = 16$